Analysis of the Packet Loss Process in an MMPP+M/M/1/K queue*

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1 Introduction

In the case of flow-type multimedia communications, as opposed to elastic traffic, the average packet loss is not the only measure of interest. The burstiness of the loss process, the number of losses in a block of packets has a great impact both on the user perceived visual quality and on the possible ways of improving it, for example by forward error correction or receiver-based error concealment.

In this report we present a model to analyze the packet loss process of a bursty source, for example VBR video, multiplexed with background traffic in a single multiplexer with a finite queue and exponentially distributed packet sizes. We model the bursty source by an L-state Markov Modulated Poisson Process (MMPP) while the background traffic is modeled by a Poisson process.

It is well known that compressed multimedia, primarily VBR video exhibits a self-similar nature [1]. Yoshihara et al. use the superposition of 2-state IPPs to model self-similar traffic in [2], and compare the loss probability of the resulting MMPP/1/D/K queue with simulations. They found that the approximation works well under heavy load conditions and gives an upper bound on packet loss probabilities. Ryu and Elwalid [3] showed that short term correlations have dominant impact on the network performance under realistic scenarios of buffer sizes for real-time traffic. Thus the MMPP may be a practical model to derive approximate results for the queuing behavior of LRD traffic such as real-time VBR video, especially in the case of small buffer sizes. Recently Cao et al. [4] showed that the traffic generated by a large number of sources tends to Poisson as the load increases due to statistical multiplexing justifying the Poisson model for the background traffic.

The report is organized as follows. Section 2 gives an overview of the previous work on the modeling of the loss process of a single server queue with exponential service times. In Section 3 we describe our model to calculate the loss probability in a block of packets. In Section 4.1 we derive the quantities used to calculate the loss probabilities.

2 Related Work

In [5], Cidon et al. present an exact analysis of the packet loss process in an M/M/1/K queue, that is the probability of loosing j packets in a block of n packets, and show that the distribution

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of losses may be more bursty compared to the independence assumption. They also consider a discrete time system describing the behavior of ATM fed with a Bernoulli arrival process. In [6], Gurewitz et al. present explicit expressions for the above quantities in interest for the M/M/1/K queue. In [7] the multidimensional generating function of the probability of j losses in a block of n packets is obtained and an easy-to-calculate asymptotic result is given under the condition that $n \leq K + j + 1$.

The waiting time and queue length distribution of the N/G/1/K queue was derived in [8] including the MMPP/G/1/K queue as a special case.

3 Model description

We consider a system with exponentially ditributed size packets having an average transmission time D. Packets arrive to the system from two sources, a Markov Modulated Poisson Process (MMPP) and a Poisson process, representing the tagged source and the background traffic respectively. The packets are stored in a buffer that can host up to K packets, and are served according to a FIFO policy. Every n consecutive packets from the tagged source form a block, and we are interested in the probability distribution of the number of lost packets in a block arriving from the MMPP in the steady state of the system. Throughout the calculations we use notations similar the those in [5].

We assume that the sources feeding the system are independent. The MMPP is described by the infinitesimal generator Q with elements r_{ij} and the arrival rate matrix $\Lambda = diag\{\lambda_1, \ldots, \lambda_L\}$, where λ_i is the average arrival rate while the underlying Markov chain is in state i. The Poisson process modeling the background traffic has average arrival rate λ. The superposition of the two sources can be described by a single MMPP with arrival rate matrix $\hat{\Lambda} = \Lambda \oplus \lambda = \Lambda + \lambda I = diag\{\hat{\lambda_1}, \ldots, \hat{\lambda_L}\}$, and infinitesimal generator $\hat{Q} = Q$, where $\oplus$ is the Kronecker sum. Packets arriving from both sources have the same length distribution, thus the same service time distribution.

Our purpose is to calculate the probability $P(j,n), n \geq 1, 0 \leq j \leq n$ of j losses in a block of n packets. We define the probability $P^a_{i,l}(j,n), 0 \leq x \leq KD, l = 1 \ldots L, n \geq 1, 0 \leq j \leq n$ as the probability of j losses in a block of n packets, given that the number of packets in the system is i just before the arrival epoch of the first packet in the block and the first packet of the block is generated in state l of the MMPP. As the first packet in the block is arbitrary,

$$P(j,n) = \sum_{l=1}^{L} \sum_{i=0}^{K} \Pi(i,l) P^a_{i,l}(j,n) \tag{1}$$

The probability $\Pi(i,l)$ of a packet arriving in state (i,l) of the queue can be calculated as outlined in Section 4.2.

