Probability and Random Processes

Lecture 8

- Topologies and metrics
- Standard spaces

Topological Spaces

- How do we measure "closeness" for objects in abstract spaces?
- Consider $\mathbb{R}$ and the collection $\mathcal{O}$ of open intervals, or more generally open sets
- $f : \mathbb{R} \to \mathbb{R}$ is continuous at b if $f(x)$ is close to $f(b)$ for all x sufficiently close to b

$\iff$ for each $\varepsilon > 0$ there is a $\delta > 0$ such that $f(x) \in (f(b) - \varepsilon, f(b) + \varepsilon)$ for all $x \in (b - \delta, b + \delta)$

$\iff$ for each $O_1 \in \mathcal{O}$ containing $f(b)$, there is a set $O_2 \in \mathcal{O}$ containing b such that $f(x) \in O_1$ for all $x \in O_2$

$\iff$ $f^{-1}(O) \in \mathcal{O}$ for all $O \in \mathcal{O}$

- Hence, the class of open sets appears to be fundamental in making statements about "closeness" and "limits"

- Fundamental properties of sets in $\mathcal{O}$ (on the real line):
  - $\mathbb{R}$ and $\emptyset$ are in $\mathcal{O}$
  - if A and B are in $\mathcal{O}$ then so is $A \cap B$
  - if $\{O_i\}$ are all open, then so is $\cup_i O_i$

$\Rightarrow$ a characterization of "open sets" in the general case

- For a given nonempty set Ω, a class $\mathcal{T}$ of subsets is a topology on Ω if
  1. $\Omega, \emptyset \in \mathcal{T}$
  2. $O_1, O_2 \in \mathcal{T} \Rightarrow O_1 \cap O_2 \in \mathcal{T}$
  3. $\mathcal{S} \subset \mathcal{T} \Rightarrow \bigcup_{O \in \mathcal{S}} O \in \mathcal{T}$
- The pair $(\Omega, \mathcal{T})$ is a topological space and the sets in $\mathcal{T}$ are called $\mathcal{T}$-open, or simply open

Continuous and Borel Measurable Functions

- Let $\mathcal{A} = (\Omega, \mathcal{T})$ and $\mathcal{B} = (\Lambda, \mathcal{S})$ be topological spaces, then a function $f : \Omega \to \Lambda$ is continuous if $O \in \mathcal{S} \Rightarrow f^{-1}(O) \in \mathcal{T}$
- Given $\mathcal{A} = (\Omega, \mathcal{T})$, the σ-algebra generated by $\mathcal{T}$ is the Borel σ-algebra on $(\Omega, \mathcal{T})$, notation $\sigma(\mathcal{A})$
- $(\Omega, \sigma(\mathcal{A}))$ is the (measurable) Borel space corresponding to $\mathcal{A} = (\Omega, \mathcal{T})$
- Given $\mathcal{A} = (\Omega, \mathcal{T})$ and $\mathcal{B} = (\Lambda, \mathcal{S})$, a function $f : \Omega \to \Lambda$ is Borel measurable if $O \in \sigma(\mathcal{B}) \Rightarrow f^{-1}(O) \in \sigma(\mathcal{A})$
  - usually the default for "measurable function" is "Borel measurable"

Metric Spaces

- For a given set Ω, a function $\rho : \Omega \times \Omega \to \mathbb{R}$ is a metric if for all $x, y, z \in \Omega$
  1. $\rho(x, y) \geq 0$ with $=$ only if $x = y$
  2. $\rho(x, y) = \rho(y, x)$
  3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$
- The pair (Ω, ρ) is a metric space

Metric Spaces as Topological Spaces

- Given (Ω, ρ), the set $B_r(x) = \{y \in \Omega : \rho(x, y) < r\}$ is called the open ball of radius r centered at x
- A set O in Ω is open if for any $x \in O$ there is an r such that $B_r(x) \subset O$,

$\Rightarrow$ defines a topology $\mathcal{T}_\rho$ on Ω; the topology induced by ρ

- Two metrics ρ_1 and ρ_2 are equivalent if $\mathcal{T}_{\rho_1} = \mathcal{T}_{\rho_2}$
- $(\Omega, \mathcal{T})$ is metrizable if there is a metric ρ such that $\mathcal{T} = \mathcal{T}_\rho$
- Example: $(\mathbb{R}^n, \mathcal{T})$ with $\mathcal{T} = \mathcal{T}_\rho$ using $\rho(x, y) = \|x - y\|$ (ordinary Euclidean distance)
  - for $\mathbb{R}^n$ we always assume this topology

Sequences and Completeness

- A topological space $(\Omega, \mathcal{T})$ and a sequence $\{x_n\}$, $x_n \in \Omega$
- The sequence converges to $x \in \Omega$ if
  - for each $O \in \mathcal{T}$ such that $x \in O$ there is an N such that $x_n \in O$ for all $n \geq N$
- In a metric space (Ω, ρ), a sequence $\{x_n\}$
  - is a Cauchy sequence if for each $\varepsilon > 0$ there is an N such that $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$
  - converges to a point x if $\lim_n \rho(x_n, x) = 0$
- (Ω, ρ) is complete if all Cauchy sequences converge to a point in Ω
- $(\Omega, \mathcal{T})$ is completely metrizable if there is a complete metric space (Γ, ρ) and a 1-to-1 mapping between $(\Omega, \mathcal{T})$ and $(\Gamma, \mathcal{T}_\rho)$ that is continuous in both directions

Limit Points, Closure

- A topological space $(\Omega, \mathcal{T})$. Given a set $E \subset \Omega$, a point $x \in \Omega$ is a limit point of E if $O \cap E \neq \emptyset$ for all $O \in \mathcal{T}$ with $x \in O$
- The set of all limit points of E = the closure of E, notation $\overline{E}$
- A set E is closed if E^c is open
- $\overline{E}$ is the smallest closed set that contains E

