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Model Reduction of Interconnected Linear Systems Using Structured Gramians ⋆

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Abstract: The problem of structure-preserving model reduction of interconnected linear systems is considered in this paper. The problem is interesting because networked models often have high order, and standard model-reduction methods usually do not preserve interconnection structure. As a tool, balanced truncation and block-diagonal generalized controllability and observability Gramians are used. Block-diagonal generalized Gramians do not exist for all interconnected systems, but a class of systems that always has such Gramians is identified. Furthermore, it is shown how general interconnected systems can be associated with interconnected systems in this class. The block-diagonal Gramians are then used to compute the reduced models and also yield asymptotic a priori approximation error bounds and stability guarantees for the reduced models.

Keywords: Model reduction; linear systems; interconnected systems; networks; linear matrix inequalities.

1. INTRODUCTION

In this paper, we consider model reduction of interconnected linear systems. The model consists of subsystems with dynamics G_i, and a network topology N that describes how the subsystems interact. An example is shown in Fig. 1. The reduced model should retain the network topology, but the subsystems should be of lower order (smaller McMillan degree). A naive approach to solve the problem is to approximate each subsystem separately, and then to interconnect the approximations. This approach does not take the dynamics of the entire system into account when approximating each subsystem, and will only work under special circumstances.

The motivation for this work is that many models that are of interest to the control community have a network structure, see Murray (2003). Examples include models of the power grid, formations of vehicles, but also control systems where controllers, actuators, and sensors are distributed over a computer network. In all of these examples there can be many subsystems that are interconnected in one way or another, and the order of the entire system can be very large. It is often desirable to obtain a model with fewer (differential) equations and whose trajectories are provably close to the original model's trajectories. There are standard methods to do this, for example balanced truncation, see Enns (1984); Glover (1984). Unfortunately,

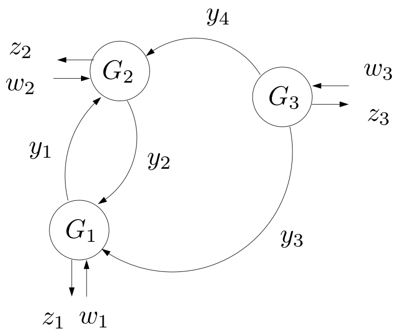

Fig. 1. An example of an interconnected system with node dynamics G_i, $i = 1, 2, 3$. We want to find low-order approximations of G_i, called $\hat{G}_i$, such that the mappings $w_i \mapsto z_i$ are preserved as well as possible, while taking the network interaction y_i into account.

the standard methods usually do not preserve the interconnection topology.

Model reduction methods that preserve network topology have been developed by Li and Paganini (2005), and by Vandendorpe and Van Dooren (2004). In Li and Paganini (2005), Linear Matrix Inequalities (LMIs) are used to find structured coordinate transformations, suitable for state truncation. In Vandendorpe and Van Dooren (2004), ideas from frequency-weighted balanced truncation and closed-loop balanced truncation are used to solve the same problem. Just as in Li and Paganini (2005), we note here the importance of finding block-diagonal matrix solutions to certain LMIs. We call these block-diagonal matrix solutions *structured (generalized) Gramians*. Structured

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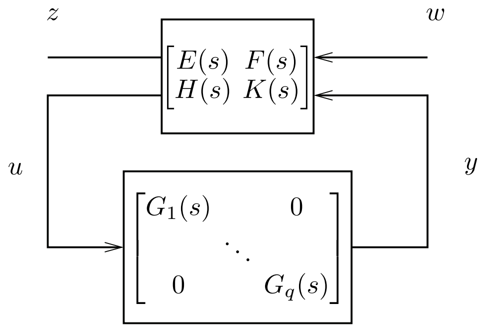

Fig. 2. The networked system modeled with a linear fractional transformation $\mathcal{F}_l(N, G)$.

Gramians have long been used in model reduction of uncertain systems, see Beck et al. (1996); Beck (2006), and for controller order reduction, see Zhou et al. (1995). A problem is that the necessary structured Gramians only exist under special circumstances.

A contribution of this paper is that a class of interconnected systems that always has structured Gramians is identified. Importantly, systems that do not belong to this class can be associated with a system in the class. This means the results can be applied to general interconnections of linear systems. Furthermore, asymptotic approximation error bounds and stability guarantees are derived for the reduced models using the structured Gramians.

