Information-Regularized Optimal LQG Control

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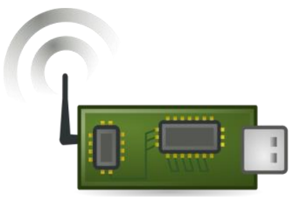
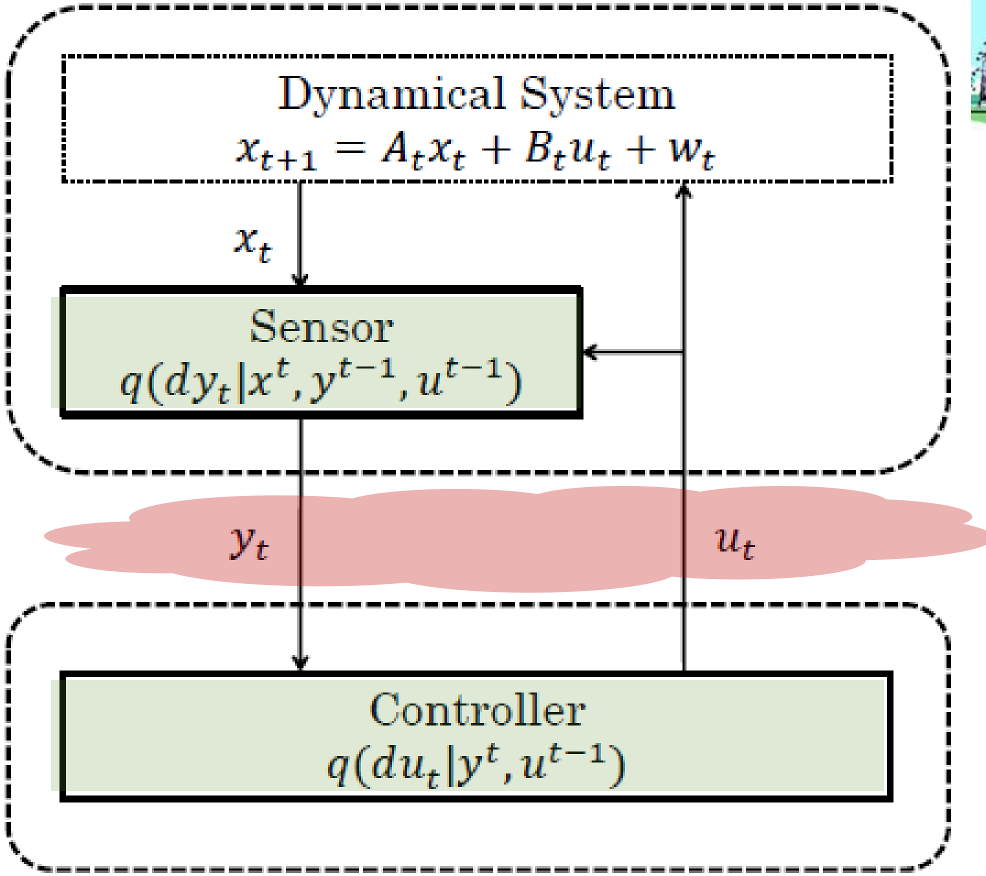
Motivation

A

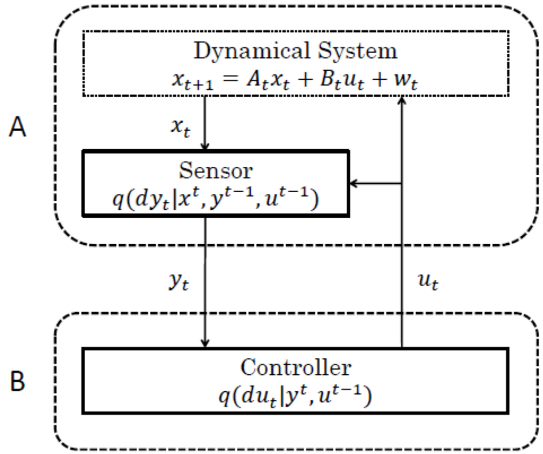




B

Optimal **sensing (A)** and **controller (B)** design in hostile or constrained networked environments

