Project Session

Jie Lu, Richard Combes, and Alexandre Proutiere
Automatic Control, KTH

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Problem 1. Strongly concave dual function.
Consider the following linearly constrained optimization problem

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax + b = 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with convexity parameter $\mu > 0$ (not necessarily differentiable) and $A \in \mathbb{R}^{p \times n}$ has full row rank. Suppose that the subgradients of f satisfy the Lipschitz condition

$$\|s(x_1) - s(x_2)\| \le L\|x_1 - x_2\|, \ \forall s(x_1) \in \partial f(x_1), \ \forall s(x_2) \in \partial f(x_2), \text{ for some } L > 0.$$

(a) Prove that the corresponding dual function $g(\nu)$ is strongly concave with concavity parameter $-\mu\lambda_{\min}(AA^T)/L^2 < 0$, where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a real symmetric matrix.
(b) Provide an algorithm which generates a sequence $\{x_k\}_{k=0}^{\infty}$ such that $\|x_k - x^\star\| \le c \cdot q^k$, where $c \in (0, \infty)$ and $q \in (0, 1)$ are some constants and $x^\star$ is the unique primal optimal solution.
Hint: Let $x^\star(\nu) = \arg\min_{x \in \mathbb{R}^n} f(x) + \nu^T(Ax + b)$ and express the (sub)gradient of $g(\nu)$ in terms of $x^\star(\nu)$.

Problem 2. Linear convergence of gradient projection method.

Consider the following constrained optimization problem

$$\min_{x\in X} f(x),$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is convex and has Lipschitz continuous gradient with Lipschitz constant $L>0$ and $X\subset\mathbb{R}^n$ is a closed convex set. Suppose the optimal set $X^\star = \arg\min_{x\in X} f(x)$ is nonempty. Let $\{x_k\}_{k=0}^{\infty}$ be a sequence generated by the gradient projection method

$$x_{k+1} = P_X[x_k - \alpha\nabla f(x_k)],\ \forall k\geq 0,\ \text{with } x_0\in X,$$

where $0<\alpha<\frac{2}{L}$. Assume that for every closed bounded set $S\subset\mathbb{R}^n$, there exists $\sigma_S>0$ such that

$$\text{dist}(x, X^\star) \leq \sigma_S \|P_X[x-\alpha\nabla f(x)] - x\|,\ \forall x\in S\cap X, \tag{1}$$

where $\text{dist}(x,X^\star) = \inf_{x^\star\in X^\star}\|x-x^\star\|$. Prove that there exists $q\in(0,1)$ such that

$$\text{dist}(x_{k+1}, X^\star) \leq q\,\text{dist}(x_k, X^\star), \quad \forall k\geq 0.$$

Hint 1: Use the fact $(x-P_X[x])^T(z-P_X[x])\leq 0$, $\forall x\in\mathbb{R}^n$, $\forall z\in X$ and the optimality condition $\nabla f(x^\star)(x-x^\star)\geq 0\ \forall x\in X$ to prove that

$$(x_k - x_{k+1})^T(x^\star - x_{k+1}) + \alpha(\nabla f(x_k) - \nabla f(x^\star))^T(x_{k+1}-x^\star) \leq 0,\ \forall x^\star\in X^\star.$$

Hint 2: $(x_k-x_{k+1})^T(x^\star - x_{k+1}) = (-\|x_k - x^\star\|^2 + \|x_k - x_{k+1}\|^2 + \|x_{k+1}-x^\star\|^2)/2$.

Hint 3: Prove that $(\nabla f(x_k) - \nabla f(x^\star))^T(x_{k+1}-x^\star) \geq -\frac{L}{4}\|x_{k+1}-x_k\|^2$. Here you need the inequality $\|y\|^2 + y^Tz \geq -\frac{1}{4}\|z\|^2$.

Hint 4: Combining the above, show that x_k remains in a closed bounded subset of X and then apply (1) to get the linear convergence rate.

Problem 3. Multiplicative-update Algorithm.

Consider a set of agents indexed by $i = 1, \dots, n$. At each step $t = 1, 2, \dots$, each of these agents may use one action selected from the set $\{1, \dots, K\}$. Agents simultaneously select an action in each step, and observe their rewards. The latter depend on the actions selected by the various agents. We denote by $X_{ik}(t)$ the reward obtained by agent i when selecting action k at step t. $X_{ik}(t)$ is a random variable whose distribution depends on the actions selected by all other agents. Assume that each agent selects at step t an action from probability distribution $p_i(t)$, independently of the actions selected by other agents. After step t, agent i updates her action distribution depending on the received reward in previous step. We consider two algorithms for these updates.

Algorithm $\times$. Agent i maintains a weight $w_{ik}(t)$ for action k, and selects at step t action k with probability $p_{ik}(t)$ proportional to $w_{ik}(t)$. After step t, the weights are updated as follows:

Let $K_i(t)$ the action selected at step t. Then:

$$w_{ik}(t+1) = w_{ik}(t) \exp\left(\frac{\gamma_t X_{ik}(t)}{K p_{ik}(t)}\right), \quad \text{if } K_i(t) = k,$$

The other weights remain unchanged. The sequence γ_t is chosen so that $\sum_t \gamma_t = \infty$ and $\sum_t \gamma_t^2 < \infty$.

Algorithm $+$. The algorithm is similar to Algorithm x, except that:

$$w_{ik}(t+1) = w_{ik}(t) + \frac{\gamma_t X_{ik}(t)}{K p_{ik}(t)}, \quad \text{if } K_i(t) = k,$$

1. Prove that Algorithm $\times$ mimics the replicator dynamics.
2. What about Algorithm $+$?