Computational Game Theory

Lecture 5

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Today´s topics

- Efficiency of equilibria
- Potential games
- Super/submodular games

Nash equilibrium vs. Social optimum

- Strategic game $G=\langle N,(A_i),(u_i)\rangle$

- Social optimum – best possible outcome

$$U = \max_{a} SWF(u_1(a), u_2(a), \ldots, u_{|N|}(a))$$

- Social welfare function *SWF* can be
  - Utilitarian $SWF=\Sigma$ *(no fairness)*
  - Bernoulli-Nash $SWF=\Pi$ *(proportional fairness)*
  - Rawls $SWF=min$ *(max-min fairness)*

Inefficiency of equilibria

- Nash equilibria a^* are in general not social optimum

- Price of Anarchy (pure)

$$PoA = \frac{\max_{a \in A} SWF(u_1(a), \ldots, u_{|N|}(a))}{\min_{a^*} SWF(u_1(a^*), \ldots, u_{|N|}(a^*))}$$

- Price of Stability (pure)

$$PoS = \frac{\max_{a \in A} SWF(u_1(a), \ldots, u_{|N|}(a))}{\max_{a^*} SWF(u_1(a^*), \ldots, u_{|N|}(a^*))}$$

- Mixed and Bayes-Nash PoA and PoS exist
- Extension to adversarial setting – Price of Malice

Steiner problem in networks

- Digraph (V,E)
  - Edge costs $c_e \geq 0 \;\; \forall e \in E$
- Set of pairs of vertices $N=(s_i,t_i)_{i=1..n}$
  - For all (s_i,t_i) t_i is reachable from s_i
  - Set of paths from s_i to t_i is A_i
  - All possible combinations of paths $A=\times_{i=1..n} A_i$
- Construct minimum weight subgraph

$$\min_{a \in A} \sum_{e \in a} c_e$$

- Applications
  - Routing in networks
  - VLSI design
- NP-hard in general

Shapley network design game

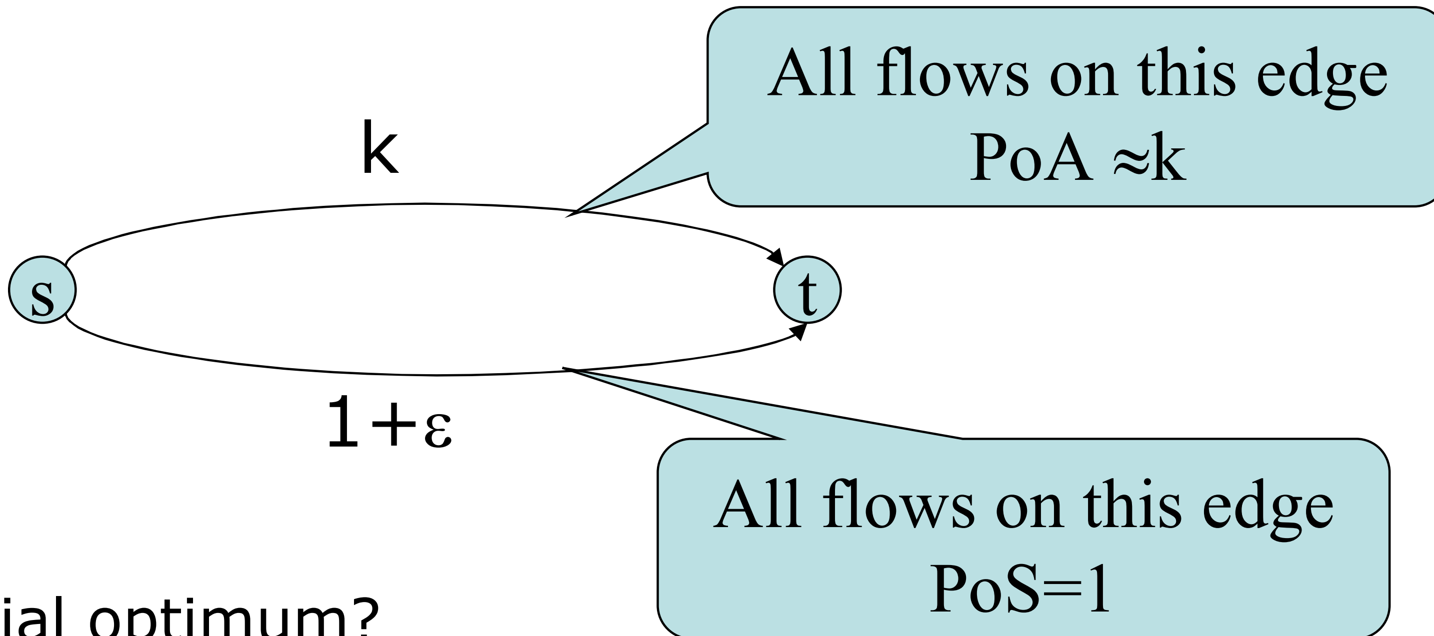
- Digraph (V,E)
  - Edge costs $c_e \geq 0 \;\; \forall e \in E$
- Set of players N
  - Player $i \in N$ wants to build a network such that t_i is reachable from s_i
- Sets of actions A_i
  - $a_i \in A_i$ is a path (s_i,t_i) in (V,E)
- Constructed network is $\cup_{i \in N} a_i$
- Cost function of player i in the constructed network