The structure of the paper is as follows. In Section 2, the model framework and preliminary results are given. In Section 3, a class of linear systems with structured Gramians is identified. In Section 4, an a priori error bound is derived and a model-reduction algorithm is presented. Finally, in Section 5 the method is applied to a simple interconnected system. The result is also compared to the method introduced in Vandendorpe and Van Dooren (2004).

Notation. Most notation in the paper is standard. To define transfer function matrices the notation $C(sI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is used. The set RH_∞ is the set of real and rational transfer function matrices in the Hardy space H_∞, see Zhou et al. (1996). Let $\|G\|_\infty$ denote the H_∞-norm of $G(s)$. With $\text{diag}\{P_1, P_2\}$ we mean the block-diagonal matrix $\left[\begin{smallmatrix} P_1 & 0 \\ 0 & P_2 \end{smallmatrix}\right]$, with $P > 0$ ($P < 0$) that P is a positive (negative) definite matrix, and with $|x|_P$ the weighted Euclidean norm $\sqrt{x^T P x}$.

2. PRELIMINARIES

The same model framework as in Sandberg and Murray (2007) is used here, and some definitions and results are repeated without proof. We model linear interconnected systems with the linear fractional transform $\mathcal{F}_l(N, G)$, where the network topology is stored in N and the subsystems in G, see Fig. 2 and equations (1)–(2) (see top of next page). The realization (2) is called a *structured realization* of $\mathcal{F}_l(N, G)$. The q subsystems that we want to model reduce are stored in the block-diagonal system

$$G(s) = \text{diag}\{G_1(s), \ldots, G_q(s)\} =: \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right]$$

where

$$A_G = \text{diag}\{A_1, \ldots, A_q\}, \quad B_G = \text{diag}\{B_1, \ldots, B_q\},$$
$$C_G = \text{diag}\{C_1, \ldots, C_q\}, \quad D_G = \text{diag}\{D_1, \ldots, D_q\},$$

and

$$A_k \in \mathbb{R}^{n_k \times n_k}, \qquad B_k \in \mathbb{R}^{n_k \times m_k},$$
$$C_k \in \mathbb{R}^{p_k \times n_k}, \qquad D_k \in \mathbb{R}^{p_k \times m_k}, \quad k = 1, \ldots, q.$$

The interconnection topology and dynamics is modeled by

$$N(s) = \begin{bmatrix} E(s) & F(s) \\ H(s) & K(s) \end{bmatrix} =: \left[\begin{array}{c|cc} A_N & B_{N,1} & B_{N,2} \\ \hline C_{N,1} & D_E & D_F \\ C_{N,2} & D_H & D_K \end{array}\right],$$

where

$$A_N \in \mathbb{R}^{n_N \times n_N}, \qquad B_{N,1} \in \mathbb{R}^{n_N \times m_N},$$
$$C_{N,1} \in \mathbb{R}^{p_N \times n_N}, \qquad D_E \in \mathbb{R}^{p_N \times m_N}.$$

The system K models how the subsystems $G_1, \ldots, G_q$ are connected to each other, and E, F, H model the external excitation and measurement on the network. Throughout the paper it is assumed that $\mathcal{F}_l(N, G)$ is a well-posed and stable feedback connection, i.e., $\|\mathcal{F}_l(N, G)\|_\infty < \infty$. In Sandberg and Murray (2007), it is shown how a mechanical system fits to this framework. The problem we want to solve is to find a new system $\hat{G}$ with the same block-diagonal structure as G, but of smaller McMillan degree, and such that $\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty$ is small. This is a structure-preserving model reduction problem. We try to solve the problem using an extension of balanced truncation. To use balanced truncation, Gramians are needed. Generalized controllability Gramians P and observability Gramians Q (non-unique) satisfy the LMIs,

$$\begin{aligned} AP + PA^T + BB^T &< 0, \quad P > 0, \\ A^T Q + QA + C^T C &< 0, \quad Q > 0, \end{aligned} \tag{3}$$

for an asymptotically stable system, where A, B, C are defined in (2). We say the system $\mathcal{F}_l(N, G)$ has *structured Gramians* if the Gramians are block diagonal,

$$P = \text{diag}\{P_N, P_1, \ldots, P_q\},$$
$$Q = \text{diag}\{Q_N, Q_1, \ldots, Q_q\},$$

such that $P_k, Q_k \in \mathbb{R}^{n_k \times n_k}$, conformally to the structured realization (2). Furthermore, we say the realization and the Gramians are *subsystem balanced* if the coordinates are such that the block-diagonal elements of the Gramians take the form

$$\begin{aligned} Q_k = P_k = \Sigma_k &= \text{diag}\{\sigma_{k,1} \ldots, \sigma_{k,n_k}\}, \\ \sigma_{k,1} \geq \ldots &\geq \sigma_{k,n_k} > 0, \quad k = 1 \ldots q. \end{aligned} \tag{4}$$

The numbers $\sigma_{k,i}$ are called *structured Hankel singular values* of the interconnected system. They are invariant under block-diagonal coordinate transformations, and can be computed as

$$\sigma_{k,i} = \sqrt{\lambda_i(P_k Q_k)}. \tag{5}$$

The following results are shown in Sandberg and Murray (2007).

