Information Theory
Spring semester, 2023

Assignment 9
Assigned: Friday, June 2, 2023
Due: Thursday, June 8, 2023 M. Skoglund

Problem 9.1: (Gallager)
(a) Compute

$$\sum_{y_i} p(y_i|0)^{1-s} p(y_i|1)^s$$

for the binary symmetric channel, the Z-channel, and the binary erasure channel (use ϵ as a parameter), and minimize the result over $s \in (0,1)$. Use this to provide an upper bound to the probability of maximum likelihood decoding error $P_{e,m}$ ($m = 1, 2$) attained using a code with two codewords $x_1 = 0^n$ (a sequence of n zeros) and $x_2 = 1^n$ (a sequence of n ones).

(b) Find *exact* expressions for the above error probabilities. Evaluate the expressions to 3 significant digits for $n = 32$ and $\epsilon = 0.1$ and compare with the bound in (a).

(c) For the BSC, *show* that

- for large even n,

$$P_{e,1} \approx \sqrt{\frac{2}{\pi n}} \left(\frac{1-\epsilon}{1-2\epsilon}\right) \left[2\sqrt{\epsilon(1-\epsilon)}\right]^n; \qquad P_{e,2} \approx P_{e,1}\left(\frac{\epsilon}{1-\epsilon}\right)$$

- for large odd n,

$$P_{e,1} = P_{e,2} \approx \sqrt{\frac{2\epsilon}{\pi n(1-\epsilon)}} \left(\frac{1-\epsilon}{1-2\epsilon}\right) \left[2\sqrt{\epsilon(1-\epsilon)}\right]^n$$

(d) Repeat parts (a) and (b) for the Z-channel for a code whose codewords are $x_1 = 0^{n/2}1^{n/2}$ and $x_2 = 1^{n/2}0^{n/2}$. Observe that this change of code will make no difference for the *other* channels.

Problem 9.2:
Prove the inequality:

$$\left[\sum_x q(x) p(y|x)^{1/(1+\rho)}\right]^{1+\rho} \leq \left[\sum_x q(x) p(y|x)^s\right]^\rho \left[\sum_x q(x) p(y|x)^{1-s\rho}\right]$$

That is, the right side is minimized over $s > 0$ by choosing $s = 1/(1+\rho)$. Use standard inequalities, do **not** take derivatives!