$$\mathrm{cost}_i(a) = \sum_{e \in a_i} c_e / k_e$$

  - k_e=#of players for which $e \in a_i$
  - Shapley cost sharing mechanism (fair)

First example

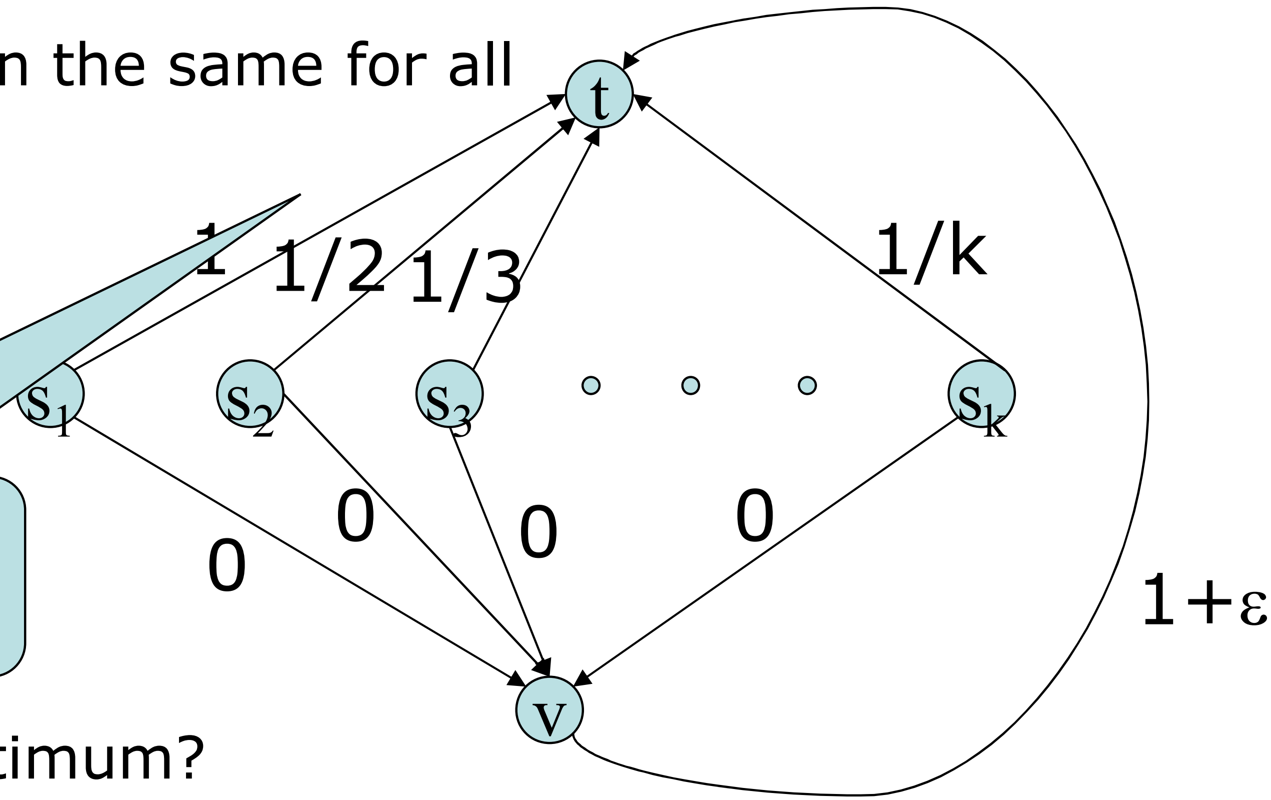
- $|N|=k$
- Source and destination the same for all



- Social optimum?
- Nash equilibria?
  - Price of Anarchy vs. Stability?

Second example

- $|N|=k$
- Destination the same for all



All flows this way
PoA=PoS $\approx H_k$

- Social optimum?
- Nash equilibria?
  - Price of Anarchy and Stability?