Proposition 1. If there exist Gramians P and Q for the interconnected system $\mathcal{F}_l(N, G)$, then there exist a block-diagonal coordinate transformation $\bar{x} = Tx$, $T = \text{diag}\{T_N, T_1, \ldots, T_q\}$, $T_N \in \mathbb{R}^{n_N \times n_N}$, $T_k \in \mathbb{R}^{n_k \times n_k}$, $k = 1, \ldots, q$, that makes the realization and the Gramians subsystem balanced (4).

Proposition 2. Assume the interconnected system $\mathcal{F}_l(N, G)$ has *structured Gramians* P and Q and is *subsystem bal-*

$$\mathcal{F}_l(N,G) = E(s) + F(s)(I - G(s)K(s))^{-1}G(s)H(s) \tag{1}$$

$$= \left[\begin{array}{cc|c} A_N + B_{N,2}LD_GC_{N,2} & B_{N,2}LC_G & B_{N,1} + B_{N,2}LD_GD_H \\ B_GMC_{N,2} & A_G + B_GMD_KC_G & B_GMD_H \\ \hline C_{N,1} + D_FD_GMC_{N,2} & D_FLC_G & D_E + D_FD_GMD_H \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{2}$$

$$L := (I - D_GD_K)^{-1}, \quad M := (I - D_KD_G)^{-1}.$$

anced (4). Let the realizations of the subsystems G_k and $\hat{G}_k$, $k = 1 \dots q$, be given by

$$G_k(s) = \left[\begin{array}{cc|c} A_{k,11} & A_{k,12} & B_{k,1} \\ A_{k,21} & A_{k,22} & B_{k,2} \\ \hline C_{k,1} & C_{k,2} & D_k \end{array}\right], \quad \hat{G}_k(s) = \left[\begin{array}{c|c} A_{k,11} & B_{k,1} \\ \hline C_{k,1} & D_k \end{array}\right],$$

where $A_{k,11} \in \mathbb{R}^{r_k \times r_k}$, $B_{k,1} \in \mathbb{R}^{r_k \times m_k}$, and $C_{k,1} \in \mathbb{R}^{p_k \times r_k}$, and the reduced-order system be $\hat{G} = \text{diag}\{\hat{G}_1, \dots, \hat{G}_q\}$. Then

$$\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty \leq 2 \sum_{k=1}^{q} \sum_{i=r_k+1}^{n_k} \sigma_{k,i}. \tag{6}$$

The way to use these results is to first find structured Gramians (if they exist) using LMI software, and then use Proposition 1 to subsystem balance them. The subsystem balanced realization is then truncated as described in Proposition 2. To get small error bounds, the generalized Gramians should be made small. In the proposed algorithm in Section 4 this is achieved by minimizing the trace of P and Q while satisfying (3) and the block-diagonal constraint. The error bound in Proposition 2 is an extension of the balanced truncation error bound in Enns (1984); Glover (1984). Similar bounds are also found in Beck et al. (1996) and Zhou et al. (1995). The main problem with Proposition 2 is that it is only for special network topologies N that there exist structured Gramians. In this paper, we look at the cases where Proposition 2 cannot be directly applied.

Remark 3. Generalized Gramians that satisfy LMIs are more flexible (but also computationally more expensive to compute) than normal Gramians that satisfy Lyapunov *equations*. The reason generalized Gramians are used here is that the block-diagonal structure is needed for the proofs to hold. One can also use the block-diagonal elements of normal Gramians for the reduction. This is done in Vandendorpe and Van Dooren (2004); Sandberg and Murray (2007) and often yields good approximations. The problem is that there are no a priori error bounds then and unstable approximations can be obtained.

3. SYSTEMS WITH STRUCTURED GRAMIANS

A problem with truncating states as suggested in Proposition 2 is that often there are no structured Gramians. In Li and Paganini (2005), it is suggested that one should try to find structured Gramians for coprime factors of the system since this increases the chances of finding them. However, for structured Gramians found from coprime factors there is no error bound like (6) available. Here we make the observation that for certain networks N of simple structure (call them M), there always exist structured Gramians. The idea is then to replace the original system $\mathcal{F}_l(N,G)$ with a suitably chosen simpler system $\mathcal{F}_l(M,G)$ and use it to find structured Gramians.

