Probability and Random Processes

Lecture 6

- Differentiation
- Absolutely continuous functions
- Continuous vs. discrete random variables
- Absolutely continuous measures
- Radon–Nikodym

Bounded Variation

- Let f be a real-valued function on $[a, b]$
- Total variation of f over $[a, b]$,

$$V_a^b f = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}$$

  over all $a = x_0 < x_1 < \cdots < x_n = b$ and n
- f is of bounded variation on $[a, b]$ if $V_a^b f < \infty$
- f of bounded variation $\Rightarrow$ f differentiable Lebesgue-a.e.

The indefinite Lebesgue integral

- Assume that f is Lebesgue measurable and integrable on $[a, b]$ and set

$$F(x) = \int_a^x f(t)dt$$

for $a \leq x \leq b$, then F is continuous and of bounded variation, and

$$V_a^b F = \int_a^b |f(x)|dx$$

(the integrals are Lebesgue integrals). Furthermore F is differentiable a.e. and $F'(x) = f(x)$ a.e. on $[a, b]$

Absolutely Continuous on $[a, b]$

Definite Lebesgue integration

- For $f : [a, b] \to \mathbb{R}$, assume that f' exists a.e. on $[a, b]$ and is Lebesgue integrable. If

$$f(x) = f(a) + \int_a^x f'(t)dt$$

for $x \in [a, b]$ then f is absolutely continuous on $[a, b]$

$\iff$ for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for any sequence $\{(a_k, b_k)\}$ of pairwise disjoint (a_k, b_k) in $[a, b]$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$

Absolutely Continuous on $\mathbb{R}$

- f is absolutely continuous on $\mathbb{R}$ if it's absolutely continuous on $[-\infty, \infty]$ and $\lim_{x \to -\infty} f(x) = 0$, i.e.

$$f(x) = \int_{-\infty}^{x} f'(t)dt$$

for all $x \in (-\infty, \infty)$

$\iff$ f is absolutely continuous on every $[a, b]$, $-\infty < a < b < \infty$, $V_{-\infty}^{\infty} f < \infty$ and $\lim_{x \to -\infty} f(x) = 0$

Discrete and Continuous Random Variables

- A probability space $(\Omega, \mathcal{A}, P)$ and a random variable X
- X is discrete if there is a countable set $K \in \mathcal{B}$ such that $P(X \in K) = 1$
- X is continuous if $P(X = x) = 0$ for all $x \in \mathbb{R}$

Distribution Functions and pdf's

- A random variable X is absolutely continuous if there is a nonnegative $\mathcal{B}$-measurable function f_X such that

$$\mu_X(B) = \int_B f_X(x)dx$$

for all $B \in \mathcal{B}$

$\iff$ The probability distribution function F_X is absolutely continuous on $\mathbb{R}$

- The function f_X is called the probability density function (pdf) of X, and it holds that $f_X = F'_X$ a.e.

Absolutely Continuous Measures

- Question: Given measures μ and ν on $(\Omega, \mathcal{A})$, under what conditions does there exist a density f for ν w.r.t. μ, such that

$$\nu(A) = \int_A f d\mu$$

for any $A \in \mathcal{A}$?

- Necessary condition: $\nu(A) = 0$ if $\mu(A) = 0$ (why?)
  - e.g. the Dirac measure cannot have a density w.r.t. Lebesgue measure
- ν is said to be absolutely continuous w.r.t. μ, notation $\nu \ll \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$
- Absolute continuity and σ-finiteness are necessary and sufficient conditions. . .

Radon–Nikodym

- The Radon–Nikodym theorem: If μ and ν are σ-finite on $(\Omega, \mathcal{A})$ and $\nu \ll \mu$, then there is a nonnegative extended real-valued $\mathcal{A}$-measurable function f on Ω such that

$$\nu(A) = \int_A f d\mu$$

  for any $A \in \mathcal{A}$. Furthermore, f is unique μ-a.e.
- The μ-a.e. unique function f in the theorem is called the Radon–Nikodym derivative of ν w.r.t. μ, notation $f = \frac{d\nu}{d\mu}$

Absolutely Continuous RV's and pdf's, again

- (Obviously), the pdf of an absolutely continuous random variable X is the Radon–Nikodym derivative of μ_X w.r.t. Lebesgue measure (restricted to $\mathcal{B}$),

$$\mu_X(B) = \int_B f_X(x)dx = \int_B \frac{d\mu_X}{dx} dx$$

Absolutely Continuous Functions vs. Measures

- If μ is a finite measure with distribution function F_μ, then μ is absolutely continuous w.r.t. Lebesgue measure, λ, iff F_μ is absolutely continuous on $\mathbb{R}$, and in this case

$$\frac{d\mu}{d\lambda} = F'_\mu \;\; \lambda\text{-a.e.}$$