Separability

- A set E is dense in Ω if $\overline{E} = \Omega$
  - c.f. the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$
- A topological space $(\Omega, \mathcal{T})$ is separable if there is a countable set $E \subset \Omega$ such that $\overline{E} = \Omega$
  - c.f. $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and $\mathbb{R} = \overline{Q}$
- $(\Omega, \mathcal{T})$ is a Polish space if it is completely metrizable and separable

Compactness

- Given a set E, a collection $\mathcal{S}$ of sets is a covering of E if $E \subset \bigcup_{S \in \mathcal{S}} S$
- Given E, if $\mathcal{S}$ is a covering of E and $\mathcal{S}' \subset \mathcal{S}$ is also a covering, then $\mathcal{S}'$ is a subcovering
- In $(\Omega, \mathcal{T})$ a covering $\mathcal{S}$ is open if $\mathcal{S} \subset \mathcal{T}$
- Given $(\Omega, \mathcal{T})$, a subset $E \subset \Omega$ is compact if every open covering of E has a finite subcovering
  - $E \subset \mathbb{R}^n$ is compact $\iff$ E is closed and bounded
- $(\Omega, \mathcal{T})$ is compact if Ω is compact
  - $\mathbb{R}^n$ is not compact...

Standard Spaces

Three kinds of "standard" (probability) spaces

- Standard Borel spaces: Borel equivalence to $([0,1],\mathcal{B}([0,1]))$
- Standard spaces as defined by Gray: The "countable extension property" (next lecture...)
- Lebesgue spaces: Isomorphic to a mixture of $([0,1],\mathcal{L}([0,1]),\lambda)$ and a countable space

Standard Borel Spaces

- Two measurable spaces $(\Omega,\mathcal{A})$ and $(\Gamma,\mathcal{G})$ are equivalent if there is a 1-to-1 mapping between them that is measurable in both directions
- If $(\Omega,\mathcal{A})$ and $(\Gamma,\mathcal{G})$ are Borel spaces corresponding to topologies on Ω and Γ, then they are called Borel equivalent if they are equivalent
- A standard Borel space is a measurable space that is Borel equivalent to either $([0,1],\mathcal{B})$ or a subspace of $([0,1],\mathcal{B})$, where $\mathcal{B}=\mathcal{B}([0,1])$ are the Borel subsets of $[0,1]$, i.e. the smallest σ-algebra that contains all the open intervals in $[0,1]$

- Uncountable standard Borel $\Rightarrow$ Borel equivalent to $([0,1], \mathcal{B})$
- Hence, by "subspace" $(\Omega, \mathcal{A})$ we need only consider
  1. $\Omega \subset [0,1]$ is finite, and $\mathcal{A} = \mathcal{P}(\Omega) \subset \mathcal{B}$
     (= the power set = collection of all subsets)
  2. $\Omega \subset [0,1]$ is countable, and again $\mathcal{A} = \mathcal{P}(\Omega) \subset \mathcal{B}$
- If $\mathcal{E} = (\Omega, \mathcal{T})$ is Polish, then $(\Omega, \sigma(\mathcal{E}))$ is standard Borel
  - sometimes used as the definition of "standard Borel"
  - this case will be our default "standard" space

Isomorphic Probability Spaces

Two probability spaces $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{G}, Q)$ are

- isomorphic if
  1. $(\Omega, \mathcal{A})$ and $(\Gamma, \mathcal{G})$ are equivalent, with 1-to-1 mapping ϕ
  2. For all $A \in \mathcal{A}$, $P(A) = Q(\phi(A))$
  3. For all $G \in \mathcal{G}$, $Q(G) = P(\phi^{-1}(G))$
- isomorphic mod 0 if
  1. $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{G}, Q)$ are not isomorphic
  2. there are sets $A_0 \in \mathcal{A}$, $G_0 \in \mathcal{G}$, with $P(A_0) = Q(G_0) = 0$
  3. $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{G}, Q)$ are isomorphic when restricted to points in $\Omega \setminus A_0$ and $\Gamma \setminus G_0$

Lebesgue Spaces

- $(\Omega, \mathcal{A}, P)$ is a Lebesgue (probability) space if P is a probability measure of the form $\alpha P_1 + (1-\alpha)P_2$, $\alpha \in [0,1]$, and
  1. $(\Omega, \mathcal{A}, P)$ is complete
  2. P_1 has no atoms and $(\Omega, \mathcal{A}, P_1)$ is isomorphic mod 0 to $([0,1], \mathcal{L}([0,1]), \lambda)$
  3. There are a countable number of points $\omega_i \in \Omega$, such that with $p_i = P(\{\omega_i\})$ we have $P_2(A) = \sum_{i:\omega_i \in A} p_i$ for all $A \in \mathcal{A}$

Some Standard Borel Spaces

- Any finite set
- The rational numbers, and the irrational numbers
- $(\mathbb{R}^n, \mathcal{B}^n)$ (with $\mathcal{B}^n$ = the Borel sets $\subset \mathbb{R}^n$)
- Separable Hilbert spaces, i.e., Hilbert spaces which admit a countable basis; for example the space of square-integrable functions with inner product

$$\langle f, g \rangle = \int f g \, dx$$

and metric $\rho(f,g) = (\langle f-g, f-g \rangle)^{1/2}$

Most abstractions corresponding to real-world phenomena *result in standard Borel spaces* $\Rightarrow$ one can almost always work with $([0,1], \mathcal{L}, \lambda)$ or $([0,1], \mathcal{B}, \lambda_{|\mathcal{B}})$, plus a finite/countable space