Consider systems of the form

$$\mathcal{F}_l(M,G) = E + FGH, \quad M = \begin{bmatrix} E & F \\ H & 0 \end{bmatrix}. \tag{7}$$

These systems are much simpler than the general ones since there is no interaction K between the subsystems in G. Still these systems are useful. A restriction is that G generally needs to be stable to ensure that $\mathcal{F}_l(M,G) \in RH_\infty$. Hence, in the following it is assumed that $G \in RH_\infty$. We have the following result that motivates the study of systems in the form $\mathcal{F}_l(M,G)$.

Theorem 4. Consider systems in the form $\mathcal{F}_l(M,G)$, (7), and assume that $E, F, H \in RH_\infty$ and $G = \text{diag}\{G_1, \dots, G_q\} \in RH_\infty$. Then there is a structured realization of $\mathcal{F}_l(M,G)$ that has structured Gramians

$$P = \text{diag}\{P_M, P_1, \dots, P_q\} > 0,$$
$$Q = \text{diag}\{Q_M, Q_1, \dots, Q_q\} > 0.$$

Proof. The proof is an extension of the results in Oh and Kim (2002). The extension lies in that we consider block-diagonal plants G with block-diagonal (structured) Gramians. Note that without loss of generality we can study networks N with $E = 0$. This is because we can split the state vector of a realization of $E + FGH$ into two parts $[x_1^T \quad x_2^T]^T$ where x_1 are the states of E. Using similar techniques as below, it can be shown that once structured Gramians P_2, Q_2 for the system FGH have been found, we can always find Gramians in the form $\text{diag}\{P_1, P_2\}$ and $\text{diag}\{Q_1, Q_2\}$ for the entire system $E + FGH$. Also note that it is enough to study strictly proper subsystems G, i.e., $D_G = 0$. If $D_G \neq 0$, we can rewrite the system as $E + FD_GH + F(G - D_G)H$, and define $E_{new} = E + FDH$ and $G_{new} = G - D_G$.

The structured realization of FGH can be chosen as

$$\left[\begin{array}{ccc|c} A_F & 0 & B_FC_G & 0 \\ 0 & A_H & 0 & B_H \\ 0 & B_GC_H & A_G & B_GD_H \\ \hline C_F & 0 & D_FC_G & 0 \end{array}\right],$$

and the controllability Lyapunov inequality takes the form (8) (see top of next page) assuming the Gramian has the structure $P = \text{diag}\{P_F, P_H, P_G\}$, $P_G = \text{diag}\{P_1, \dots, P_q\}$.

Choose a $\gamma > 0$ such that

$$\left\| \left[\begin{array}{c|c} A_H^T & C_H^TB_G^T \\ \hline B_H^T & D_H^TB_G^T \end{array}\right] \right\|_\infty^2 < \gamma. \tag{9}$$

Since $H \in RH_\infty$ by assumption, there is always such a γ. Then solve $A_GP_G + P_GA_G^T + \gamma I = 0$. Because of the block-diagonal and stable structure of A_G, the solution $P_G > 0$ has clearly the required block-diagonal structure. What remains to show is that there always exist P_F and P_H that satisfy (8) with this P_G.

$$\begin{bmatrix} A_F P_F + P_F A_F^T & 0 & B_F C_G P_G \\ 0 & A_H P_H + P_H A_H^T + B_H B_H^T & P_H C_H^T B_G^T + B_H D_H^T B_G^T \\ P_G C_G^T B_F^T & B_G C_H P_H + B_G D_H B_H^T & A_G P_G + P_G A_G^T + B_G D_H D_H^T B_G^T \end{bmatrix} < 0 \tag{8}$$

The first step is to study the lower right 2×2 block matrix of (8). Let $R := \gamma I - B_G D_H D_H^T B_G^T$. Note that $R > 0$, by the choice of γ. Given P_G and R, we want to find a P_H such that

$$\begin{bmatrix} A_H P_H + P_H A_H^T + B_H B_H^T & P_H C_H^T B_G^T + B_H D_H^T B_G^T \\ B_G C_H P_H + B_G D_H B_H^T & -R \end{bmatrix} < 0. \tag{10}$$

By the bounded real lemma, see Zhou et al. (1996), this LMI has a solution $P_H > 0$ if, and only if, (9) holds. Denote the matrix on the left hand side in (10) by $-L$.