Problem Formulation



- Linear stochastic plant (**given**):

$$\mathbf{x}_{t+1} = A_t\mathbf{x}_t + B_t\mathbf{u}_t + \mathbf{w}_t,\ t = 1, \cdots, T$$
$$\mathbf{x}_1 \sim \mathcal{N}(0, P_{1|0})$$
$$\mathbf{w}_t \sim \mathcal{N}(0, W_t)$$

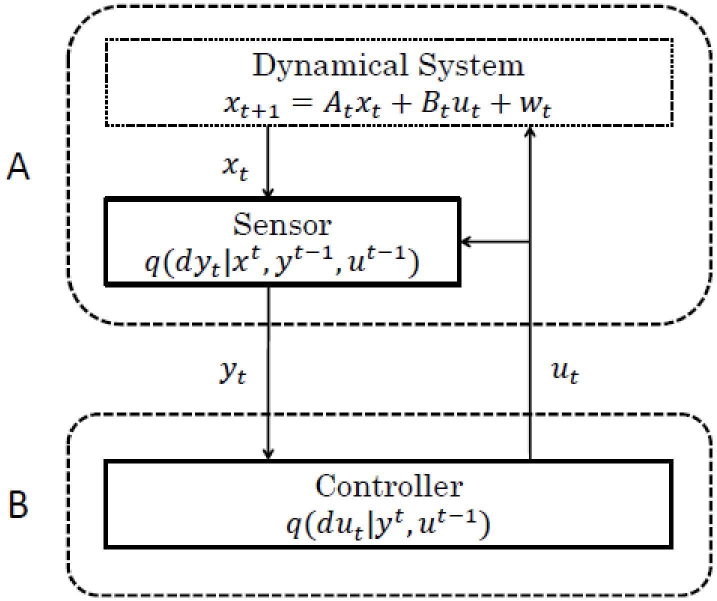
- *Linear* sensing policy in set $\Pi_s{}^{\mathrm{lin}}$ (**to be determined**):

$$\mathbf{y}_t = C_t\mathbf{x}_t + \mathbf{v}_t,\ \mathbf{v}_t \sim \mathcal{N}(0, V_t)$$

- Control policy in set Π_c (**to be determined**):

$$\{q(du_t|y^t, u^{t-1})\}_{t=1}^{T}$$
$$y^t \triangleq (y_1, y_2, \ldots, y_t),\ \text{etc.}$$

Performance Criteria



- Overall control objective:

$$\min_{\pi_s^{\text{lin}} \times \pi_c} J_{\text{cont}} + J_{\text{info}}$$

- Control performance:

$$J_{\text{cont}} \triangleq \sum_{t=1}^{T} \frac{1}{2} \mathbb{E} \left(\|\mathbf{x}_{t+1}\|_{Q_t}^2 + \|\mathbf{u}_t\|_{R_t}^2 \right)$$

- Price of communication (∝ transmitted "bits" A→B):

$$J_{\text{info}} \triangleq \sum_{t=1}^{T} \gamma_t I(\mathbf{x}_t; \mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{u}^{t-1})$$

Comparison to Regular LQG Control

- In regular LQG control, sensing policy is given:

$$\mathbf{y}_t = C_t \mathbf{x}_t + \mathbf{v}_t, \; \mathbf{v}_t \sim \mathcal{N}(0, V_t)$$

- Control performance alone

$$J_{\text{cont}} \triangleq \sum_{t=1}^{T} \frac{1}{2} \mathbb{E}\left(\|\mathbf{x}_{t+1}\|_{Q_t}^2 + \|\mathbf{u}_t\|_{R_t}^2 \right)$$

is optimized with LQR, i.e., $\mathbf{y}_t = \mathbf{x}_t$

- …but leads to infinite number of transmitted bits:

