Probability and Random Processes
Spring semester, 2024

Assignment 11
Assigned: Thursday, April 18, 2024
Due: Friday, April 26, 2024 M. Skoglund

Problem 11.1: Define and describe the concept of a measurable dynamical system and relate it to possible ways of modeling random processes.

Problem 11.2: Define, describe and relate the concepts AMS, stationary, recurrent and ergodic.

Problem 11.3: State and explain the ergodic theorem.

Problem 11.4: Given a measurable dynamical system, let $\mathcal{I}$ be the class of invariant sets. Prove that $\mathcal{I}$ is a σ-algebra.

Problem 11.5: Show that if $(\Omega, \mathcal{A}, P_1, \phi)$ and $(\Omega, \mathcal{A}, P_2, \phi)$ are both stationary and ergodic, then either $P_1 = P_2$ or $P_1 \perp P_2$ (that is, there is a set $A \in \mathcal{A}$ such that $P_1(A^c) = 0$ and $P_2(A) = 0$).

Problem 11.6: Let $\Omega = [0, 1)$ and define $\phi : \Omega \to \Omega$ by

$$\phi(x) = \begin{cases} 2x, & 2x < 1 \\ 2x - 1, & 2x \geq 1 \end{cases}$$

(i.e., $\phi(x) = 2x \bmod 1$). For $x \in [0, 1)$, prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi^n x = \frac{1}{2} \quad \text{Leb. a.e}$$

Problem 11.7: Consider a discrete-time random process $\{X_n\}_{n=1}^{\infty}$, $X_n \in \mathbb{R}$, modeled as: X_n and X_m are independent for $n \neq m$, and for each n, X_n is Gaussian with variance σ^2. Also, at time $n = 0$ the mean-value of X_n for future n's is drawn from a zero-mean Gaussian distribution with variance 1 (drawn once, to hold for all $n \geq 1$). Characterize the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_n$$

and relate it to concepts discussed in class.