Dynamical System Decomposition Using Dissipation Inequalities

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***Abstract*— In this paper we investigate stability and interaction measures for interconnected systems that have been produced by decomposing a large-scale linear system into a set of lower order subsystems connected in feedback. We begin by analyzing the requirements for asymptotic stability through generalized dissipation inequalities and storage functions. Using this insight we then describe various metrics based on a system's energy dissipation to determine how strongly the subsystems interact with each other. From these metrics a decomposition algorithm is described.**

I. INTRODUCTION

In this work we describe a set of algorithms that can be used to analyze the stability and characterize the interconnection strength of Linear Time Invariant (LTI) dynamical systems. The methods proposed are based on the notion of dynamical system decomposition [1] and dissipation inequalities with quadratic supply rates [2].

It is frequently the case that many systems have an underlying network structure. If the network structure or connection topology is known *a priori* then this information can be used to help design scalable algorithms for interrogating the system of interest. When the network structure is not known it is important to impose an interconnection topology (decomposition) in order to facilitate further analysis.

In this paper two issues are addressed. We begin by deriving stability criteria for interconnected LTI subsystems using dissipation inequalities and quadratic supply rate functions [2], [3]. In the sequel a method for decomposing networks using the supply rate as a metric for interconnection strength is described and illustrated on an RC network.

System decomposition was first suggested as a framework for handling large-scale systems by Šiljak [4]. However the framework did not provide any insight on how to produce the system decomposition. In recent work [1], [5] an algorithmic method for producing decompositions based on representing the system as a graph and minimizing the worst case "energy flow" between states was presented. In [6] an alternative approach using Hankel-norm based lumping technique for decomposition was presented.

This work extends the decomposition framework developed in [1] in two significant ways; i) nodes are allowed to represent subsystems and ii) edges represent interaction strengths derived from generic quadratic supply rate functions. We investigate the physical interpretation of a class of supply rates and conclude by presenting a clustering based decomposition algorithm.

The paper is organized as follows: In Section II we introduce the necessary background material. Section III uses quadratic supply rate functions to derive various stability criteria. In Section IV edge weight metrics are described and a decomposition algorithm presented. A numerical example is given in Section V and the paper is concluded in Section VI.

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II. PRELIMINARIES

A. Notation

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. If $M \in \mathbb{R}^{n\times n}$ and $M = M^\top$ then $M > 0$, $M \geq 0$ denote that M is positive definite, positive semidefinte respectively. The maximum singular value of M is denoted by $\bar{\sigma}(M)$. Given k matrices $M_1, \dots, M_k$, $\text{diag}(M_1, \dots, M_k)$ denotes the concatenated block diagonal matrix.

B. Storage Functions and Quadratic Supply

We consider dissipation inequalities containing quadratic storage functions, quadratic supply rates, and the notion of (Q, S, R) dissipativity described in [3]. Consider the LTI system

$$\frac{d}{dt}x(t) \triangleq \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $y(t) \in \mathbb{R}^m$ are the state, input and output vectors, respectively. For simplicity we will omit the time argument and refer to the former as simply x, u, and y.

System (1) is said to be *dissipative* with respect to the supply rate $w(u, y)$ if there exists a continuously differentiable *storage* function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following *dissipation inequality*

$$\dot{V}(x) \leq w(u, y), \tag{2}$$

with $V(0) = 0$ and $V(x) \geq 0$ for all $x \neq 0$. We are interested in quadratic supply rate functions of the form

$$w(u, y) = y^\top Qy + 2u^\top Sy + u^\top Ru, \tag{3}$$

with $Q = Q^\top$ and $R = R^\top$, where u and y form input-output pairs (u, y) and Q, S and R are of appropriate dimensions. System (1) is said to be dissipative if

$$\begin{bmatrix} A^\top P + PA - C^\top QC & PB - C^\top S \\ B^\top P - S^\top C & -R \end{bmatrix} \leq 0, \tag{4}$$

for $P > 0$, in which case $V(x) = x^\top Px$.

C. Stability of Interconnected Dissipative Systems

We study the interconnection of N subsystems, Σ_i, where

$$\Sigma_i \begin{cases} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i, \end{cases} \tag{5}$$

and the interconnection between subsystems is

$$u_i = -\sum_{j=1}^{N} H_{ij} y_j. \tag{6}$$

Defining $x = [x_1^\top \cdots x_N^\top]^\top$, $u = [u_1^\top \cdots u_N^\top]^\top$, and $y = [y_1^\top \cdots y_N^\top]^\top$, the interconnection may be written as $u = -Hy$ and the global system dynamics are described by

$$\dot{x} = Ax - BHCx, \quad y = Cx, \tag{7}$$

where $A = \operatorname{diag}(A_1, \cdots, A_N)$, $B = \operatorname{diag}(B_1, \cdots, B_N)$, and $C = \operatorname{diag}(C_1, \cdots, C_N)$.

Assumption 1: Each subsystem Σ_i is dissipative with respect to a given supply rate (Q_i, S_i, R_i).