Claim

- Pure strategy equilibria always exist in the Shapley network design game

Exact potential games

- Let $G=<N,(A_i),(u_i)>$ be a finite strategic game and $A=\times_{i\in N}A_i$.

  A function $\Psi:A\rightarrow R$ is an exact potential for G if

$$\psi(a_{-i},b_i)-\psi(a_{-i},a_i)=u_i(a_{-i},b_i)-u_i(a_{-i},a_i)$$
$$\forall a\in A,\forall a_i,b_i\in A_i$$

- A game $G=<N,(A_i),(u_i)>$ is called an exact potential game if it admits an exact potential.

An example

- Prisoner's dilemma

	Do not confess	Confess
Do not confess	6,6	0,9
Confess	9,0	1,1

- And its exact potential

0	3
3	4

Weighted potential games

- Let $G=<N,(A_i),(u_i)>$ be a finite strategic game and $A=\times_{i\in N}A_i$.
- A function $\Psi:A\rightarrow R$ is a weighted potential for G if

$$\psi(a_{-i},b_i)-\psi(a_{-i},a_i)=w_i(u_i(a_{-i},b_i)-u_i(a_{-i},a_i))$$
$$\forall a\in A,\forall a_i,b_i\in A_i,w_i>0$$

- A game $G=<N,(A_i),(u_i)>$ is called a weighted potential game if it admits a weighted potential.

- Inclusion
  - Every exact potential game is a weighted potential game.

Ordinal potential games

- Let $G=<N,(A_i),(u_i)>$ be a finite strategic game and $A=\times_{i\in N}A_i$.
- A function $\Psi{:}A\to R$ is an ordinal potential for G if

$$\psi(a_{-i},b_i)-\psi(a_{-i},a_i)>0 \Leftrightarrow u_i(a_{-i},b_i)-u_i(a_{-i},a_i)>0$$
$$\forall a\in A, \forall a_i,b_i\in A_i$$

- A game $G=<N,(A_i),(u_i)>$ is called an ordinal potential game if it admits an ordinal potential.

- Inclusion
  - Every weighted potential game is an ordinal potential game.

Another example

- Battle of the Sexes

	Theatre	Sports
Sports	3,2	0,0
Theatre	0,0	2,3

- And its ordinal potential

2	0
0	2

Existence of equilibria

- Let Ψ be an ordinal potential for $G=<N,(A_i),(u_i)>$. The equilibrium set of G coincides with that of $<N,(A_i),(\Psi)>$.

  That is,

  $a \in A$ is a NE of $G \Leftrightarrow \Psi(a_{-i},a_i) \geq \Psi(a_{-i},a_i')$ for $a_i' \in A_i$

  If Ψ admits a maximum value in A, then G possesses a pure strategy Nash equilibrium.

- Proof

  $$\psi(a_{-i},b_i)-\psi(a_{-i},a_i)>0 \Leftrightarrow u_i(a_{-i},b_i)-u_i(a_{-i},a_i)>0$$
  $$\forall a \in A, \forall a_i, b_i \in A_i$$

  Consider $a \in A$ for which $\Psi(a)$ is maximal.

  For any $a'=(a_{-i},a_i')$ we have $\Psi(a_{-i},a_i) \geq \Psi(a_{-i},a_i')$ and hence $u_i(a_{-i},a_i) \geq u_i(a_{-i},a_i')$

- Consequence:
  - Every finite ordinal potential game possesses a pure-strategy Nash equilibrium

Example continued: SND game

- Consider the SND game <$N,(A_i),(u_i)$>
- Define for each $e \in E$

$$\Psi_e(a) = c_e H_{k_e}$$

k_e=#of players for which $e \in a_i$

$$H_k = \sum_{j=1}^{k} \frac{1}{j}$$

- Define the function

$$\Psi(a) = \sum_e \Psi_e(a)$$

- $\Psi(a)$ is an exact potential for the SND game

Example continued: SND game II

- Let $a=(a_i)_{i=1..k}$, $a_i' \neq a_i$ be an alternate path for player i, and $a'=(a_{-i},a_i')$. Then
$$\Psi(a)-\Psi(a')=u_i(a')-u_i(a)$$
- Proof
$$e\in a_i, e\in a_i' \text{ or } e\notin a_i, e\notin a_i' \to \begin{cases} \psi_e(a)=\psi_e(a') \\ c_e/k_e|_{a_i}=c_e/k_e|_{a_i'} \end{cases}$$
$$e\in a_i, e\notin a_i' \to \begin{cases} \psi_e(a')=\psi_e(a)-c_e/k_e \\ u_i(a')=u_i(a)+c_e/k_e \end{cases}$$
$$e\notin a_i, e\in a_i' \to \begin{cases} \psi_e(a')=\psi_e(a)+c_e/(k_e+1) \\ u_i(a')=u_i(a)-c_e/(k_e+1) \end{cases}$$
- Furthermore
$$\mathrm{cost}(a)\le\Psi(a)\le H_k\mathrm{cost}(a)$$

Price of stability

- Let $G=<N,(A_i),(u_i)>$ be a finite strategic game with exact potential Ψ such that

$$\frac{\text{cost}(a)}{C} \leq \Psi(a) \leq D\,\text{cost}(a)$$

for some constants $C,D>0$. Then PoS$\leq C\times D$.

