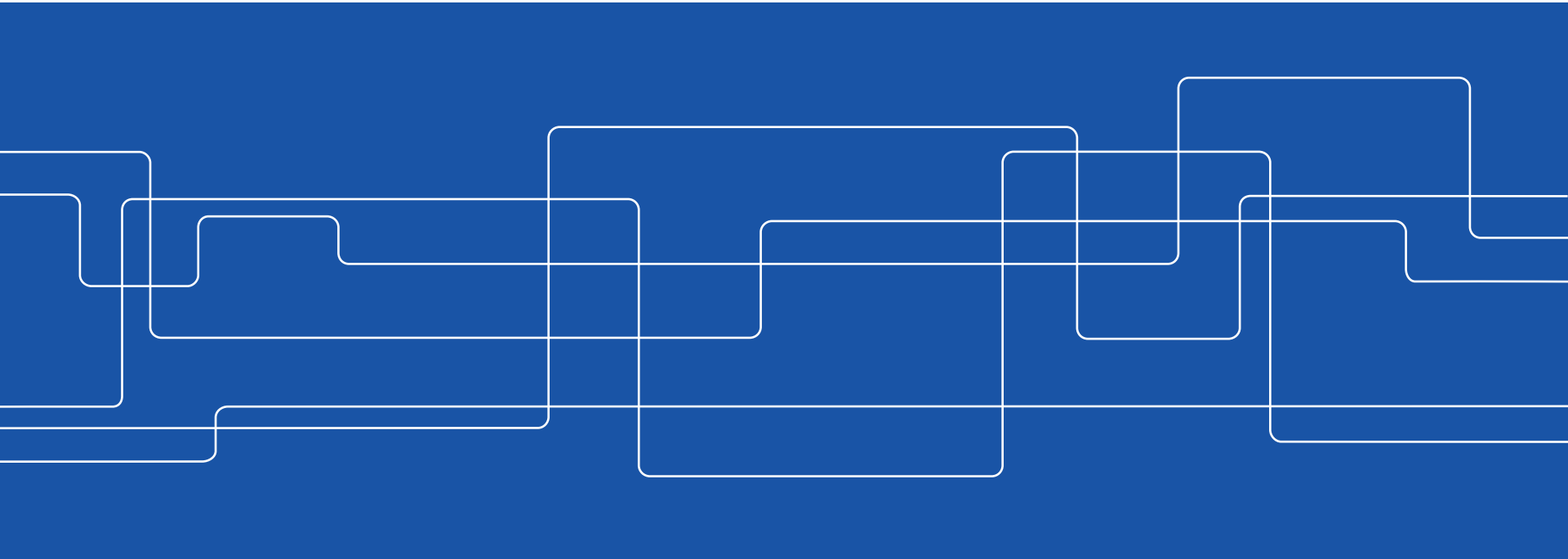




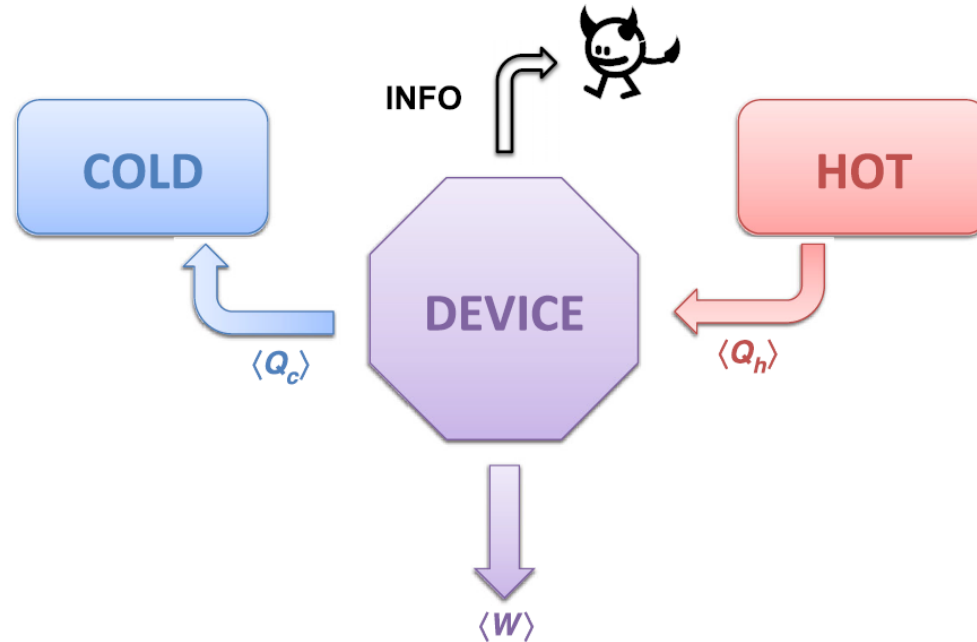
Thermodynamic Costs in Implementing Kalman-Bucy Filters