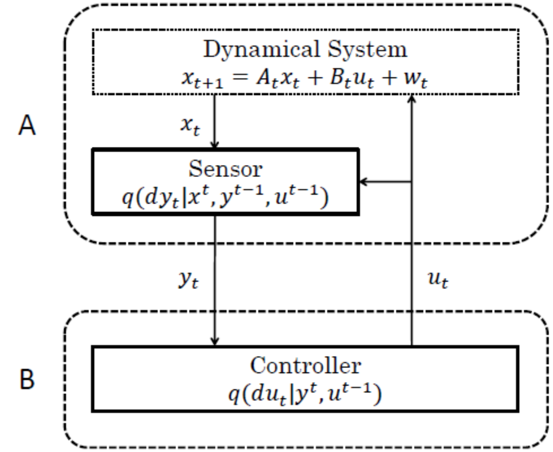
$$J_{\text{info}} \triangleq \sum_{t=1}^{T} \gamma_t I(\mathbf{x}_t; \mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{u}^{t-1}) \to \infty$$

Contributions and Related Work

- Jointly optimal control and (linear) sensing policies by means of a semi-definite program
- $\Rightarrow$ trade-offs between control performance and information loss

- Large body of literature on optimal quantization and encoding for the LQG problem [Lewis and Tou, 1965; Curry, 1969; Borkar, 1993; Borkar and Mitter, 1997; Nair *et al*., 2007; Bao *et al*., 2011; Yüksel and Basar, 2013; Yüksel, 2014,…]
- **Here:** Focus on the optimal "test channel" design (channel is not given), and *quantization is not considered*

Step-by-Step Solution



Separation principle:

1. Optimal state feedback → $\mathbf{u}_t = K_t \mathbf{x}_t$
2. Covariance scheduling → $P_{t|t}$
3. Sensor design → $\{C_t, V_t\}$ (incl. dimension)
4. Filter design → Kalman gain L_t
5. Policy construction

$$\hat{\mathbf{x}}_{t+1|t} = A_t \hat{\mathbf{x}}_t + B_t \mathbf{u}_t$$
$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t-1} + L_t(\mathbf{y}_t - C_t \hat{\mathbf{x}}_{t|t-1})$$
$$\mathbf{u}_t = K_t \hat{\mathbf{x}}_t$$

Proof Idea (1)

- For fixed sensor-controller policy in $\Pi_s{}^{\text{lin}} \times \Pi_c$:

$$p(dx_t|y^{t-1}, u^{t-1}) \sim \mathcal{N}(\hat{\mathbf{x}}_{t|t-1}, P_{t|t-1})$$
$$p(dx_t|y^{t}, u^{t-1}) \sim \mathcal{N}(\hat{\mathbf{x}}_{t}, P_{t|t})$$

- Transmitted information independent of $\mathbf{u}_t$:

$$\begin{aligned} I(\mathbf{x}_t;\mathbf{y}_t|\mathbf{y}^{t-1},\mathbf{u}^{t-1}) &= h(\mathbf{x}_t|\mathbf{y}^{t-1},\mathbf{u}^{t-1}) - h(\mathbf{x}_t|\mathbf{y}^{t},\mathbf{u}^{t-1}) \\ &= \frac{1}{2}\log\det P_{t|t-1} - \frac{1}{2}\log\det P_{t|t}. \end{aligned}$$

- Two-player Stackelberg game between sensor agent A (leader) and controller agent B (follower):

$$\min_{\pi_s^{\text{lin}}\times\pi_c} J_{\text{cont}} + J_{\text{info}} \quad = \quad \min_{\pi_s^{\text{lin}}} \left(J_{\text{info}} + \min_{\pi_c} J_{\text{cont}} \right)$$

Proof Idea (2)