Given the previous assumption, for each subsystem there exists a symmetric matrix $P_i > 0$ such that the dissipation inequality (2) holds with the (Q_i, S_i, R_i) supply rate. Therefore the inequality

$$\sum_{i=1}^{N} \dot{V}_i(x_i) \leq \sum_{i=1}^{N} w_i(u_i, y_i) \tag{8}$$

holds regardless of the interconnection.

Remark 1: Defining $P = \operatorname{diag}(P_1, \cdots, P_N)$ and $V(x) = x^\top Px$, we have $\dot{V}(x) = x^\top(A^\top P + PA)x + u^\top B^\top Px + x^\top PBu = \sum_{i=1}^{N} \dot{V}_i(x_i)$. Furthermore, $w(u, y) = \sum_{i=1}^{N} w_i(u_i, y_i)$ is given by (3) with $Q = \operatorname{diag}(Q_1, \cdots, Q_N)$, $S = \operatorname{diag}(S_1, \cdots, S_N)$, and $R = \operatorname{diag}(R_1, \cdots, R_N)$.

Inequality (8) may then be rewritten as

$$\sum_{i=1}^{N} \dot{V}_i(x_i) \leq y^\top \hat{Q} y, \tag{9}$$

where $\hat{Q} = Q - S^\top H - H^\top S + H^\top RH$. A sufficient condition for stability of the global interconnected system follows from (9):

Lemma 1 ([3]): Assume each system Σ_i is observable. The global interconnected system is asymptotically stable if $\hat{Q}$ is negative definite. Furthermore, $V(x) = \sum_{i=1}^{N} V_i(x_i) = x^\top Px$ is a Lyapunov function for the global system, with $P = \operatorname{diag}(P_1, \cdots, P_N)$.

D. Algebraic Graph Theory

Consider an *undirected weighted* graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{Z})$ where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is the set of N vertices or nodes, $\mathcal{E} = \{e_1, \ldots, e_M\} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge-set and $\mathcal{Z} = \{z_1, \ldots, z_M\}$ where $z_j > 0$ is the weight of edge j. Associated with $\mathcal{G}$ is a symmetric weighted adjacency matrix $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ where $[\mathcal{A}(\mathcal{G})]_{ij} > 0$ if there exists an edge connecting v_i to v_j. The weighted $N \times N$ Laplacian matrix is defined by $\mathcal{L}(\mathcal{G}) = \operatorname{diag}(\mathcal{A}(\mathcal{G})\mathbf{1}) - \mathcal{A}(\mathcal{G})$.

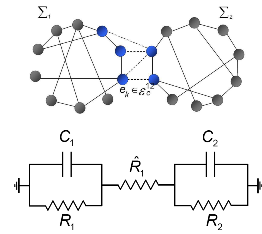

Fig. 1. Top: Graph of interconnected subsystems Σ_1, Σ_2. Boundary nodes $\mathcal{V}_b^{12}$ are blue. Dashed edges, $\mathcal{E}_c^{12}$, connect the subsystems via the boundary nodes. Bottom: Two nodes in an RC network, edge $e_k \in \mathcal{E}_c^{12}$ corresponds to resistor $\widehat{R}_1$ and the vertices correspond to the resistor and capacitors connected in parallel.

The incidence matrix, $\mathcal{C}(\mathcal{G}) \in \mathbb{R}^{N \times M}$ of an undirected graph is defined by assigning an arbitrary direction to each $e_i \in \mathcal{E}$ and setting $[\mathcal{C}(\mathcal{G})]_{ij} = 1$ if e_i enters v_j, -1 if e_i leaves v_j and 0 otherwise. The weighted Laplacian can then be equivalently defined by $\mathcal{L}(\mathcal{G}) = \mathcal{C}(\mathcal{G})W(\mathcal{G})\mathcal{C}(\mathcal{G})^\top$ where $\mathcal{W}(\mathcal{G}) \in \mathbb{R}^{M \times M}$ is a diagonal matrix and $[\mathcal{W}(\mathcal{G})]_{ii} = z_i$ with $z_i \in \mathcal{Z}$. Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{Z})$, consider a subset of the vertices $\mathcal{V}_j \subset \mathcal{V}$; we call the graph $\mathcal{G}_j = \mathcal{G}/\mathcal{V}_j$ an *induced subgraph* of $\mathcal{G}$.

Assume that we have a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{Z})$ which has been partitioned such that $\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2$ and $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \emptyset$. If there exists an edge $e_k = (v_i, v_j) \in \mathcal{E}$ such that $v_i \in \mathcal{V}_1$ and $v_i \in \mathcal{V}_2$ (or vice-versa) then v_i and v_j are called boundary nodes and belong to the set $\mathcal{V}_b^{12}$. The set of edges that connect boundary nodes in $\mathcal{V}_1, \mathcal{V}_2$ is given by $\mathcal{E}_c^{12}$.

When it is clear from the context we will omit the graph argument and refer to the matrices simply as $\mathcal{A}, \mathcal{C}, \mathcal{W}$ and $\mathcal{L}$. For a thorough overview of algebraic graph theory see [7].