The second step is to note that the Lyapunov controllability inequality (8) has a solution if (by Schur complements)

$$A_F P_F + P_F A_F^T + [0 \quad B_F C_G P_G] L^{-1} [0 \quad B_F C_G P_G]^T < 0,$$

and $L > 0$. Since A_F is a Hurwitz matrix ($F \in RH_\infty$ by assumption) there is always a positive definite solution P_F. This concludes the proof and shows there always exist a structured Gramian $P = \text{diag}\{P_F, P_H, P_G\}$. The proof that there is a structured observability Gramian is the dual to the above proof. ■

Thus Proposition 2 can always be applied to systems $\mathcal{F}_l(M, G)$. Next, it is shown how this result can be used to deal with general network topologies N with $K \neq 0$.

General network topologies N. Consider a Taylor approximation of $\mathcal{F}_l(N, \hat{G})$ around $\hat{G} = G$. Using the perturbation model $\hat{G} = G + \Delta$,

$$\begin{aligned} &\mathcal{F}_l(N, \hat{G}) - \mathcal{F}_l(N, G) \\ &\quad = F(I - \hat{G}K)^{-1}\hat{G}H - F(I - GK)^{-1}GH \\ &\quad = F(I - GK)^{-1}\Delta(I - KG)^{-1}H + \text{H.O.T.} \\ &\quad = \mathcal{F}_l(\hat{N}_1, \hat{G}) - \mathcal{F}_l(\hat{N}_1, G) + \text{H.O.T.}, \end{aligned}$$

if $\|\Delta K(I - GK)^{-1}\|_\infty < 1$, where

$$\begin{aligned} &\hat{N}_1 := \begin{bmatrix} 0 & F_1 \\ H_1 & 0 \end{bmatrix}, \\ &F_1 = F(I - GK)^{-1}, \quad H_1 = (I - KG)^{-1}H. \end{aligned} \tag{11}$$

Hence, the first-order term in $\mathcal{F}_l(N, \hat{G}) - \mathcal{F}_l(N, G)$ can be made small by choosing a reduced model $\hat{G}$ that makes $\|\mathcal{F}_l(\hat{N}_1, G) - \mathcal{F}_l(\hat{N}_1, \hat{G})\|_\infty$ small. Note that $\hat{N}_1$ belongs to the class M in (7). For the Taylor series to converge it is essential that $\|\Delta K(I - GK)^{-1}\|_\infty < 1$. By the small-gain theorem this condition is also sufficient for stability of $\mathcal{F}_l(N, \hat{G})$ if $\mathcal{F}_l(N, G)$ is stable. Notice that

$$\mathcal{F}_l(\hat{N}_2, G) - \mathcal{F}_l(\hat{N}_2, \hat{G}) = \Delta K(I - GK)^{-1},$$

using

$$\hat{N}_2 := \begin{bmatrix} 0 & F_2 \\ H_2 & 0 \end{bmatrix}, \quad F_2 = I, \quad H_2 = K(I - GK)^{-1}. \tag{12}$$

Note that $\hat{N}_2$ belongs to the class M in (7). By reducing the subsystems in G with respect to the system $\mathcal{F}_l(\hat{N}_2, G)$, we can ensure that $\|\Delta K(I - GK)^{-1}\|_\infty < 1$ is true by making the error bound (6) in Proposition 2 smaller than 1.

The following lemma is useful to us, since it shows that $\hat{N}_1$ and $\hat{N}_2$ can be augmented into a larger M.

Lemma 5. Let

$$M = \begin{bmatrix} E & F \\ H & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & F_1 \\ E_{21} & E_{22} & F_2 \\ H_1 & H_2 & 0 \end{bmatrix},$$

and assume $\|\mathcal{F}_l(M, G)\|_\infty \le \gamma$. Then $\|\mathcal{F}_l(M_i, G)\|_\infty \le \gamma$, where $M_i = \begin{bmatrix} E_{ii} & F_i \\ H_i & 0 \end{bmatrix}$, $i = 1, 2$.