The probabilities $P^a_{i,l}(j,n)$ can be derived according to the following recursion. The recursion is initiated for n = 1 with the following relations

$$P^a_{i,l}(j,1) = \begin{cases} 1 & j = 0 \\ 0 & j \geq 1 \end{cases} \qquad i \leq K-1$$
$$P^a_{i,l}(j,1) = \begin{cases} 0 & j = 0, j \geq 2 \\ 1 & j = 1 \end{cases} \qquad K-1 < i. \tag{2}$$

Using the notation $p_m = \frac{\lambda_m}{\lambda_m+\lambda}$ and $\overline{p}_m = \frac{\lambda}{\lambda_m+\lambda}$, for $n \geq 2$ the following equations hold

$$P^a_{i,l}(j,n) = \sum_{m=1}^{L}\sum_{k=0}^{i+1} Q^{l,m}_{i+1}(k)\{p_m P^a_{i+1-k,m}(j,n-1) + \overline{p}_m P^s_{i+1-k,m}(j,n-1)\} \tag{3}$$

for $0 \leq i \leq K-1$ and

$$P^a_{i,l}(j,n) = \sum_{m=1}^{L}\sum_{k=0}^{K} Q^{l,m}_{M}(k)\{p_m P^a_{K-k,m}(j-1,n-1) + \overline{p}_m P^s_{K-k,m}(j-1,n-1)\} \tag{4}$$

for $i = K$. $P^s_{i,l}(j,n)$ is given by

$$P^s_{i,l}(j,n) = \sum_{m=1}^{L}\sum_{k=0}^{i+1} Q^{l,m}_{i+1}(k)\{p_m P^a_{i+1-k,m}(j,n) + \overline{p}_m P^s_{i+1-k,m}(j,n)\} \tag{5}$$

for $0 \leq i \leq K-1$ and

$$P^s_{i,l}(j,n) = \sum_{m=1}^{L}\sum_{k=0}^{K} Q^{l,m}_{M}(k)\{p_m P^a_{K-k,m}(j,n) + \overline{p}_m P^s_{M-k,m}(j,n)\} \tag{6}$$

for $i = K$. The probability $P^s_{i,l}(j,n), 0 \leq i \leq K, l = 1\ldots L, 0 \leq j \leq n$ is the probability of j losses in a block of n packets, given that the number of packets in the system is i just before the arrival of a packet from the background traffic and the MMPP is in state l. In (3) to (6) $Q^{lm}_i(k)$ denotes the joint conditional probability of that out of i packets k leave during an interarrival time and the next arrival occures in state m of the underlying Markov chain, given that the last arrival occured in state l and is calculated in Section 4.1.

The procedure of computing $P^a_{i,l}(j,n)$ is as follows. First we calculate $P^a_{i,l}(j,1), i = 0\ldots KN$ from the initial conditions (2). Then in iteration k we first calculate $P^s_{i,l}(j,k), k = 1\ldots n-1$ using equations (5) and (6) and the probabilities $P^a_{i,l}(j,k)$, which have been calculated during iteration $k-1$. Then we calculate $P^a_{i,l}(j,k+1)$ using equations (3) and (4).

4 Derivation of $Q^{l,m}_i(k)$ and $\Pi_{i,l}$

In this section we show how to calculate the quantity $Q^{l,m}_i(k)$ and the steady state probability of the MMPP+M/M/1/K queue.

4.1 Calculation of $Q^{l,m}_i(k)$

The probability of k service completions during an interarrival time from the joint arrival process, $Q^{l,m}_i(k)$, is given by

$$\begin{aligned} Q^{l,m}_i(k) &= P^{l,m}(k) \quad\quad ifk < i \\ Q^{l,m}_i(k) &= \textstyle\sum_{j=i}^{\infty} P^{l,m}(j) \; ifk = i, \end{aligned} \tag{7}$$

where $P^{l,m}(k)$ denotes the joint probability of having k service completions with exponentially distributed service times between two arrivals and the next arrival coming in state m of the MMPP given that the last arrival came in state l.

The z-transform $P^{l,m}(z)$ of $P^{l,m}(k)$ is given by

$$P^{l,m}(z) = \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{\lambda_t^k}{k!} e^{-\lambda t} f^{l,m}(t) dt \right) z^k$$
$$= f^{l,m*}(\mu - \mu z), \qquad (8)$$

where $f^{l,m}(t)$ is the interarrival time distribution given that the next arrival is in state m and the last arrival was in state l of the MMPP. The Laplace transform of $f^{l,m}(t)$ is denoted with $f^{l,m}*(s)$ and is given by

$$f^{l,m*}(s) = \mathcal{L}\left\{ e^{(\hat{Q}-\hat{\Lambda})x} \hat{\Lambda} \right\} = (sI - \hat{Q} + \hat{\Lambda})^{-1} \hat{\Lambda}. \qquad (9)$$

The inverse z-transform of (8) can be expressed analytically by partial fraction decomposition as long as $L \leq 4$, and has the form

$$P^{l,m}(k) = \sum_{j=1}^{L} A_j^{lm} \frac{1}{\alpha_j^k}, \qquad (10)$$

where α_j is the j^{th} root of $T(z) = det[(\mu - \mu z)I - \hat{Q} + \hat{\Lambda}]$ and can be calculated algebraically for $L \leq 4$.