E. Illustrative Example

Consider the undirected weighted graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{Z})$. Vertices $v_i \in \mathcal{V}$ correspond to a capacitor C_i in parallel with a resistor R_i connected to ground and each edge $e_k \in \mathcal{E}$ represents a resistor $\widehat{R}_k$ with $z_k = 1/\widehat{R}_k \in \mathcal{Z}$ connecting the free terminals of two vertices, see Figure 1. Let all capacitors have unit capacitance and denote $x_i \in \mathbb{R}$ the voltage in capacitor i and $u_i \in \mathbb{R}$ the current entering node i. Each node is modeled by the first-order system

$$\dot{x}_i = -g_i x_i + u_i, \quad y_i = x_i, \tag{10}$$

where $g_i = 1/R_i$. Defining $G = \operatorname{diag}(g_1, \cdots, g_N)$ the dynamics of the global network are given by

$$\dot{x} = -(\mathcal{L} + G)x, \quad y = x. \tag{11}$$

Let Σ_j be a subnetwork described by the induced graph $\mathcal{G}_j(\mathcal{V}_j, \mathcal{E}_j, \mathcal{Z}_j)$. The dynamics of Σ_j are described by

$$\Sigma_j \begin{cases} \dot{x}_j = -(\mathcal{L}_j + G_j)x_j + B_j u_j \\ y_j = B_j^\top x_j \end{cases} \tag{12}$$

where u_j represents the input current to Σ_j from the rest of the network that enters through the boundary nodes whose dynamics are described by the matrix B_j. To construct B_j: i) Define the following sets $\mathcal{V}_i, \mathcal{V}_j, \mathcal{E}_i, \mathcal{E}_j, \mathcal{V}_b^{ij}$ and $\mathcal{E}_c^{ij}$; ii)

From these sets construct an incidence matrix corresponding to $\mathcal{V}_b^{ij}, \mathcal{E}_c^{ij}$ such that all edges enter $\mathcal{V}_j$. Then, B_j corresponds to the part of the incidence matrix with nodes belonging to $\mathcal{V}_j$.

Defining the storage function for this system as $V_j(x_j) = \frac{1}{2}x_j^\top x_j$ we have $\dot{V}_j(x_j) = -x_j^\top \mathcal{L}_j x_j - x_j^\top G_j x_j + u_j^\top y_j$, where the power dissipated on the internal edge resistors, the power dissipated on the node resistors, and the input power to Σ_j correspond to $-x_j^\top \mathcal{L}_j x_j$, $-x_j^\top G_j x_j$, and $u_j^\top y_j$, respectively.

Given $\dot{V}_j(x_j)$, two interesting supply rate functions for which the system is dissipative can be immediately identified.

Observation 1: If Σ_j is stable, then Σ_j is $(0, I, 0)$-dissipative.
Furthermore, consider $x_j^\top G_j x_j = x_j^\top \tilde{G}_j x_j + y_j^\top G_j^b y_j$, where the first term is the power dissipated in the internal node resistors, while the second term corresponds to power dissipated on the boundary node resistors.

Observation 2: If $\mathcal{L}_i + \tilde{G}_j$ is positive semidefinite, then Σ_j is $(-G_j^b, I, 0)$-dissipative.

III. STABILITY ANALYSIS

Here we describe some stability results based on composite Storage functions. We assume that a decomposition algorithm has already been applied to obtain the subsystems. In Section IV we present a clustering algorithm for decomposition.

Consider the LTI system Σ:

$$\Sigma \left\{ \begin{array}{rcl} \dot{x} & = & Ax, \quad x(0) = x_0, \\ y & = & x \end{array} \right. \tag{13}$$

with $x \in \mathbb{R}^n$. Now assume that it has been decomposed into two subsystems connected in feedback

$$\Sigma_1 \left\{ \begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + u_1 \\ u_1 & = & A_{12}x_2 \\ y_1 & = & x_1 \end{array} \right. , \Sigma_2 \left\{ \begin{array}{rcl} \dot{x}_2 & = & A_{22}x_2 + u_2 \\ u_2 & = & A_{21}x_1 \\ y_2 & = & x_2 \end{array} \right. \tag{14}$$

where the state vector has been permuted such that $x = [x_1^\top, x_2^\top]^\top$ and $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ and no state belongs to multiple subsystems.

Remark 2: Assume system Σ has been decomposed into Σ_1 and Σ_2 which are dissipative w.r.t. the quadratic supply rates $w_1(u_1, y_1)$ and $w_2(u_2, y_2)$ of the form (3) respectively. If $w_1(u_1, y_1) + w_2(u_2, y_2) < 0$ for all input output pairs then the sum of the Storage functions $V_1(x_1) + V_2(x_2)$ is a Lyapunov function that proves that the equilibrium point of (13) is asymptotically stable. This is a direct application of Lemma 1.
Note that this and all further results generalize to the case where Σ is decomposed into multiple subsystems. For the sake of clarity we focus here on the case of two subsystems.