Proof. We have

$$\mathcal{F}_l(M, G) = \begin{bmatrix} \mathcal{F}_l(M_1, G) & E_{12} + F_1 G H_2 \\ E_{21} + F_2 G H_1 & \mathcal{F}_l(M_2, G) \end{bmatrix}.$$

If the induced L_2-norm of $\mathcal{F}_l(M, G)$ is less than γ, then the induced L_2-norms of all the block elements are less than γ. ■

Hence, if we want to make both $\|\mathcal{F}_l(\hat{N}_1, G) - \mathcal{F}_l(\hat{N}_1, \hat{G})\|_\infty$ and $\|\mathcal{F}_l(\hat{N}_2, G) - \mathcal{F}_l(\hat{N}_2, \hat{G})\|_\infty$ small, we should find structured Gramians for $\mathcal{F}_l(\hat{N}, G)$ where

$$\hat{N} := \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & F_2 \\ H_1 & H_2 & 0 \end{bmatrix}, \tag{13}$$

and F_1, H_1, F_2, H_2 are taken from $\hat{N}_1$ and $\hat{N}_2$. Using this method, a structure-preserving model reduction method for general interconnected linear systems is suggested in the next section.

4. ERROR BOUNDS AND ALGORITHM

Using the network approximation $\hat{N}$ for the general interconnected system $\mathcal{F}_l(N, G)$, we can derive the following theorem.

Theorem 6. Assume that $\mathcal{F}_l(N, G)$ is internally stable, and that $G, E, F, H \in RH_\infty$. Furthermore, let the *structured* and *subsystem balanced* Gramians to the system $\mathcal{F}_l(\hat{N}, G)$ be

$$\begin{aligned} P &= \text{diag}\{P_{\hat{N}}, \Sigma_1, \ldots, \Sigma_q\}, \\ Q &= \text{diag}\{Q_{\hat{N}}, \Sigma_1, \ldots, \Sigma_q\}, \\ \Sigma_k &= \text{diag}\{\sigma_{k,1} \ldots, \sigma_{k,n_k}\} > 0, \quad k = 1 \ldots q, \end{aligned}$$

where $\hat{N}$ is defined in (11)–(13). Then the reduced-order model $\hat{G} = \text{diag}\{\hat{G}_1, \ldots, \hat{G}_q\}$, defined by

$$G_k(s) = \left[\begin{array}{cc|c} A_{k,11} & A_{k,12} & B_{k,1} \\ A_{k,21} & A_{k,22} & B_{k,2} \\ \hline C_{k,1} & C_{k,2} & D_k \end{array}\right], \quad \hat{G}_k(s) = \left[\begin{array}{c|c} A_{k,11} & B_{k,1} \\ \hline C_{k,1} & D_k \end{array}\right],$$

where $A_{k,11} \in \mathbb{R}^{r_k \times r_k}$, $B_{k,1} \in \mathbb{R}^{r_k \times m_k}$, and $C_{k,1} \in \mathbb{R}^{p_k \times r_k}$, satisfies

(i) $\hat{G} \in RH_\infty$;
(ii) $\mathcal{F}_l(N, \hat{G}) \in RH_\infty$, if $2\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k} \sigma_{k,i} < 1$;
(iii)

$$\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty \le (2 + \text{o}(1)) \sum_{k=1}^{q} \sum_{i=r_k+1}^{n_k} \sigma_{k,i},$$

as $\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i} \to 0$.

Proof. Using $\hat{N}$, Lemma 5, and Proposition 2, we have that

(a) $\|\Delta K(I-GK)^{-1}\|_\infty \le 2\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i}$;
(b) $\|\mathcal{F}_l(\hat{N}_1,\hat{G}) - \mathcal{F}_l(\hat{N}_1,G)\|_\infty \le 2\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i}$.

Now we can prove the statements of the theorem:

(i) From the (3,3)-block of the LMI (8) in the proof of Theorem 4, it holds that $A_k\Sigma_k + \Sigma_k A_k^T < 0$, $k = 1,\ldots,q$. Since $\Sigma_k > 0$ is diagonal, this proves that the block $A_{k,11}$ on the diagonal of A_k has its eigenvalues in the left complex half plane.
(ii) The small-gain theorem gives that $\mathcal{F}_l(N,\hat{G}) = \mathcal{F}_l(N,G+\Delta) \in RH_\infty$, assuming $\mathcal{F}_l(N,G)$ and G are in RH_∞ and $\|\Delta K(I-GK)^{-1}\|_\infty < 1$. Using (a) the result follows.
(iii) Keeping the second-order term of the Taylor series expansion, we have

$$\begin{aligned}
&\mathcal{F}_l(N,\hat{G}) - \mathcal{F}_l(N,G)\\
&\quad= F(I-\hat{G}K)^{-1}\hat{G}H - F(I-GK)^{-1}GH\\
&\quad= F[(I-GK)^{-1} + (I-GK)^{-1}\Delta K(I-GK)^{-1}\\
&\qquad+ \text{H.O.T.}](G+\Delta)H - F(I-GK)^{-1}GH\\
&\quad= F(I-GK)^{-1}\Delta(I-KG)^{-1}H\\
&\qquad+ F(I-GK)^{-1}\Delta K(I-GK)^{-1}\Delta H + \text{H.O.T.}
\end{aligned}$$