In the following we show how the calculation proceeds for $L = 3$. To calculate the roots α_j we first calculate the roots β_j of $t(s) = det[sI - \hat{Q} + \hat{\Lambda}]$ in (9). To do so, we rewrite it to the form

$$t(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0, \qquad (11)$$

where

$$a_3 = 1$$
$$a_2 = r_{12} + r_{13} + \lambda_1 + r_{31} + r_{32} + \lambda_3 + r_{21} + r_{23} + \lambda_2$$
$$a_1 = \lambda_2 * r_{31} + r_{13} * \lambda_3 - r_{21}^2 + r_{13} * r_{23} + r_{13} * r_{32} + r_{13} * r_{21} + r_{21} * r_{31} + \qquad (12)$$
$$r_{21} * r_{32} + r_{21} * \lambda_3 + r_{23} * r_{31} + r_{23} * \lambda_3 + \lambda_2 * r_{32} + \lambda_2 * \lambda_3 + r_{12} * r_{31} + \qquad (13)$$
$$r_{12} * r_{32} + r_{12} * \lambda_3 + r_{12} * r_{21} + r_{12} * r_{23} + r_{12} * \lambda_2 + r_{12} * \lambda_2 + r_{13} * \lambda_2 + \qquad (14)$$
$$\lambda_1 * r_{31} + \lambda_1 * r_{32} + \lambda_1 * \lambda_3 + \lambda_1 * r_{21} + \lambda_1 * r_{23} + \lambda_1 * \lambda_2$$
$$a_0 = r_{12} * \lambda_2 * r_{32} + r_{12} * \lambda_2 * \lambda_3 + r_{13} * r_{21} * \lambda_3 - r_{21}^2 * r_{31} - r_{21}^2 * r_{32} - r_{21}^2 * \lambda_3 + \qquad (15)$$
$$r_{12} * r_{21} * r_{31} + r_{12} * r_{21} * r_{32} + r_{12} * r_{21} * \lambda_3 + r_{12} * r_{23} * r_{31} + r_{12} * r_{23} * \lambda_3 + \qquad (16)$$
$$r_{13} * r_{23} * \lambda_3 + r_{13} * \lambda_2 * r_{32} + r_{13} * \lambda_2 * \lambda_3 + \lambda_1 * r_{21} * r_{31} + \lambda_1 * r_{21} * r_{32} + \qquad (17)$$
$$\lambda_1 * r_{21} * \lambda_3 + \lambda_1 * r_{23} * r_{31} + \lambda_1 * r_{23} * \lambda_3 + \lambda_1 * \lambda_2 * r_{31} + \qquad (18)$$
$$\lambda_1 * \lambda_2 * r_{32} + \lambda_1 * \lambda_2 * \lambda_3 - r_{31} * r_{21} * r_{23}. \qquad (19)$$

We denote the roots of (11) with $\beta_j, j = 1,2,3$. Knowing β_j we can perform the partial fraction decomposition of (9) with respect to s

$$f^{l,m*}(s) = \sum_{j=1}^{L} \frac{B_j^{lm}}{s + \beta_j}, \qquad (20)$$

where B_j^{lm} can be calculated as

$$\begin{aligned}
B_1^{lm} &= (c_2^{lm} * \beta_1^2 - c_1^{lm} * \beta_1 + c_0^{lm})/(\beta_2 - \beta_1)/(\beta_3 - \beta_1) \\
B_2^{lm} &= (c_2^{lm} * \beta_2^2 - c_1^{lm} * \beta_2 + c_0^{lm})/(\beta_1 - \beta_2)/(\beta_3 - \beta_2) \\
B_3^{lm} &= (c_2^{lm} * \beta_3^2 - c_1^{lm} * \beta_3 + c_0^{lm})/(\beta_2 - \beta_3)/(\beta_1 - \beta_3).
\end{aligned} \tag{21}$$