If we assume a generic interconnection structure for Σ_1, Σ_2 of the form

$$\left[\begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left[\begin{array}{cc} H_{11} & H_{12} \\ H_{21} & H_{22} \end{array} \right] \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right] \tag{15}$$

then the right hand side of $\dot{V}_1(x_1) + \dot{V}_2(x_2) \leq w_1(u_1, y_1) + w_2(u_2, y_2)$ with (u_i, y_i) obtained from the decomposition and interconnection matrix (15) is given by (16) on the next page. By appropriate choice of the Q, S, R matrices, the supply functions (3) can represent passivity, finite-gain etc. each of which alters the structure of (16). The remainder of this section provides stability tests for (13) based on its decomposed subsystems (14) using passivity and finite gain arguments. Other properties are tested in [8].

A. Passivity

An LTI system of the form (5) is said to be passive if it is dissipative with respect to supply rate (3) with $(Q_i, S_i, R_i) = (0, I, 0)$ and LMI (4) is feasible. Assume that Σ has been decomposed into Σ_1, Σ_2 (which is equivalent to (5)). Substituting the appropriate matrices into the supply rate functions, we see from Equation (16) that we require

$$\left[\begin{array}{cc} H_{11}^\top S_1 + S_1^\top H_{11} & S_1^\top H_{12} + H_{21}^\top S_2 \\ \star & H_{22}^\top S_2 + S_2^\top H_{22} \end{array} \right] < 0, \tag{17}$$

where from (14) we have that

$$H = \left[\begin{array}{cc} 0 & A_{12} \\ A_{21} & 0 \end{array} \right].$$

With this interconnection structure the diagonal block entries in (17) are zero and the off diagonal blocks are given by $A_{12} + A_{21}^\top$ and its transposition. In this form (17) cannot be negative definite as its eigenvalues will be real and symmetric about the imaginary axis. This problem can be alleviated if we consider a slight modification to the decomposition described by (14) by imposing a further decomposition on the drift matrices A_{ii} and include a feedback term. The new decomposition for Σ_1 is

$$\widehat{\Sigma}_1 \left\{ \begin{array}{rcl} \dot{x}_1 & = & \epsilon_1 A_{11}x_1 + u_1 \\ u_1 & = & A_{12}x_2 + \delta_1 A_{11}x_1 \\ y_1 & = & x_1 \end{array} \right. \tag{18}$$

where $\epsilon_1 + \delta_1 = 1$ and we assume all matrices are of compatible dimension. In the same manner $\widehat{\Sigma}_2$ can be constructed. LMI (17) is then replaced by

$$\left[\begin{array}{cc} \delta_1(A_{11} + A_{11}^\top) & A_{12} + A_{21}^\top \\ \star & \delta_2(A_{22} + A_{22}^\top) \end{array} \right] < 0 \tag{19}$$

When $\widehat{\Sigma}_1, \widehat{\Sigma}_2$ are dissipative with respect to $(0, I, 0)$ and LMI (19) is feasible the original system (13) is stable as verified by the Lyapunov function $V(x) = V_1(x_1) + V_2(x_2)$. An alternative approach is to select ϵ_i, δ_i arbitrarily (ensuring $\epsilon_1 + \delta_1 = 1$) and using the modified decomposition $\widehat{\Sigma}$ solve LMI (17) where the decision variables are the diagonal matrices $S_i > 0$. Such an approach is possible because any system that is dissipative with respect to $(0, I, 0)$ is also dissipative with respect to any $(0, X, 0)$ supply rate with $X > 0$ diagonal.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} (H_{11}^\top R_1 H_{11} + H_{11}^\top S_1 + S_1^\top H_{11} + \\ +H_{21}^\top R_2 H_{21} + Q_1) & (H_{11}^\top R_1 H_{12} + S_1^\top H_{12} + \\ & +H_{21}^\top S_2 + H_{21}^\top R_2 H_{22}) \\ \star & (H_{22}^\top R_2 H_{22} + H_{22}^\top S_2 + S_2^\top H_{22} + \\ & +H_{12}^\top R_1 H_{12} + Q_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (16)$$

B. Finite Gain

For LTI systems the $\mathcal{L}_2$ gain from input to output of a system in the form of (5) can be calculated by solving:

$$\begin{aligned} \min \quad & \gamma_i \\ \text{s.t.} \quad & \begin{bmatrix} A_i^\top P_i + P_i A_i + C_i^\top C_i & P_i B_i \\ B_i^\top P_i & -\gamma_i^2 I \end{bmatrix} \leq 0 \quad (20) \\ & P_i > 0, \quad \gamma_i > 0. \end{aligned}$$

The $\mathcal{L}_2 \to \mathcal{L}_2$ gain is then given by γ_i [9]. For two systems connected in feedback if $\gamma_1\gamma_2 < 1$ then the feedback connection is stable [10]. A generalization of the small gain theorem for networks is given in [11]. Following from LMI (20) it can be seen that the supply rate functions associated with finite gain are given by $(-I, 0, \gamma_i^2 I)$ for $i = 1, 2$.