- Proof
  - Let $a^* \in A$ be a local maximizer of $\Psi \Rightarrow a^*$ is NE
  - Let $\hat{a}$ be a global maximizer of Ψ

$$\left.\begin{array}{l} D\,\text{cost}(\hat{a}) \geq \Psi(\hat{a}) \\ \Psi(\hat{a}) \geq \Psi(a^*) \\ \Psi(a^*) \geq \dfrac{\text{cost}(a^*)}{C} \end{array}\right\} \Longrightarrow \begin{array}{c} D\,\text{cost}(\hat{a}) \geq \Psi(\hat{a}) \geq \Psi(a^*) \geq \dfrac{\text{cost}(a^*)}{C} \\ C\times D\,\text{cost}(\hat{a}) \geq \text{cost}(a^*) \\ C\times D \geq \dfrac{\text{cost}(a^*)}{\text{cost}(\hat{a})} \end{array}$$

Improvement path

- A path in A is a sequence $\gamma=(a^0,a^1,\ldots)$ such that for every $k\geq 1$ there is a **unique** player i such that $a^k=(a_{-i}^{k-1},a_i^{'})$ for some $a_i^{'}\neq a_i^{k-1}$

- Initial point of γ is a^0
- For finite γ last element called terminal point

- A path $\gamma=(a^0,a^1,\ldots)$ is an improvement path w.r.t. game $G=<N,(A_i),(u_i)>$ if for all $k\geq 1$ $u_i(a^k)>u_i(a^{k-1})$, where player i is the unique deviator at step k.
  - path generated by *myopic* players
  - "Nash" or "asynchronous better reply" dynamics

Finite improvement property

- The strategic game $G=<N,(A_i),(u_i)>$ has the finite improvement property (FIP) if every improvement path $\gamma=(a^0,a^1,\ldots)$ is finite.

- Every finite ordinal potential game has the FIP.
- Proof
  - By definition $\psi(a^0)<\psi(a^1)<\ldots$
  - Since A is finite, the improvement path must be finite

- In any finite ordinal potential game the asynchronous better reply dynamic always converges to a Nash equilibrium

Generalized Ordinal Potential

- Let $G=\langle N,(A_i),(u_i)\rangle$ be a finite strategic game, and $A=x_{i\in N}A_i$. A function $\Psi:A\to R$ is a generalized ordinal potential for G if

$$u_i(a_{-i},b_i)-u_i(a_{-i},a_i)>0 \Rightarrow \psi(a_{-i},b_i)-\psi(a_{-i},a_i)>0$$
$$\forall a\in A, \forall a_i,b_i\in A_i$$

	L	R
T	1,0	2,0
B	2,0	0,1

0	3
1	2

- Let $G=\langle N,(A_i),(u_i)\rangle$ be a finite strategic game. G has the FIP property iff G has a generalized ordinal potential.

Infinite potential games

- A strategic game $G=<N,(A_i),(u_i)>$ is a bounded game if $(u_i)_{i\in N}$ are bounded
- Every bounded infinite weighted potential game possesses an ε-equilibrium point for every $\varepsilon>0$
- Proof:
  - Ψ is bounded because u_i is bounded, hence $\exists a' \in A \;\; s.t. \; \Psi(a') > \sup_{a\in A} \Psi(a) - \varepsilon$

Approximate finite improvement

- A path $\gamma=(a^0,a^1,\ldots)$ is an ε-improvement path for the strategic game $G=<N,(A_i),(u_i)>$ if for all $k\geq 1$ $u_i(a^k)>u_i(a^{k-1})+\varepsilon$, where i is the unique deviator at step k.
  - ε-Nash dynamics
- The strategic game $G=<N,(A_i),(u_i)>$ has the approximate FIP property if for $\forall\varepsilon>0$ every ε-improvement path is finite.
- Every bounded infinite potential game has the approximate FIP property.

Continuous potential games

- A strategic game $G=<N,(A_i),(u_i)>$ is continuous if A_i are topological spaces, and u_i are continuous w.r.t $A=\times_{i\in N}A_i$.