The coefficients $c_2^{lm}, c_1^{lm}, c_0^{lm}$ are the following:

$$\begin{aligned}
c_2^{11} &= \lambda_1 \\
c_1^{11} &= \lambda_1(r_{31} + r_{32} + \lambda_3 + r_{21} + r_{23} + \lambda_2) \\
c_0^{11} &= \lambda_1(r_{21}r_{31} + r_{21}r_{32} + r_{23}r_{31} + r_{21}\lambda_3 + r_{23}\lambda_3 + \lambda_2 r_{31} + \lambda_2 r_{32} r_{31} + r3 + \lambda_2\lambda_3) \\
c_2^{12} &= 0 \\
c_1^{12} &= \lambda_2 r_{21} \\
c_0^{12} &= \lambda_2(r_{21}r_{31} + r_{21}r_{32} + r1\lambda_3 + r_{13}r_{32}) \\
c_2^{13} &= 0 \\
c_1^{13} &= \lambda_3 r_{13} \\
c_0^{13} &= \lambda_3(r_{21}r_{23} + r_{13}r_{21} + r_{13}r_{23} + r_{13}\lambda_2) \\
c_2^{21} &= 0 \\
c_1^{21} &= \lambda_1 r_{21} \\
c_0^{21} &= \lambda_1(r_{21}r_{31} + r_{21}r_{32} + r_{21}\lambda_3 + r_{23}r_{31}) \\
c_2^{22} &= \lambda_2 \\
c_1^{22} &= \lambda_2(r_{31} + r_{12} + r_{13} + \lambda_1 + r_{32} + \lambda_3) \\
c_0^{22} &= \lambda_2(r_{12}r_{31} + r_{12}r_{32} + r_{12}\lambda_3 + r_{13}r_{32} + r_{13}\lambda_3 + \lambda_1 r_{31} + \lambda_1 r3 + \lambda_1\lambda_3) \\
c_2^{23} &= 0 \\
c_1^{23} &= \lambda_3 r_{23} \\
c_0^{23} &= \lambda_3(r_{23}r_{12} + r_{13}r_{23} + r_{23}\lambda_1 + r_{13}r_{21}) \\
c_2^{31} &= 0 \\
c_1^{31} &= \lambda_1 r_{31} \\
c_0^{31} &= \lambda_1(r_{21}r_{32} + r_{21}r_{31} + r_{23}r_{31} + \lambda_2 r_{31}) \\
c_2^{32} &= 0 \\
c_1^{32} &= \lambda_2 r_{32} \\
c_0^{32} &= \lambda_2(r_{32}r_{12} + r_{13}r_{32} + r_{32}\lambda_1 + r_{21}r_{31}) \\
c_2^{33} &= \lambda_3 \\
c_1^{33} &= \lambda_3(r_{13} + r_{12} + \lambda_1 + r_{21} + r_{23} + \lambda_2) \\
c_0^{33} &= \lambda_3(r_{13}r_{23} + r_{13}r_{21} - r_{21}^2 + r_{12}r_{21} + r_{12}r_{23} + r_{13}\lambda_2 + r_{12}\lambda_2 + \lambda_1 r_{21} + \lambda_1 r_{23} + \lambda_1\lambda_2).
\end{aligned} \tag{22}$$

Thus the Laplace transform of the conditional probability $f^{l,m}(t)$ has the form

$$f^{l,m}(s) = \sum_{j=1}^{L} B_j^{lm} \frac{1}{s - \beta_j} \tag{23}$$

and $f^{l,m}(t)$ is

$$f^{l,m}(t) = \sum_{j=1}^{L} B_j^{lm} e^{\beta_j t}. \tag{24}$$

From the Laplace transform of the interarrival time distribution we get the z-transform of the number of departures by substituting $s = (\mu - \mu z)$. Thus the roots α_j of $T(z)$ can be calculated as

$$\alpha_j = 1 + \beta_j/\mu. \tag{25}$$

The coefficients $A_j^{l,m}$ in (10) can be calculated as

$$A_j^{l,m} = B_j^{l,m}/(\mu\alpha_j). \tag{26}$$

Given the probability $P^{l,m}(k)$ one can express $Q_i(k)$ as

$$Q_i(k) = \begin{cases} \sum_{j=1}^{L} A_j^{lm} \left(\frac{1}{\alpha_j}\right)^k & 0 \le k < i \\ \sum_{j=1}^{L} \frac{A_j^{lm}}{1-1/\alpha_j} \left(\frac{1}{\alpha_j}\right)^i & k = i. \end{cases} \tag{27}$$

4.2 Calculation of the steady state probability

In this section we show how to calculate the steady state probablity $\Pi(i,l)$ of that an arrival from the MMPP arrives in state (i,l) in the MMPP+M/M/1/K queue. We suppose that the steady state probability $\pi(i,l)$ of the MMPP+M/M/1/K queue has been calculated by some matrix-geometric approach [8].

Then the probabilities $\Pi(i,l)$, $0 \le i \le K, 1 \le l \le L$ can be calculated by applying the conditional PASTA property

$$\Pi(i,l) = \frac{\pi(i,l)\lambda_l}{\sum_{l=1}^{L} \lambda_l \sum_{i=0}^{K} \pi(i,l)}. \tag{28}$$

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