Substituting the appropriate Q, S, R matrices and interconnections into (16) gives the following stability requirement:

$$\begin{aligned} \begin{bmatrix} \gamma_2^2 A_{21}^\top A_{21} & 0 \\ 0 & \gamma_1^2 A_{12}^\top A_{12} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &< 0 \\ \Leftrightarrow \bar{\sigma}\left(\begin{bmatrix} \gamma_2 I & 0 \\ 0 & \gamma_1 I \end{bmatrix} \begin{bmatrix} A_{21} & 0 \\ 0 & A_{12} \end{bmatrix} \right) &< 1 \quad (21) \end{aligned}$$

The stability condition (21) is stated formally below.

Lemma 2: Assume system (13) has been decomposed into the subsystems given in (14). Further assume that the subsystems are dissipative with respect to $S_i = 0, Q_i = -I$ and $R_i = \gamma_i^2 I$ where γ_i denotes the $\mathcal{L}_2$-norm of subsystem i. Then if max $\{\gamma_2\bar{\sigma}(A_{21}), \gamma_1\bar{\sigma}(A_{12})\} < 1$ system (13) is asymptotically stable as verified by the Lyapunov function $V_1(x_1) + V_2(x_2)$.

Lemma 2 and Equation (21) provide a *nominal* stability test for the decomposed subsystems. What would be desirable is to determine the maximum $\mathcal{L}_2$ gains (i.e. γ_i's) such that (21) holds. Such a characterization would provide a robustness measure for the decomposed system. From the equivalence relation in (21) the maximum achievable γ's denoted by $\hat{\gamma}$ that satisfy the stability requirement in Lemma 2 are given by $\hat{\gamma}_1 = \bar{\sigma}(A_{12})^{-1}$ and $\hat{\gamma}_2 = \bar{\sigma}(A_{21})^{-1}$.

If we consider more generic supply rates with $(-\kappa I, 0, \kappa\gamma^2 I)$, $\kappa > 0$ instead of $(-I, 0, \gamma^2 I)$ then it is possible to strengthen Lemma 2. Observe that if a system is dissipative w.r.t. $(-I, 0, \gamma^2 I)$ then it is also dissipative w.r.t. $(-\kappa I, 0, \kappa\gamma^2 I)$ for any $\kappa > 0$.

Lemma 3: If there exists a scalar $\kappa > 0$ such that Σ_1 and Σ_2 are dissipative w.r.t. $(-\kappa I, 0, \kappa\gamma_1^2 I)$ and $(-\kappa I, 0, \kappa\gamma_2^2 I)$ respectively then system (13) is stable if max $\left\{\kappa\gamma_1\bar{\sigma}(A_{21}), \kappa^{-1}\gamma_2\bar{\sigma}(A_{12})\right\} < 1$.

In [8] using the same framework we describe how input and output strict passivity of coupled systems can be determined and provide numerical examples.

IV. DECOMPOSITION

In this section we study the decomposition of interconnected dissipative subsystems. Consider a dissipative subsystem Σ_i described by (5) with a nonnegative storage function $V_i(x_i)$. Such storage is a scalar measure of the subsystem's state, which could be thought of as the amount of "abstract energy" stored by the subsystem in its internal state x_i.

The supply rate upper bounds the rate of change of the storage in the system and thus, indirectly, the change of the subsystem's state. By estimating various forms of supply interchanges between subsystems we can evaluate which subsystems interact most strongly with each other using the supply rate upper bound as an indication of the worst case (strongest) interaction.

A. Undirected Supply Measure

With the interconnection of subsystems Σ_1 and Σ_2 and when Assumption 1 holds, having the total supply rate given by $w_1(u_1, y_1) + w_2(u_2, y_2) = -y^\top \hat{Q} y \approx 0$ implies that, over the trajectories of the global system, the interconnections H supply to the subsystems is small. Hence $w_1(u_1, y_1) + w_2(u_2, y_2)$ could be seen as a measure of *undirected interaction*, indicating how relevant the interconnection is to the global system dynamics. Additionally, from Lemma 1 having $w_1(\cdot) + w_2(\cdot) < 0$ implies stability of the global system.

Remark 3: Σ_1 and Σ_2 could be connected to several other subsystems. For the previous discussion to hold, one should constrain the supply rate to be separable along the different edges.

Assumption 2: Define $\mathcal{E}_c \subset \mathcal{E}$ as the set of edges interconnecting K subsystems. For each subsystem Σ_i we assume the supply rate is separable along the edges, which implies $w_i(u_i, y_i) = \sum_{e_k \in \mathcal{E}_c} w^i_{e_k}(u_i, y_i)$.

Remark 4: Define $\mathcal{E}_c^{ij} \subseteq \mathcal{E}_c$ as the set of edges connecting Σ_i and Σ_j. The measure for the undirected interaction between Σ_i and Σ_j is then $\sum_{e_k \in \mathcal{E}_c^{ij}} \left[w^i_{e_k}(u_i, y_i) + w^j_{e_k}(u_j, y_j) \right]$.

Take the RC-network described previously. The metric discussed in this section corresponds to the electric energy dissipated in the interconnecting resistors for Observation 1 and to the electric energy dissipated on the interconnecting and boundary resistors for Observation 2.

Remark 5: A *directed supply* measure can also be derived using similar arguments as above, resulting in $w_1(u_1, y_1)$ being a measure for *directed interaction* from Σ_2 and the interconnection to Σ_1, see [8].