Lemma 1: For every fixed $\{q_{\mathbf{y}_t|\mathbf{x}_t}\}_{t=1}^{T} \in \pi_s^{\text{lin}}$, the certainty equivalence controller $\mathbf{u}_t = K_t\hat{\mathbf{x}}_t$ where $\hat{\mathbf{x}}_t = \mathbb{E}(\mathbf{x}_t|\mathbf{y}^t, \mathbf{u}^{t-1})$ is an optimizer of $\min_{\pi_c} J_{\text{cont}}$. Moreover,

$$\min_{\pi_c} J_{\text{cont}} = \frac{1}{2}\text{Tr}(N_1 P_{1|0}) + \frac{1}{2}\sum_{k=1}^{T}\big(\text{Tr}(W_k S_k) + \text{Tr}(\Theta_k P_{k|k})\big).$$

Standard backward Riccati recursion for optimal LQR:

$$S_t = \begin{cases} Q_t & \text{if } t = T \\ Q_t + N_{t+1} & \text{if } t = 1, \cdots, T-1 \end{cases}$$

$$M_t = B_t^\top S_t B_t + R_t$$

$$N_t = A_t^\top (S_t - S_t B_t M_t^{-1} B_t^\top S_t) A_t$$

$$K_t = -M_t^{-1} B_t^\top S_t A_t$$

$$\Theta_t = K_t^\top M_t K_t$$

Proof Idea (3)

$$J_{\text{info}} + \min_{\pi_c} J_{\text{cont}} = \sum_{t=1}^{T-1} \left(\frac{1}{2}\text{Tr}(\Theta_t P_{t|t}) + \frac{\gamma_{t+1}}{2} \log\det P_{t+1|t} - \frac{\gamma_t}{2} \log\det P_{t|t} \right) + \frac{1}{2}\text{Tr}(\Theta_T P_{T|T}) - \frac{\gamma_T}{2} \log\det P_{T|T} + c$$

Optimize the estimation covariance $\{P_{t|t}\}$, $\{P_{t|t-1}\}$ ⇒ Semi-definite program:

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} \left(\frac{1}{2}\text{Tr}(\Theta_t P_{t|t}) - \frac{\gamma_t}{2} \log\det \Pi_t \right) + C \\
\text{s.t.} \quad & \Pi_t \succ 0, \quad t = 1, \cdots, T \\
& P_{t+1|t+1} \preceq A_t P_{t|t} A_t^\top + W_t, \quad t = 1, \cdots, T-1 \\
& P_{1|1} \preceq P_{1|0}, P_{T|T} = \Pi_T \\
& \begin{bmatrix} P_{t|t} - \Pi_t & P_{t|t} A_t^\top \\ A_t P_{t|t} & W_t + A_t P_{t|t} A_t^\top \end{bmatrix} \succeq 0, \quad t = 1, \cdots, T-1
\end{aligned}$$

Proof Idea (4)

Sensor policy:

1. Set $r_t = \mathrm{rank}(P_{t|t}^{-1} - P_{t|t-1}^{-1})$ where

$$P_{t|t-1} \triangleq A_{t-1} P_{t-1|t-1} A_{t-1}^\top + W_{t-1}, t = 2, \cdots, T$$

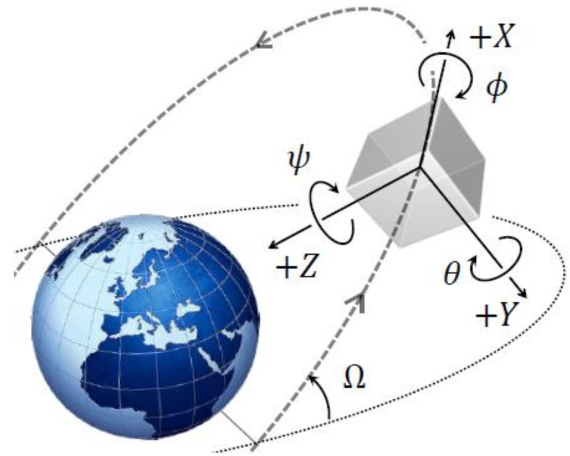
2. Choose $C_t \in \mathbb{R}^{r_t \times n_t}$ and $V_t \in \mathbb{S}_{++}^{r_t}$ such that

$$C_t^\top V_t^{-1} C_t = P_{t|t}^{-1} - P_{t|t-1}^{-1}$$

(use singular value decomposition, for example)

Done! We have determined $\{K_t, C_t, V_t, L_t\}$!