Using the triangle inequality, (a), and (b), it follows that

$$\begin{aligned}
&\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty\\
&\le \|\mathcal{F}_l(\hat{N}_1,\hat{G}) - \mathcal{F}_l(\hat{N}_1,G)\|_\infty\\
&\quad+ \|\Delta K(I-GK)^{-1}\|_\infty \|F(I-GK)^{-1}\|_\infty \|\Delta H\|_\infty\\
&\quad+ \cdots\\
&\le 2\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i} + \mathrm{o}\left(2\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i}\right).
\end{aligned}$$

Notice that $\|\Delta H\|_\infty \to 0$ as $\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i} \to 0$. This is because $P > 0$ and $Q > 0$, and that the structured singular values are strictly positive, meaning that the only way $\sum_{k=1}^{q}\sum_{i=r_k+1}^{n_k}\sigma_{k,i}$ goes to zero is that if $r_k \to n_k$, implying that $\Delta \to 0$.

■

The theorem shows that structured Gramians for $\mathcal{F}_l(\hat{N},G)$ (existence guaranteed) can be used for model reduction of G, and some guarantees on the quality of approximation of the original interconnected system $\mathcal{F}_l(N,G)$ hold. The structured Hankel singular values $\sigma_{k,i}$ can be used to choose approximation order r_k. For each subsystem G_k, there is in practice often a significant drop in size of the singular values $\sigma_{k,i}$ for some i. A good initial guess for r_k is to keep the dominant singular values.

Remark 7. It is also possible to compute structured Gramians for $\mathcal{F}_l(\hat{N}_1,G)$ or $\mathcal{F}_l(\hat{N}_2,G)$ alone, and use them to reduce G. This has some computational advantages since the dimensions of the LMIs are smaller. Notice that if the Gramians come from $\mathcal{F}_l(\hat{N}_1,G)$, then Theorem 6(i) still holds, and if the Gramians come from $\mathcal{F}_l(\hat{N}_2,G)$, then Theorem 6(i)–(ii) still hold.

Remark 8. The model $\hat{N}$ is useful for understanding what the important dynamics in the subsystems in G are. One way to interpret Theorem 6 is as a frequency-weighted model reduction problem, where $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}(G-\hat{G})\,[H_1\ \ H_2]$, is made small. When the F- and H- components are "large", the approximation error $G-\hat{G}$ has to be "small". Hence, by studying the weights, we can see what frequencies are amplified and attenuated and gain insight about the interconnected system.

Model reduction algorithm. An algorithm that implements the proposed model-reduction method is described next. Three different methods are suggested, denoted by (MA)–(MC).

0. Form a structured realization of the interconnected system $\mathcal{F}_l(N,G)$ and $G = \operatorname{diag}\{G_1,\ldots,G_q\}$, and choose approximation tolerance ϵ.
1. Choose a model M with structure (7). Three choices, (MA)–(MC), are suggested in this paper:
   (MA) $M = \hat{N}_1$ from (11);
   (MB) $M = \hat{N}_2$ from (12);
   (MC) $M = \hat{N}$ from (13).
2. Form a structured realization of $\mathcal{F}_l(M,G) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.
3. Compute structured Gramians (existence guaranteed by Theorem 4) by solving
$$\min \sum_{k=1}^{q} \operatorname{trace} P_k, \quad \text{s.t.} \quad AP + PA^T + BB^T < 0$$
$$P = \operatorname{diag}\{P_M, P_1, \ldots, P_q\}$$
and
$$\min \sum_{k=1}^{q} \operatorname{trace} Q_k, \quad \text{s.t.} \quad A^TQ + QA + C^TC < 0$$
$$Q = \operatorname{diag}\{Q_M, Q_1, \ldots, Q_q\}.$$
4. Compute structured Hankel singular values $\sigma_{k,i} = \sqrt{\lambda_i(P_kQ_k)}$ and apply subsystem balancing coordinate transformations T_k to G, see Proposition 1. Choose approximation order r_k. If (MC), r_k can be chosen using Theorem 6(iii) and the tolerance ϵ.
5. Truncate the realization of G as described in Proposition 2 and Theorem 6 to obtain $\hat{G}$. Form $\mathcal{F}_l(N,\hat{G})$.
6. If $\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\hat{G})\|_\infty \le \epsilon$ stop, else increase r_k by including the largest previously truncated structured Hankel singular value, and goto 5.