B. Computing Edge Weights for Stability

We now discuss how the previously described measures of interaction, which are time varying functions of the system state, can be condensed to a representative static value for use in a decomposition algorithm such as the one in [5]. Ideally we would like to find "good" decompositions that satisfy the stability criteria defined in Section III.

Consider the global interconnected subsystem (7), with no assumption of stability. Define for each edge e_i the supply rate function

$$w_{e_i}(u,y) = y^\top Q_{e_i} y + 2u^\top S_{e_i} y + u^\top R_{e_i} u = y^\top \hat{Q}_{e_i} y \quad (22)$$

where u has been eliminated using the interconnection $u = -Hy$ and $\hat{Q}_{e_i}$ symmetric, corresponding to either the directed or the undirected interaction measure. For instance, taking the RC-network in Figure 1 and considering the undirected interaction measure for the supply rate from Observation 1, $w_i(u_i, y_i) = u_i^\top y_i$, we have

$$w_{e_1}(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\widehat{R}_1} & -\frac{1}{\widehat{R}_1} \\ -\frac{1}{\widehat{R}_1} & \frac{1}{\widehat{R}_1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{(y_1 - y_2)^2}{\widehat{R}_1},$$

which corresponds to the electric power dissipated by $\widehat{R}_1$.

As mentioned in Section III and IV-A, the undirected interaction measure is also related to stability. In fact given a cut $\mathcal{E}_c$ and Assumption 2, if the subsystems Σ_1 and Σ_2 are dissipative with nonnegative storage functions characterized by P_1 and P_2 respectively, then we have

$$x^\top \left[(A - BHC)^\top P + P(A - BHC)\right] x \leq y^\top \Big(\sum_{e_k \in \mathcal{E}_c} \hat{Q}_{e_k}\Big) y, \quad (23)$$

with $P = \mathrm{diag}(P_1,\, P_2)$, from which it follows that the global system is stable if $y^\top (\sum_{e_k \in \mathcal{E}_c} \hat{Q}_{e_k}) y < 0$ (see Lemma 1).

As such a cut is not known *a priori*, we provide heuristics to compute appropriate edge weights. Let y_{e_k} be the output of the two nodes incident to the edge e_k and define $\tilde{Q}_{e_k}$ such that $y^\top \hat{Q}_{e_k} y = y_{e_k}^\top \tilde{Q}_{e_k} y_{e_k}$. For a suitable permutation yielding $\Pi y = [y_{e_k}^\top \; y_{\mathcal{E}/e_k}^\top]^\top$ we have $\tilde{Q}_{e_k}$ as the first diagonal block of $\Pi \hat{Q}_{e_k} \Pi^{-1}$. For instance, in the RC-network in Figure 1 we have $\tilde{Q}_{e_k} = \hat{Q}_{e_k}$. A sufficient condition for $\hat{Q}_{\mathcal{E}_c} = \sum_{e_k \in \mathcal{E}_c} \hat{Q}_{e_k} < 0$ is to require $\tilde{Q}_{e_k} < 0 \quad \forall e_k \in \mathcal{E}$. Furthermore, note that (23) may be thought as the inclusion of an ellipsoid, $-x^\top \left[(A - BHC)^\top P + P(A - BHC)\right] x$, by another ellipsoid $-x^\top C^\top (\sum_{e_k \in \mathcal{E}_c} \hat{Q}_{e_k}) C x$, where the latter corresponds to a sum of ellipsoids. Since the former ellipsoid is only known after the cut, one would like the latter ellipsoid to be as large as possible, as this would increase the set of matrices P_1 and P_2 for which such inclusion holds. Therefore, assuming $\hat{Q}_{\mathcal{E}_c} < 0$ and denoting $\mathcal{P}_{\mathcal{E}_c} = \{y\colon -y^\top \hat{Q}_{\mathcal{E}_c} y \leq 1\}$ as the ellipsoid associated with a given cut $\mathcal{E}_c$, a suitable partitioning algorithm would solve $\max_{\mathcal{E}_c} \mathrm{vol}(\mathcal{P}_{\mathcal{E}_c})$ where $J(\mathcal{E}_c) = \mathrm{vol}(\mathcal{P}_{\mathcal{E}_c})$ is the utility of a cut $\mathcal{E}_c$. Combining these two features, and denoting $\mathcal{P}_{e_k}$ as the ellipsoid defined by $\tilde{Q}_{e_k}$, the edge weights $J(e_k) = \mathrm{vol}(\mathcal{P}_{e_k}^*)$ may be computed by solving

$$\max_{Q_{e_k} < 0, R_{e_k} = R_{e_k}^\top, S_{e_k}} \quad \mathrm{vol}(\mathcal{P}_{e_k})$$

which is not a convex problem in Q_{e_k}, S_{e_k}, and R_{e_k}. The partitioning algorithm would then choose a set of edges forming a cut $\mathcal{E}_c$ such that $\sum_{e_k \in \mathcal{E}_c} J(e_k)$ is maximized.