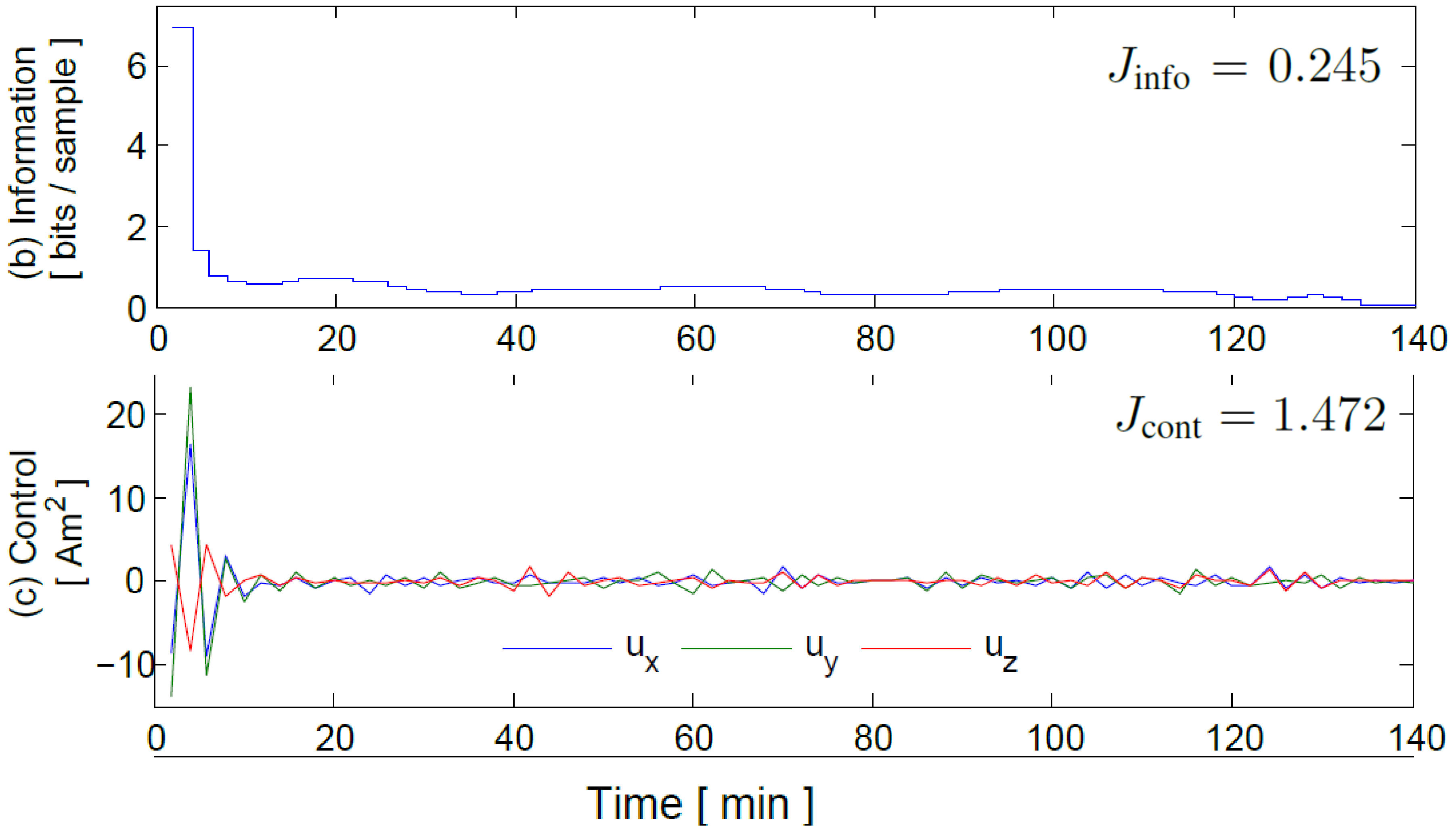
Example:
Satellite Control with Limited Communication