- Let $G=<N,(A_i),(u_i)>$ be a continuous exact potential game with compact action sets. G possesses a pure strategy Nash-equilibrium.

Construction of the potential

- Let $G=<N,(A_i),(u_i)>$, $A_i \subset R$ compact, u_i continuously differentiable and $\Psi: A \to R$.

  Then Ψ is a potential for G iff Ψ is continuously differentiable and

$$\frac{\partial u_i}{\partial a_i} = \frac{\partial \Psi}{\partial a_i} \quad \forall i \in N$$

Congestion games

- Set of players $N=\{1,\ldots,n\}$
- Primary factors $T=\{1,\ldots,t\}$
- Action set $A_i=\{1,\ldots,a_{iJ}\}\subseteq 2^T$
  - Action $a_i\subseteq T$
- Cost of action a_i

Same for all players!

$$\mathrm{cost}_i(a_{-i},a_i)=\sum_{\tau\in a_i} c_\tau(k_\tau),$$

where k_τ= # of players using factor τ in a

Congestion games

- Every congestion game is an exact potential game with potential

$$\Psi(a) = \sum_{\tau \in T} \sum_{y=1}^{k_\tau} c(y)$$

- Every finite potential game is isomorphic to a congestion game.

R.W. Rosenthal, "A Class of Games Possessing Pure-Strategy Nash Equilibria," vol. 2, Int. J. Game Theory, pp. 65–67, 1973
D. Monderer, L.S. Shapley, "Potential Games", Games and Economic Behavior vol. 14., pp. 124-143, 1996

Examples of congestion games

- Selfish routing games
  - Non-atomic
    - Multipath allowed
  - Atomic non-weighted
    - Single path only
    - Same amount of traffic for all players
- Market sharing games

- Load balancing games

Minimum cut problem

- Network of nodes $V \cup \{s\} \cup \{t\}$, $|V|=m$
- Capacity $c(w,z) \geq 0$ for every pair of nodes

$$(w,z) \in V \cup \{s\} \cup \{t\} \times V \cup \{s\} \cup \{t\}$$

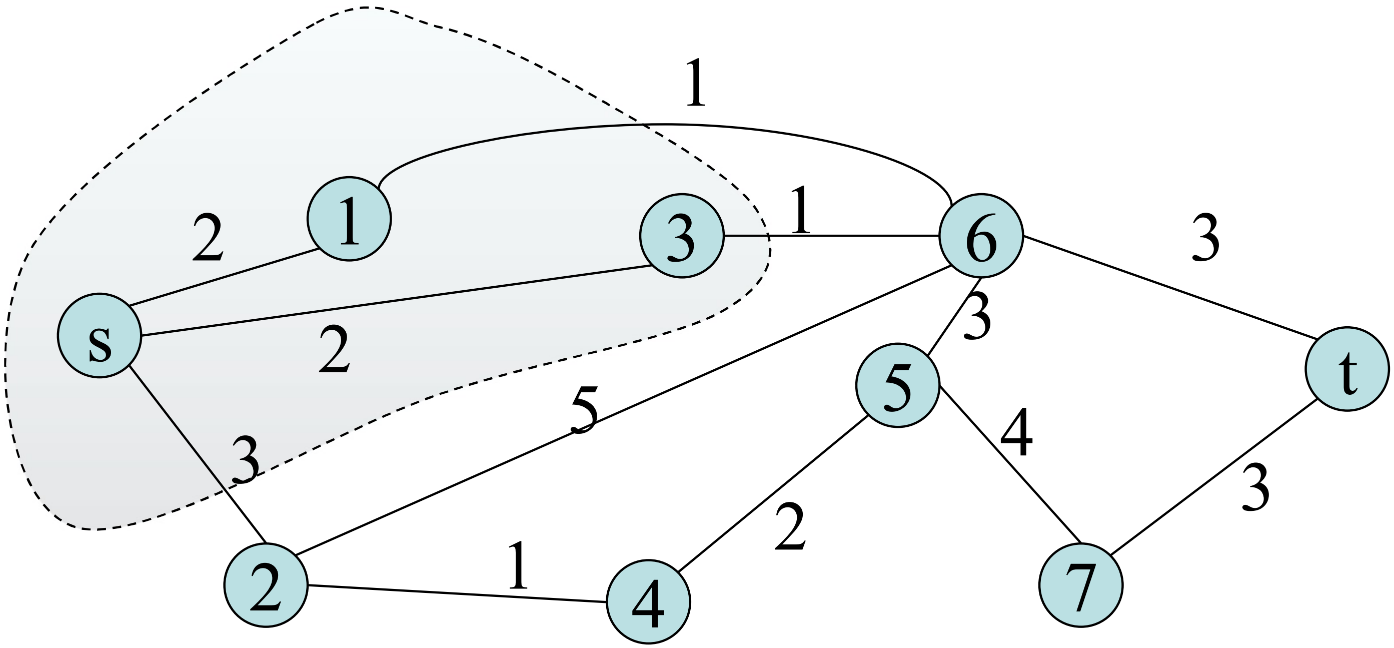
- $X \subseteq V$ then $X \cup \{s\}$ is a *cut*
- Cut capacity

