State-Space Representations of Transfer Function Systems

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February 2, 2013

1 State-Space Representation in Canonical Forms

We here consider a system defined by

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \; , \tag{1}$$

where u is the control input and y is the output. We can write this equation as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \; . \tag{2}$$

Later, we shall present state-space representation of the system defined by (1) and (2) in controllable canonical form, observable canonical form, and diagonal canonical form.

1.1 Controllable Canonical Form

We consider the following state-space representation, being called a controllable canonical form, as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{3}$$

$$y = \begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \dots & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \tag{4}$$

Note that the controllable canonical form is important in dicsussing the pole-placement approach to the control system design.

1.2 Observable Canonical Form

We consider the following state-space representation, being called an observable canonical form, as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & \dots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ b_{n-2} - a_{n-2} b_0 \\ \dots \\ b_1 - a_1 b_0 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \tag{6}$$

1.3 Diagonal Canonical Form

We here consider the transfer function system given by (2). We have the case where the dominator polynomial involves only distinct roots. For the distinct root case, we can write (2) in the form of

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{7}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}\ . \tag{8}$$

The diagonal canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \ddots & & \\ & & & -p_{n-1} & \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u \tag{9}$$

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u \tag{10}$$

2 Numerical Examples

Example 1: Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \ , \tag{11}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ . \tag{12}$$

Solution: From (11) and (12), we determine the following parameters: $b_0 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, $a_1 = 6$, $a_2 = 11$, $a_3 = 6$. Thus, the resulting transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 11s + 6} \ .$$

Example 2: Find the state-space representation of the following transfer function system (13) in the diagonal canonical form.

$$G(s) = \frac{2s + 3}{s^2 + 5s + 6} \ . \tag{13}$$

Solution: Partial fraction expansion of (13) is

$$\frac{2s + 3}{s^2 + 5s + 6} = \frac{A}{s + 2} + \frac{B}{s + 3} \ .$$

Hence, we get $A = -1$ and $B = 3$. We now have two distinct poles. For this, we can write the transfer function (13) in the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{14}$$

$$y = \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{15}$$

Example 3: Obtain the state-space representation of the transfer function system (16) in the controllable canonical form.

$$G(s) = \frac{s^2 + 3s + 3}{s^2 + 2s + 1} \tag{16}$$

Solution: From the transfer function (16), we obtain the following parameters: $b_0 = 1$, $b_1 = 3$, $b_2 = 3$, $a_1 = 2$, and $a_2 = 1$. The resulting state-space model in controllable canonical form is obtained as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ , \tag{17}$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \ . \tag{18}$$

Example 4: Consider the following state equations:

$$\dot{x}_1 = x_2(t) \ , \tag{19}$$

$$\dot{x}_2 = x_3(t) \ , \tag{20}$$

$$\dot{x}_3 = -6x_1(t) - 11x_2(t) - 6x_3(t) + 6u(t) \ , \tag{21}$$

$$y = x_1(t) \ , \tag{22}$$

and determine the controllable canonical form.

Solution: Using the state equations (19), (20), (21), and (22), we write the following high order differential equation:

$$\frac{d^3}{dt^3}y(t) + 6\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) + 6y(t) = 6u(t) \ .$$

The state variables $x(t) = y$, $x_2(t) = \dot{y}$, and $x_3 = \ddot{y}$. Hence, we get

$$\begin{aligned} \dot{x}_1 &= x_2(t) \ , \\ \dot{x}_2 &= x_3(t) \ , \\ \dot{x}_3 &= \frac{d^3}{dt^3}y(t) = -6\ddot{y}(t) - 11\dot{y}(t) - 6y(t) + 6u(t) \\ &= -6x_3(t) - 11x_2(t) - 6x_1(t) + 6u(t) \ . \end{aligned}$$

In matrix form, we have

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u(t) \ , \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \ . \end{aligned}$$

Example 5: Consider the following state equations

$$\dot{x}_1(t) = -x_1(t) + x_3(t) + 4u(t) \ , \tag{23}$$

$$\dot{x}_2(t) = -3x_1(t) + 2u(t) \ , \tag{24}$$

$$\dot{x}_3(t) = -5x_1(t) + x_2(t) + u(t) \ , \tag{25}$$

$$y(t) = x_1(t) \ , \tag{26}$$

and determine the observable canonical form.

Solution: Using the state equations (23), (24), (25), and (26), we write the following high order differential equation:

$$\frac{d^3}{dt^3}y(t) + \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 3y(t) = 4\frac{d^2}{dt^2}u(t) + \frac{d}{dt}u(t) + 2u(t)\ .$$

We introduce $x_1(t) = y(t)$ in the equation, and collect all terms without differentiation on the right hand side, we get

$$\frac{d^3}{dt^3}x_1(t) + \frac{d^2}{dt^2}x_1(t) + \frac{d}{dt}x_1(t) - \frac{d^2}{dt^2}u(t) - \frac{d}{dt}u(t) = -3x_1(t) + 2u(t)\ ,$$

i.e.,

$$\frac{d}{dt}\left[\frac{d^2}{dt^2}x_1(t) + \frac{d}{dt}x_1(t) + x_1(t) - \frac{d}{dt}u(t) - u(t)\right] = -3x_1(t) + 2u(t)\ .$$

Now introduce the expression within the paranthesis as a new state variable

$$x_2(t) = \frac{d^2}{dt^2}x_1(t) + \frac{d}{dt}x_1(t) + 5x_1(t) - 4\frac{d}{dt}u(t) - u(t)\ ,$$

i.e.,

$$\dot{x}_2(t) = -3x_1(t) + 2u(t)\ . \tag{27}$$

Repeating this precedure yields

$$\frac{d}{dt}\left[\frac{d}{dt}x_1(t) + x_1(t) - 4u(t)\right] = x_2(t) - 5x_1(t) + u(t)\ , \tag{28}$$

and we introduce

$$x_3(t) = \frac{d}{dt}x_1(t) + x_1(t) - 4u(t)\ ,$$

i.e.,

$$\dot{x}_1(t) = x_3(t) - x_1(t) + 4u(t)\ . \tag{29}$$

From (27), (28), and (29), we define the state-space form of

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 0 \\ -5 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} u(t)\ ,$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)\ .$$

3 References

1. Katsuhiko Ogata, *Modern Control Engineering*, 4^{th} Ed., Prentice Hall Inc., New Jersey, 2002.
2. KTH Reglerteknik, *Reglerteknik Allmän Kurs*, Del 2, 2007.