5. EXAMPLE

In this section, we apply the model-reduction methods (MA)–(MC) to an interconnected mechanical system, see Fig. 3. The model is further described in Sandberg and Murray (2007). The model has two subsystems G_1 and G_2 consisting of two models of elastic masses of order $n_1 = 8$ and $n_2 = 10$. The masses are interconnected with a linear spring with spring constant k. As the spring constant is varied, different dynamics get excited in the masses. This means that the reduced models will depend on the spring constant. The structured realization of $\mathcal{F}_l(N,G)$ does not have structured Gramians when $k \ge 1$. Hence, we use approximate network models in the class M. Note that if

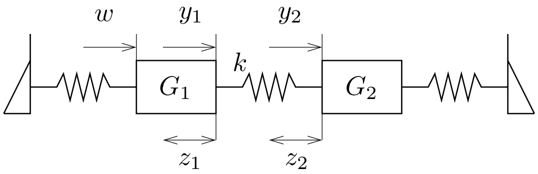

Fig. 3. The interconnected mechanical system. The first mass is perturbed by the force w and we measure the positions z_1 and z_2. The forces y_1 and y_2 are interaction forces between the systems, determined by the spring k and positions z_1 and z_2.

standard balanced truncation is applied to $\mathcal{F}_l(N, G)$, then it will generally not be possible to extract reduced models $\hat{G}_1$ and $\hat{G}_2$ of the individual mass models.

An extension of the closed-loop balanced truncation method, see Vandendorpe and Van Dooren (2004), is also applied to this example in Sandberg and Murray (2007). This method is denoted (MO) here. Method (MO) does in general not have an a priori error bound like Theorem 6 and it relies on Gramians that solve Lyapunov *equations*, see Remark 3. Here we compare the methods (MO) and (MA)–(MC). The results are summarized in Table 1.

Table 1. The approximation errors $\|\mathcal{F}_l(N, G) - \mathcal{F}_l(N, \hat{G})\|_\infty$ are listed for different k, r_1, r_2, along with the total computation time (CPU) on a 2GHz MacBook. $0.35 \leq \|\mathcal{F}_l(N, G)\|_\infty \leq 0.5$ in all cases.

k	r_1	r_2	(MO)	(MA)	(MB)	(MC)
1	6	2	0.035	0.044	0.044	0.044
10	6	6	0.036	0.036	0.036	0.036
20	6	6	0.044	0.075	0.076	0.076
40	6	8	∞	0.10	0.10	0.10
100	6	6	0.12	0.032	0.032	∞
CPU [sec]			7	96	48	217

From Table 1, we can draw a number of conclusions:

- The method (MO) is the computationally cheapest method. This is not surprising since it does not require solutions of LMIs but solutions of Lyapunov equations.
- For small spring constants ($k \leq 20$), the method (MO) also delivers the best approximations but the difference to (MA)–(MC) is very small. However, when $k = 40$, the two masses interact heavily and (MO) fails to deliver a stable approximation, i.e., $\mathcal{F}_l(N, \hat{G}) \notin RH_\infty$.
- The methods (MA)–(MC) all deal well with the most difficult case, $k = 40$. The methods (MA)–(MB) yield good approximations for all spring constants.
- The methods (MA)–(MC) all run into numerical problems for large k. This is especially true for (MC) that is not able to deliver a stable approximation when $k = 100$ because of numerical problems.

Based on the results, it is suggested that (MO) is the first method of choice for computing a structured approximation. If it fails to deliver a good approximation then (MB) is able to deliver a good and stable approximation within a reasonable amount of time. Notice that when (MB) is used, we can use Theorem 6 to guarantee stability using the structured singular values. The method (MC) that is the preferred method from a theoretical standpoint is rather computationally heavy. This is due to the high order of the network $\hat{N}$ and the resulting LMIs.

6. CONCLUSION

We have characterized a class of interconnected systems $\mathcal{F}_l(M, G)$ that always has structured Gramians. Interconnected systems that are not in this class can be approximated by systems in the form $\mathcal{F}_l(M, G)$. The structured Gramians can be used for model reduction that preserves the interconnection structure. It was also shown how asymptotic a priori error bounds are obtained. In the example studied, the introduced method compared favorably to the method in Vandendorpe and Van Dooren (2004); Sandberg and Murray (2007) in the sense that it always delivered stable approximations with small approximation error. However, this came at a higher computational cost.

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