The volume of an ellipsoid $\mathcal{P}_{e_k}$ is proportional to $\sqrt{\det(-\tilde{Q}_{e_k}^{-1})}$, hence the previous problem will yield a solution such that $\det(-\tilde{Q}_{e_k})$ is minimized, which implies $y_{e_k}^\top \tilde{Q}_{e_k} y_{e_k} \approx 0$. Therefore, since $\det(-\tilde{Q}_{e_k})$ is the product of the eigenvalues of $-\tilde{Q}_{e_k}$, we can instead consider the convex problem

$$\min_{Q_{e_k} < 0, R_{e_k} = R_{e_k}^\top, S_{e_k}} \quad \lambda_{max}(-\tilde{Q}_{e_k})$$

which is related to maximizing the diameter of $\mathcal{P}_{e_k}$, and take $J(e_k) = 1/\lambda_{max}(-\tilde{Q}_{e_k}^*)$. Note that this also relates to finding weakly interacting subsystems based on the undirected measure. Therefore, large values of $J(e_k)$ indicate that this is a good edge to cut in a decomposition algorithm and will also help in verifying stability using the criteria in Section III.

C. Computing Edge Weights for Weakly Connected Systems

Assume now the global system is stable and we want to decompose it into subsystems that interact weakly over time, for example to facilitate the design of distributed controllers. Consider the system (7) and define for each edge the supply rate function (22). Since there are no external inputs, w_{e_i} is a function of the initial condition x_0 and time. Hence one needs to evaluate these functions to compute a static value measuring the interactions over time. Instead the *total supply* defined as $W_{e_i}(x_0) \triangleq \int_0^\infty w_{e_i}(t)\, dt$ is used and the edge weight is computed by evaluating $W_{e_i}(x_0)$ for the relevant initial conditions. The following result allows us to compute $W_{e_i}(x_0)$ for a given initial condition:

Proposition 1: Assuming the global system (7) is stable, for a given initial condition x_0 we have $W_{e_i}(x_0) = x_0^\top T_{e_i} x_0$, where T_{e_i} is the Gramian matrix satisfying the Lyapunov equation $(A - BHC)^\top T_{e_i} + T_{e_i}(A - BHC) + C^\top \hat{Q}_{e_i} C = 0$.

Proof: We have $W_{e_i}(x_0) = \int_0^\infty y(t)^\top \hat{Q}_{e_i} y(t)\, dt = x_0^\top T x_0$, where $\bar{A} = A - BHC$ and $T = \int_0^\infty e^{(\bar{A}^\top t)} C^\top \hat{Q}_{e_i} C e^{(\bar{A} t)}\, dt$. Note that the expression for T resembles the well-known observability Gramian. The rest of the proof follows the characterization of the observability Gramian found in [12]. ■

Note that for finite time horizons $W_{e_i}(x_0)$ can be computed by means of simulation as an alternative to solving the Gramian.

Recalling the decomposition's objective, a partitioning algorithm would select a set of edges $\mathcal{E}_c$ forming a cut such that $\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)$ is close to zero, thus solving $\min_{\mathcal{E}_c} \left|\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)\right|$ where we define $J(\mathcal{E}_c) = |\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)|$ as the cost of a cut for a given initial condition. We now analyze the evaluation of $W_{e_i}(x_0)$ for two different sets of initial conditions.

1) Worst-case initial condition: For a given cut $\mathcal{E}_c$, the worst-case initial condition is the one maximizing $|\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)|$ and the cut cost would be given by $J(\mathcal{E}_c) = \max_{\|x_0\|=1} |\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)|$. Since we do not know the set $\mathcal{E}_c$ *a priori*, the edge weights are also unknown, which would require a combinatorial approach to solve this partitioning algorithm. A possible relaxation decoupling the edge weights from the cut can be made based on the following inequality $\max_{x_0} |\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)| \leq \sum_{e_i \in \mathcal{E}_c} \max_{x_{0_i}} |W_{e_i}(x_{0_i})|$. Note that in the right hand side the initial condition x_{0_i} is dependent only the edge e_i. Therefore, by defining the new cost function $\bar{J}(\mathcal{E}_c) = \sum_{e_i \in \mathcal{E}_c} \max_{x_{0_i}} |W_{e_i}(x_{0_i})|$, we obtain weights that only depend on each particular edge, $\bar{J}(e_i) = \max_{x_{0_i}} |W_{e_i}(x_{0_i})|$, and an upper bound on the edge cost $J(\mathcal{E}_c)$. The weight $\bar{J}(e_i)$ can be computed by solving $\max_{\|x_0\|=1} \; |x_0^\top T_{e_i} x_0|$ where T_{e_i} is given by Proposition 1.

Remark 6: Since $C^\top \hat{Q}_{e_i} C$ is symmetric, T is also symmetric and thus we have $\max_{x_0} |x_0^\top T x_0| = \max_i |\lambda_i(T)| = |\lambda^*(T)|$, where $\{\lambda_i(T)\}$ are the eigenvalues of T. Computing the eigenvalue value decomposition of T, we conclude x_0^* is given by the eigenvector associated with $\lambda^*(T)$.

