IT for Statistics and Learning
2023

Assignment 4
Assigned: Fri Dec 1, 2023
Due: Fri Dec 8, 2023 M. Skoglund

Problem 4.1: For any test $P_{Z|\omega}$, prove that

$$P_Z(\{0\}) - Q_Z(\{0\}) \leq \sqrt{\frac{1}{2} D(P\|Q)}$$

(with $D(P\|Q)$ in nats) where $P_Z = P_{Z|\omega} \circ P$ and $Q_Z = P_{Z|\omega} \circ Q$, and assuming $P \ll Q$.

Problem 4.2: We proved in class that the LLR test is optimal in the sense that it acheives $\beta_\alpha(P, Q)$. Prove that the test can always be implemented, that is, for any $0 < \alpha < 1$ and

$$\beta_\alpha(P, Q) = \inf_{P_{Z|\omega}: P_Z(\{0\}) \geq \alpha} Q_Z(\{0\}),$$

where $P_Z = P_{Z|\omega} \circ P$ and $Q_Z = P_{Z|\omega} \circ Q$, there is a τ and a λ such that $\alpha = P(\{L > 0\}) + \lambda P(\{L = \tau\})$. Furthermore, prove also that the LLR test is unique, in the sense that any test that achieves $\beta_\alpha(P, Q)$ must be an LLR test. That is, it has the form

$$P_{Z|\omega}(\{0\}|\omega) = \begin{cases} 1 & L(\omega) > \tau \\ \lambda & L(\omega) = \tau \\ 0 & L(\omega) < \tau \end{cases}$$

(assuming $P \ll Q$ and except possibly for $\omega \in \Omega$ in a set of zero probability under both P and Q).

Problem 4.3: Assume $\Theta = \mathbb{R}$ and let $R_\theta(\hat{\theta})$ denote the risk corresponding to the rule $P_{\hat{\theta}|X=x}$. Assume that T is a sufficient statistic of X for θ described by the kernel $P_{T|X}$. Prove that for any decision rule $P_{\hat{\theta}|X=x}$ there is a rule $P_{\hat{\theta}|T=t}$ that results in the same risk. That is, there is no loss in basing the decision on T instead of X.

Problem 4.4: For $\Theta = \mathbb{R}$ and $T(\theta) = \theta$ prove that the Bayes estimator $P^*_{\hat{\theta}|X=x}$ always can be taken to correspond to a deterministic function $\hat{\theta}(x)$ without loss, by using the data processing inequality for Bayes risk to demonstrate that randomization cannot decrease the risk.

Problem 4.5: Let $\Theta = \mathbb{R}$ and assume $\ell(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$. Specify the Bayes estimator for θ given $X = x$ when X is absolutely continuous with density

$$f_\theta(x) = e^{-|x-\theta|/2}$$

and the prior π for θ has density $\exp(-|\theta - \eta|/2)$ for some $\eta \in (0, \infty)$.

Problem 4.6: Assume that $X \in \{0, 1, \ldots, n\}$ is a binomial random variable, that is

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

and that n is fixed and known but $0 < p < 1$ is unknown. Determine the minimax estimator for p based on $X = x$ for the loss function

$$\ell(p, \hat{p}) = \frac{(p - \hat{p})^2}{p(1-p)}$$