Computational Game Theory

Lecture 3

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Topics for today

- IEASDA
- Computing NE
  - Zero-sum games
  - Two player games
- Cardinality
  - Quadratic games

Alternative solution concept

- Simple reasoning
  - Player *i* should not choose an action that is strictly worse than some other action

- Solution
  - Iteratively remove the actions that are worse than some other actions

	L	C	R
U	0,2	3,1	2,3
M	1,4	2,1	4,1
D	2,1	4,4	3,2

Beliefs

- For a strategic game <N,(A_i),(u_i)> the belief μ_i of player *i* is a probability measure on A_{-i}
  - $\mu_i : \times_{j \in N \setminus \{i\}} A_j \rightarrow \Re$

- Assigns probability to the actions of the other players
  - Does *not* assume independence
  - Does *not* have to be correct

- An action a_i of player *i* is a best response to the belief μ_i if

$$u_i(\mu_i, a_i) \geq u_i(\mu_i, a'_i) \quad a'_i \in A_i$$

Never best response

- Action of player *i* in a strategic game is a never best response if it is not a best response to any belief of player *i*
  - in pure strategies

	L	M	R
T	(4,12)	(6,4)	(2,5)
C	(8,3)	(2,6)	(4,5)
B	(6,5)	(5,9)	(3,8)

- $\delta_M \rightarrow T$, $\delta_L \rightarrow C$, $0.3\delta_L + 0.6\delta_M + 0.1\delta_R \rightarrow B$
- $\delta_T \rightarrow L$, $\delta_B \rightarrow M$
- R is a never best response

Strictly dominated action

- In a strategic game $<N, (A_i), (u_i)>$ the action $a_i \in A_i$ of player *i* is strictly dominated if there is a mixed strategy α_i such that $U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$ for all $a_{-i} \in A_{-i}$
  - Mixed strategy α_i is better for any pure belief of player *i*
- An action of a player in a 2 person finite strategic game is a never best response $\Leftrightarrow$ it is strictly dominated
  - Note the difference in the definitions!

Iterated elimination of strictly dominated actions

- The set $X \subseteq A$ of outcomes of a finite strategic game $<N,(A_i),(u_i)>$ survives iterated elimination of strictly dominated actions if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^{T}$ of sets that satisfies the following conditions for each $j \in N$
  - $X_j^0 = A_j$ and $X_j^T = X_j$
  - $X_j^{t+1} \subseteq X_j^t$ for each $t=0,..T-1$
  - for each $t=0,\ldots,T-1$ every action $a_j \in X_j^t \setminus X_j^{t+1}$ is strictly dominated in the game $<N,(X_i^t),(u_i^t)>$, where u_i^t for each $i \in N$ is the function u_i *restricted to* $\times_{j \in N} X_j^t$
  - no $a_j \in X_j^T$ is strictly dominated in the game $<N,(X_i^T),(u_i^T)>$

Example

	L	C	R
U	0,2	3,1	2,3
M	1,4	2,1	4,1
D	2,1	4,4	3,2

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

Example – Cont'd

	L	C
M	1,4	2,1
D	2,1	4,4

⇓

	L	C
D	2,1	4,4

⇓

	C
D	4,4

Remarks

- Strategic game is solvable by IESDA if only one outcome survives ($|X|=1$)
  - Order of elimination does not matter

Example

- Consider the following strategic game

G^0	c_1	c_2	c_3	c_4
r_1	(2,3)	(2,4)	(2,3)	(4,2)
r_2	(4,2)	(3,3)	(0,2)	(2,1)
r_3	(1,4)	(1,2)	(0,0)	(3,1)
r_4	(1,0)	(2,1)	(5,5)	(3,2)

- Apply iterated elimination of strictly dominated actions

Example – step 1

- r_3 is strictly dominated by r_1

G^1	c_1	c_2	c_3	c_4
r_1	(2,3)	(2,4)	(2,3)	(4,2)
r_2	(4,2)	(3,3)	(0,2)	(2,1)
r_4	(1,0)	(2,1)	(5,5)	(3,2)

Example – step 2

- c_1 is strictly dominated by c_2
- c_4 is strictly dominated by c_3

G^2	c_2	c_3
r_1	(2,4)	(2,3)
r_2	(3,3)	(0,2)
r_4	(2,1)	(5,5)

Example – step 3

- α_1=(0,1/2,1/2) dominates r_1

G^3	c_2	c_3
r_2	(3,3)	(0,2)
r_4	(2,1)	(5,5)

- Can we eliminate more actions?
  - Rational player will only choose among actions in G^3
- Can we tell the NE of G^3?
  - what about the NE of the original game G^0?