$$f(X) = \sum_{w \in X \cup s} \sum_{z \notin X \cup s} c(w,z)$$

- $X^* \cup \{s\}$ is *minimum cut* if $X^* \subseteq V$ *and*

$$f(X^*) \leq f(X) \quad \forall X \subseteq V$$

Minimum cut problem



- Min-cut: {1,3,s}

Minimum cut game

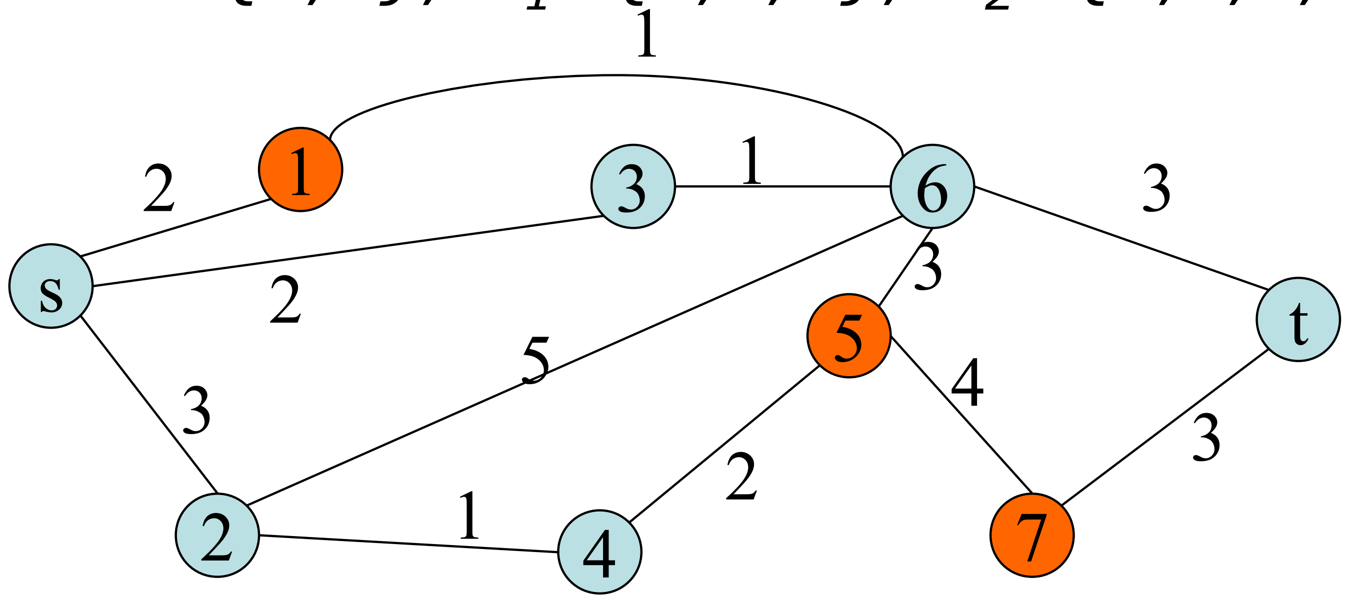
- Set of players N, $|N|=n$
  - $V=\cup_{i=1..n}V_i$, $V_i \cap V_j=\emptyset$ for $\forall i \neq j$, $|V_i|=m_i$
- Action set $A_i=\{X_i : X_i \subseteq V_i\}$
- Capacity function $c_i(w,z)$ (player specific)
- Objective of player i

$$\min_{a_i} f_i(X) = f_i(X_i \cup (\bigcup_{i \neq j} X_j))$$

- Claim: The minimum cut game has a pure strategy Nash equilibrium.