$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\omega}_\phi \\ \dot{\omega}_\theta \\ \dot{\omega}_\psi \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-4\omega_0^2\sigma_x & 0 & 0 & 0 & 0 & \omega_0(1-\sigma_x) \\
0 & 3\omega_0^2\sigma_y & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_0^2\sigma_z & -\omega_0(1+\sigma_z) & 0 & 1
\end{bmatrix}
\begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_\phi \\ \omega_\theta \\ \omega_\psi \end{bmatrix}
$$

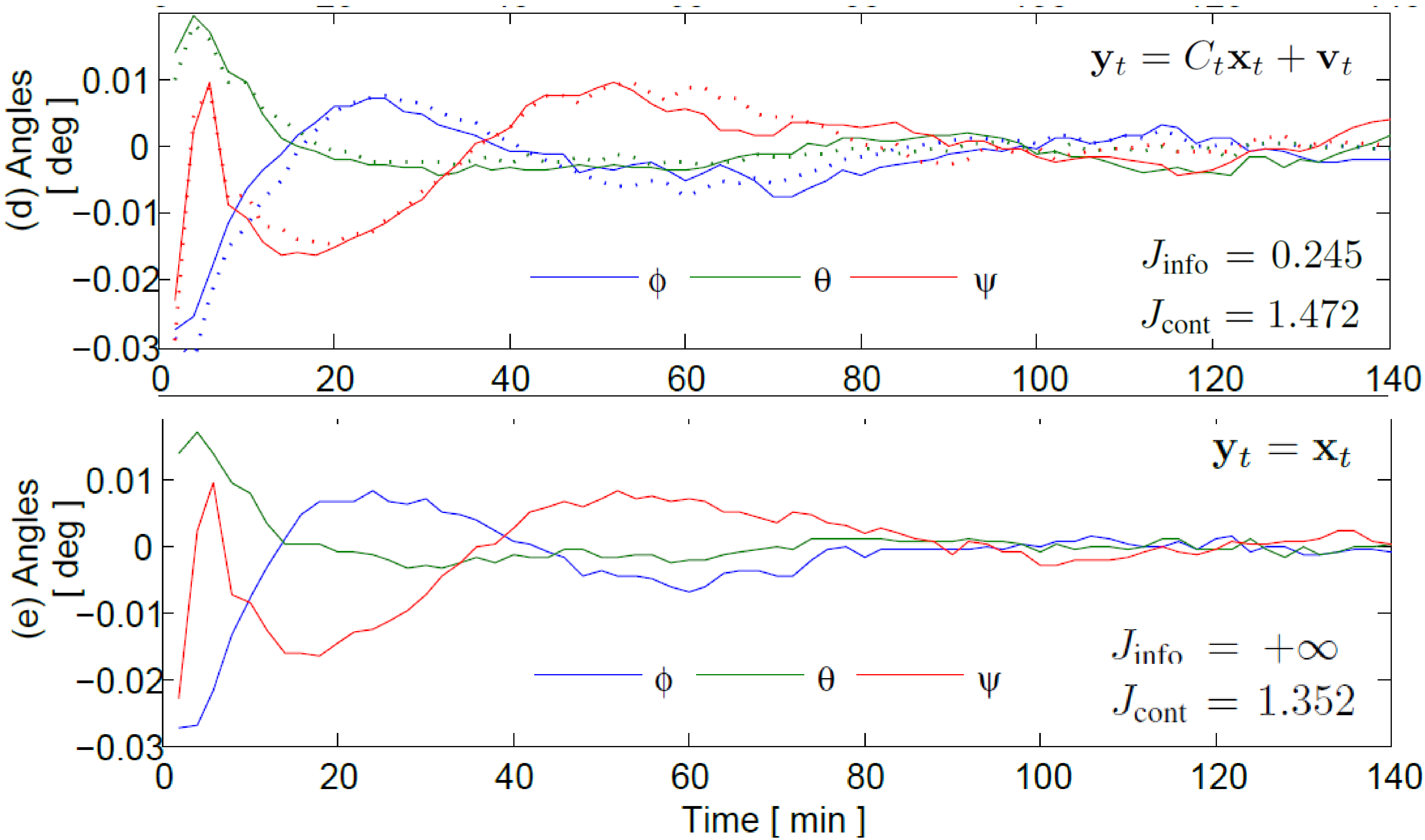
$$
+
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & b_z(t)/I_x & -b_y(t)/I_x \\
-b_z(t)/I_y & 0 & b_x(t)/I_y \\
b_y(t)/I_z & -b_x(t)/I_z & 0
\end{bmatrix}
\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}
+
\begin{bmatrix} w_\phi \\ w_\theta \\ w_\psi \\ n_\phi \\ n_\theta \\ n_\psi \end{bmatrix}
$$