2) Gaussian initial condition: We now consider a stochastic description of the initial condition for the global system. Let $x_0 \sim \mathcal{N}(\bar{x}, \Omega)$. From Proposition 1 it follows that the total supply $W_{e_i}(x_0) = x_0^\top T_{e_i} x_0$ is a random variable. Hence the cost of a given cut $\mathcal{E}_c$ is $J(\mathcal{E}_c) = |\mathbb{E}_{x_0}[\sum_{e_i \in \mathcal{E}_c} W_{e_i}(x_0)]|$. Using the triangle inequality we obtain the following upper bound of the cut cost $J(\mathcal{E}_c) \leq \sum_{e_i \in \mathcal{E}_c} |\mathbb{E}_{x_0}[W_{e_i}(x_0)]| = \tilde{J}(\mathcal{E}_c)$. Hence we assign $\tilde{J}(e_i) = |\mathbb{E}_{x_0}[x_0^\top T_{e_i} x_0]|$ as the weight for e_i, which may be computed using the following result:

Proposition 2: Given $x_0 \sim \mathcal{N}(\bar{x}, \Omega)$ we have $\mathbb{E}_{x_0}[x_0^\top T_{e_i} x_0] = \bar{x}^\top T_{e_i} \bar{x} + \text{trace}(T_{e_i} \Omega)$.

Proof: Direct application of Lemma 3.3 in [13]. ■

D. Decomposition Methods

Given the aforementioned methods to compute static edge weights, a system decomposition algorithm based on the directed and undirected interaction measures is described.

The directed interaction measure provides two weights, one for each edge direction. Hence it is suitable for clustering algorithms where a given set of nodes $\mathcal{V}_0$ is of interest and we want to find $\mathcal{V}_i$ such that $\mathcal{V} = \mathcal{V}_i \cup \mathcal{V}_j$, $\mathcal{V}_0 \subseteq \mathcal{V}_i$, and Σ_i is not affected much by Σ_j. A possible algorithm to accomplish this task proceeds as follows:

1) Set $\mathcal{V}_i = \mathcal{V}_0$ and define $\mathcal{E}_c$ as the edge set connecting nodes from $\mathcal{V}_j$ to $\mathcal{V}_i$;
2) Compute the directed weight from $\mathcal{V}_j$ to $\mathcal{V}_i$ for each edge $e_k \in \mathcal{E}_c$;
3) Pick the set of nodes from $\mathcal{V}_j$ that have the largest directed weight, $\bar{\mathcal{V}}_j$, and set $\mathcal{V}_i^+ = \mathcal{V}_i + \bar{\mathcal{V}}_j$;
4) Set $\mathcal{V}_i = \mathcal{V}_i^+$, define the new cut set $\mathcal{E}_c$, and repeat from 2 until the interaction measure is below the tolerance level.

The undirected interaction measure provides a single weight, W_{e_k} which can be readily incorporated into the framework presented in [5].

V. EXAMPLE

Consider an RC-network described by the graph in Figure 1 with dynamics given by (11). Let each node have unit capacitance and resistance and let $[\mathcal{W}]_{ii} = 1/\hat{R}_i = 0.1 \, \forall e_i \in \mathcal{E}_c^{12}$ and $[\mathcal{W}]_{ii} = 1 \, \forall e_i \notin \mathcal{E}_c^{12}$. For each node v_i, consider the supply rate defined in Observation 1, $w_i(u_i, y_i) = u_i^\top y_i$. Recalling that u_i is the input current to node i and $y_i = x_i$ is the voltage at the corresponding capacitor, from Kirchhoff's Current Law we conclude that the supply rate is separable along the edges connected to v_i, since $w_i(u_i, y_i)$ is the sum of the input power from each edge. Hence Assumption 2 holds. Following the steps in Section IV-A for the *undirected* measure and the *worst-case* initial condition approach we compute the edge weights, which correspond to the electric power dissipated at each edge resistor. For the dashed edges we obtain the weights $\bar{J}(e_i \in \mathcal{E}_c^{12}) = [0.0579, 0.0625, 0.0693, 0.0623]^\top$, while $\min_i \bar{J}(e_i \notin \mathcal{E}_c^{12}) = 0.4016$. Applying a spectral graph decomposition algorithm with these edge weights we obtain $\mathcal{E}_c^{12}$ as the cut set, as shown in Figure 1.

Considering instead the undirected measure with Gaussian initial condition $x_0 \sim \mathcal{N}(0, I)$, we obtain the following weights $\tilde{J}(e_i \in \mathcal{E}_c^{12}) = [0.0636, 0.0670, 0.0727, 0.0671]^\top$, and $\min_i \tilde{J}(e_i \notin \mathcal{E}_c^{12}) = 0.4278$. As before, the cut set obtained after spectral decomposition is $\mathcal{E}_c^{12}$.

VI. CONCLUSIONS

It has been shown how the supply rates of dissipative dynamical systems can be used as a metric for measuring subsystem interaction strength in a networked system. Furthermore, based upon this metric an algorithm for decomposing a networked system was presented and illustrated on a 20 node RC circuit. We also provided stability criteria for the decomposed system based on passivity and bounded gain.

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