Remarks

- Let α^* be a mixed strategy NE of the game $G=<N,(A_i),(u_i)>$ then

  $\alpha^*_i(a_i)=0$ for all $a_i \in A_i \backslash X_i$ and α^* is a mixed strategy NE of G^T

Alternative solution concepts

- Rationalizability
- Iterated elimination of weakly dominated actions

Minimax in mixed strategies

- Consider
  - payoff matrix $A=[a_{ij}]_{mxn}$
  - mixed strategy profiles α_1 and α_2

- Player 1 aims to maximize its payoff

$$U_1(\alpha) \leq \max_{\alpha_1} \min_{\alpha_2} \alpha_1 A \alpha_2^T$$

- Player 2 aims to minimize its loss (the payoff of player 1)

$$U_1(\alpha) \leq \min_{\alpha_2} \max_{\alpha_1} \alpha_1 A \alpha_2^T$$

Minimax and LP

- Optimization formulation of the problem
- Player 1's objective

$$\max_{\alpha_1}\min_{\alpha_2}\alpha_1 A\alpha_2^T = \max_{\alpha_1}\min_{\alpha_2}\sum_{j=1}^{n}\alpha_{2j}\sum_{i=1}^{m}a_{ij}\alpha_{1i}$$
$$= \max_{\alpha_1}\min_{j=1..n}\left\{\sum_{i=1}^{m}a_{ij}\alpha_{1i}\right\}$$

- To maximize the payoff, the minimum (*s*) should be maximized

$$\sum_{i=1}^{m}a_{ij}\alpha_{1i} \geq s \quad for\ j=1,\ldots,n$$

Minimax and LP

- Primal problem

$$\max \quad s$$

$$s.t. \quad \sum_{i=1}^{m} \alpha_{1i} A_{ij} \geq s \quad j = 1,\dots,n$$

$$\sum_{i=1}^{m} \alpha_{1i} \leq 1$$

$$\alpha_{1i} \geq 0 \quad i = 1,\dots,m$$

- Dual problem

$$\min \quad t$$

$$s.t. \quad \sum_{j=1}^{n} \alpha_{2j} A_{ij} \leq t \quad i = 1,\dots,m$$

$$\sum_{j=1}^{n} \alpha_{2j} \geq 1$$

$$\alpha_{2j} \geq 0 \quad j = 1,\dots,n$$

Minimax theorem

- Strong duality theorem (*s* bounded and feasible)
  - If the primal problem has an optimal solution $\alpha^*{}_1$ then the dual also has an optimal solution $\alpha^*{}_2$, and $s=t$.
- Two person zero-sum game solvable in polynomial time
- Minimax theorem (von Neumann, 1928, 1944)
  - For every two-person, zero-sum game with finite strategies there exists an equilibrium strategy α^* and

$$\max_{\alpha_1} \min_{\alpha_2} \alpha_1 A \alpha_2^T = \min_{\alpha_2} \max_{\alpha_1} \alpha_1 A \alpha_2^T$$

J. von Neumann, "Zur Theorie der Gesellschaftsspiele", Mathematische Annalen, 100, pp. 295–300, 1928

J. von Neumann, O. Morgenstern, ”Theory of Games and Economic Behavior,” Princeton University Press, 1944

Unique pure NE for ZSG

- Consider a ZSG, and let $u_1(a_1,a_2)$ strictly concave in a_1 and strictly convex in a_2. Then there exists a unique SP in pure strategies.
    - Follows from Rosen’s theorem with $L=u_1$

Computing Nash equilibria

György Dán, https://people.kth.se/~gyuri

Quadratic Game

$$u_i(a_i, a_{-i}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} a_j R_{j,k}^{(i)} a_k + \sum_{j=1}^{N} r_j^{(i)} a_j + c_i$$

If $R_{ii}^{(i)}$ negative definite

- U_i concave in a_i
- NE is in pure strategies

If R is invertible

- Unique pure NE: $a^* = -R^{-1}r$