Optimal Joint Control and Sensing



(Sampling period = 2 min)

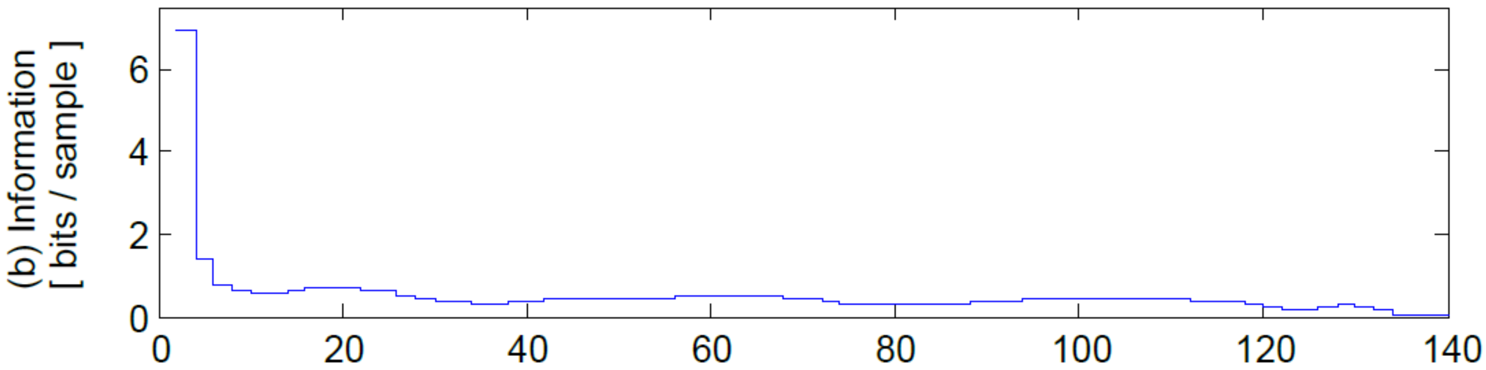
Deviations from Desired Trajectory



(Sampling period = 2 min)

Observations

- Optimal to acquire lots of information initially, and then demand drops rapidly



- Very little loss in control performance by penalizing information transfer in this example

- Due to small process noise in space…

Summary

- Characterization of optimal joint sensor and controller design for LQG control
- Need to solve a semi-definite program
- Close connection to the sequential rate-distortion problem [Tatikonda, 2000]

- Future work:
  - Nonlinear sensing, quantization, and encoding
  - Applications in security and privacy

To appear at IEEE CDC 2015 (preprint arXiv:1503.01848)

Closing of ACCESS-FORCES Workshop on Cyber-Physical Systems

- Slides will be made available on the workshop webpage
- Group work: Follow-up meeting at KTH in February 2016 (TBA)
  - Activities until then: Continued group discussions, writing white papers, MSc theses, etc.
- Upcoming ACCESS Industrial Workshop in May 2016
  - Contact: James Gross (jamesgr@kth.se)
- **Thank you all for attending and for making the workshop a success!**
- Feedback or remaining questions? hsan@kth.se