Minimum cut game

- $N=\{1,2\}$, $V_1=\{1,5,7\}$, $V_2=\{2,3,4,6\}$



- Min-cut: {1,3,s}
- NE: $X_1=\{1\}$, $X_2=\{3\}$

Lattices and Sublattices

- A partially ordered set $(A,\geq)$ is a lattice if
  - for $a,b \in A$ $\exists c \in A$ s.t. $a \vee b = c$ ($c \geq a$, $c \geq b$, *join*)
  - for $a,b \in A$ $\exists c \in A$ s.t. $a \wedge b = c$ ($a \geq c$, $b \geq c$, *meet*)
- A sublattice of a lattice L is a subset of L and itself a lattice with respect to the same $\wedge$ and $\vee$ operators.
- If A is a nonempty compact sublattice of R^m, it has a greatest and a least element.
  - the sublattice A is bounded
  - componentwise partial ordering

G. Birkhoff, "Lattice theory", American Mathematical Society, 1967

Increasing Differences

- Let X,T be posets, $S \subseteq X \times T$, $S_t = \{x | (x,t) \in S\}$
  - $u_i(a)$ has increasing differences in (a_i, a_j) if for $(a'_i, a'_j) \in S$ such that $a'_i \geq a_i$ and $a'_j \geq a_j$

$$u_i(a_i^{'}, a_j^{'}) - u_i(a_i, a_j^{'}) \geq u_i(a_i^{'}, a_j) - u_i(a_i, a_j)$$

- Let A_i be poset, $A \subseteq \times A_i$
  - $u_i(a)$ has increasing differences on A if it has increasing differences in all (a_i, a_j) for $i \neq j$ and fixed $a_{-i,j}$

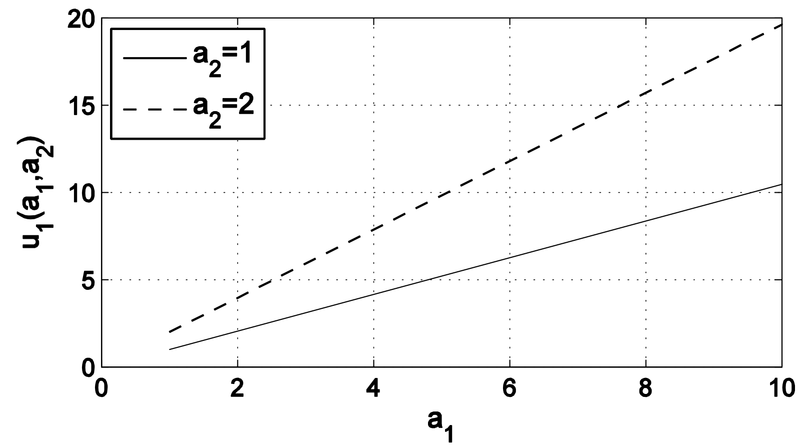
- Twice differentiable

$$\frac{\partial^2 u_i}{\partial a_l \partial a_k} \geq 0 \ \forall k,l$$

- Example
  - $f: R \rightarrow R$ convex

$$u_i(a) = f(\Pi_{i=1}^{|N|} a_i)$$

    - *Strictly* increasing



Supermodular Functions

- $u_i(a_{-i},a_i)$ is supermodular on A_i (lattice) if for $a_i,a^*_i \in A_i$ and $\forall a_{-i} \in A_{-i}$

$$u_i(a_{-i},a_i)+u_i(a_{-i},a_i^*) \le u_i(a_{-i},a_i \wedge a_i^*)+u_i(a_{-i},a_i \vee a_i^*)$$

- $u_i(a_{-i},a_i)$ is strictly supermodular on A_i if for $a_i,a^*_i \in A_i$ and $\forall a_{-i} \in A_{-i}$

$$u_i(a_{-i},a_i)+u_i(a_{-i},a_i^*) < u_i(a_{-i},a_i \wedge a_i^*)+u_i(a_{-i},a_i \vee a_i^*)$$

whenever a_i and a^*_i are not comparable w.r.t $\ge$

- $u_i(a)$ is supermodular on A (lattice) if for $a,a^* \in A$

$$u_i(a)+u_i(a^*) \le u_i(a \wedge a^*)+u_i(a \vee a^*)$$

Substitute:
$a=(a_{-i},a_i)$,
$a^*=(a_{-i}, a^*_i)$

Submodular Functions

- $u_i(a_{-i},a_i)$ is submodular on A_i (lattice) if for $a_i,a^*_i \in A_i$ and $\forall a_{-i} \in A_{-i}$

$$u_i(a_{-i},a_i)+u_i(a_{-i},a_i^*) \geq u_i(a_{-i},a_i \wedge a_i^*)+u_i(a_{-i},a_i \vee a_i^*)$$

- Alternative definition
  - Let f bet set function defined on S, and $X \subseteq Y \subseteq S$. Then f is submodular if $\forall x \in S \backslash Y$

$$f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$$

- Example
  - Let Q matrix with column set B. For $X \subseteq B$ let r(X) be the rank of matrix formed by X. r(x) is submodular.

Supermodularity $\Rightarrow$ Increasing differences

- Let A_i lattice, A sublattice of $\times A_i$
- If $u_i(a)$ is supermodular on A then it has increasing differences on A

take $a'_i \geq a_i$ and $a'_{-i} \geq a_{-i}$ and $x=(a_{-i}, a'_i)$, $y=(a'_{-i}, a_i)$

$$u_i(x)+u_i(y) \leq u_i(x \wedge y)+u_i(x \vee y)$$

$$x \vee y=(a'_{-i}, a'_i) \qquad x \wedge y=(a_{-i}, a_i)$$

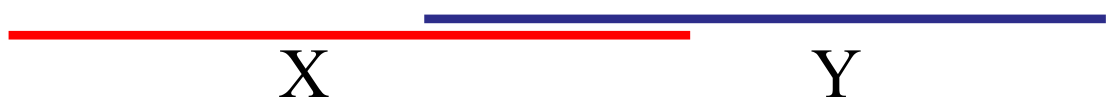
$$\Downarrow$$

$$u_i(a'_{-i}, a'_i)-u_i(a'_{-i}, a_i) \geq u_i(a_{-i}, a'_i)-u_i(a_{-i}, a_i)$$

Partial Ordering of Sublattices

- Let X, Y be nonempty sublattices of E^n
  - Partial ordering $\trianglelefteq^p$

$$X \leq^p Y \quad \text{if} \quad x \wedge y \in X \text{ and } x \vee y \in Y \quad \forall x \in X, y \in Y$$



- Let X_y be collection of nonempty sublattices of E^n for $y \in Y \subseteq E^m$
  - X_y is ascending on Y if $X_y \trianglelefteq^p X_w$ for $y \leq w$

- Let X_y be lower/upper contour set on sublattice of E^n
  - X_y is ascending in y

D.M. Topkis, "Equilibrium points in nonzero-sum n-person submodular games", SIAM J. Control and Optimization 17(6), pp.773-787, 1979.

Topkis's Theorem

- Let D be a lattice (independent of θ, or ascending in θ). If f has increasing differences in (x,θ) and is supermodular in x then

$$x^* = \arg\max_{x\in D} f(x,\theta)$$

is increasing in the strong set order.

D.M. Topkis, "Equilibrium points in nonzero-sum n-person submodular games", SIAM J. Control and Optimization 17(6), pp.773-787, 1979.

Supermodular games

- Strategic game $G=<N,(A_i),(u_i)>$ is (strictly) supermodular if
  - A_i is a non-empty sublattice of a Euclidean space
  - u_i has (strictly) increasing differences in (a_{-i},a_i)
  - u_i is (strictly) supermodular on A_i

Existence of equilibria

- Let $G=\langle N,(A_i),(u_i)\rangle$ be a supermodular game,
  - A_i compact, and
  - u_i upper-semicontinuous in a_i for each a_{-i},

  then the set of pure strategy NE is nonempty and possesses greatest and least elements.

$$\limsup_{x \to x_0} f(x) \le f(x_0)$$

D.M. Topkis, "Equilibrium points in nonzero-sum n-person submodular games", SIAM J. Control and Optimization 17(6), pp.773-787, 1979.

Example – Min-cut game rev.

- Set of actions $A_i = 2^{V_i}$ (power set of V_i)
  - Lattice with respect to inclusion, union, intersection

- *f(X)* is submodular on *A*

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$$

$$f(S) + f(T) - f(S \cap T) - f(S \cup T) =$$
$$= c(A(S:T)) + c(A(T:S)) \geq 0$$
$$\text{where } A(X:Y) = \{(i,j) \in E : i \in X, j \in Y\}$$

  - $f_i(X)$ is submodular on X_i

D.M. Topkis, "Ordered optimal solutions", PhD thesis, U. of Stanford, 1968

Convergence to Equilibria

- Let G be a supermodular game and let
  - A_i compact,
  - u_i upper-semicontinuous on $A_i(a_{-i})$ $\forall a_{-i} \in A_i$
  - (the best response correspondences $B_i(a_{-i})$ have the ascending property)

  then the best response dynamic converges to a pure Nash equilibrium (starting from least element)

- Similar result holds for submodular games (descending property)

D.M. Topkis, "Equilibrium points in nonzero-sum n-person submodular games", SIAM J. Control and Optimization 17(6), pp.773-787, 1979.

Super- and submodular games

- Supermodular games
  - Strategic complements
  - Minimum cut game (e.g., choosing activities)
  - Facility location problem
  - Steiner tree in a graph (minimum spanning tree)
- Submodular games
  - Strategic substitutes
- Mixture of submodular and supermodular
  - S-modular

Literature

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