Henrik Sandberg (KTH)
Nigel J. Newton (Essex)

Jean-Charles Delvenne (UCL)
Sanjoy K. Mitter (MIT)



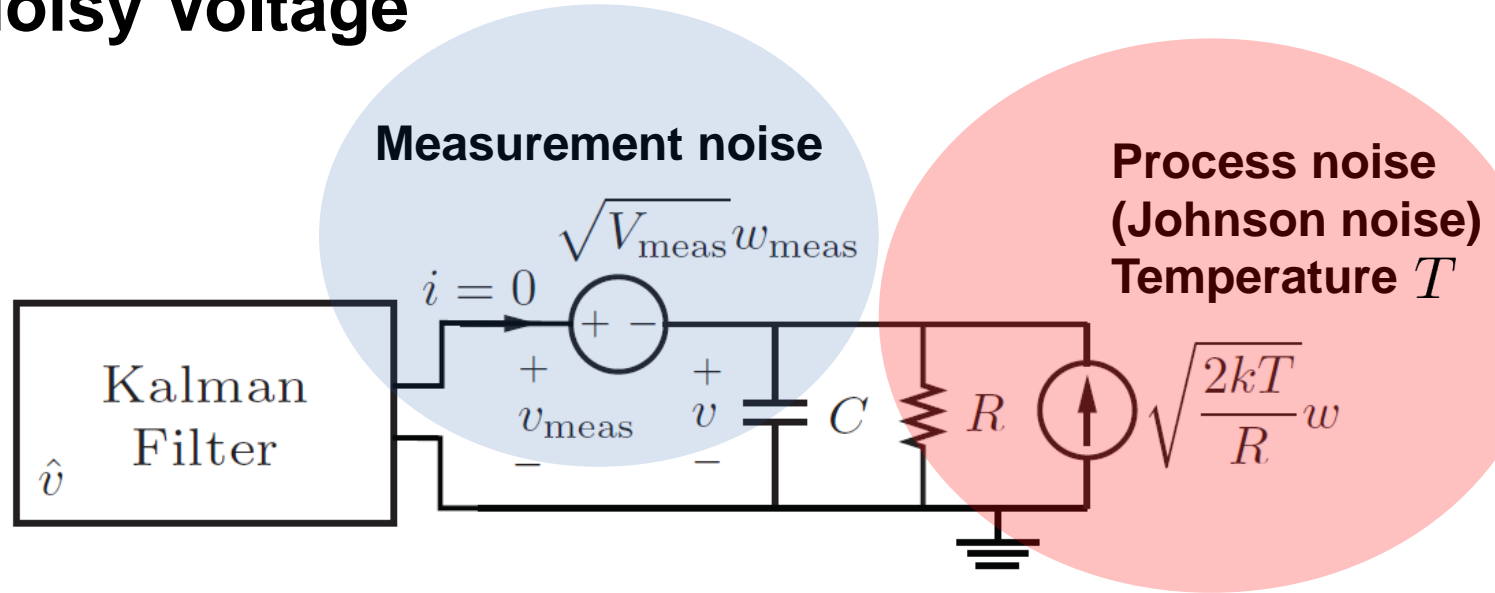
Background: Information Processing and the 2nd Law of Thermodynamics (Maxwell's Demon)



$$\langle W \rangle \leq -\Delta F + kT \langle I \rangle$$

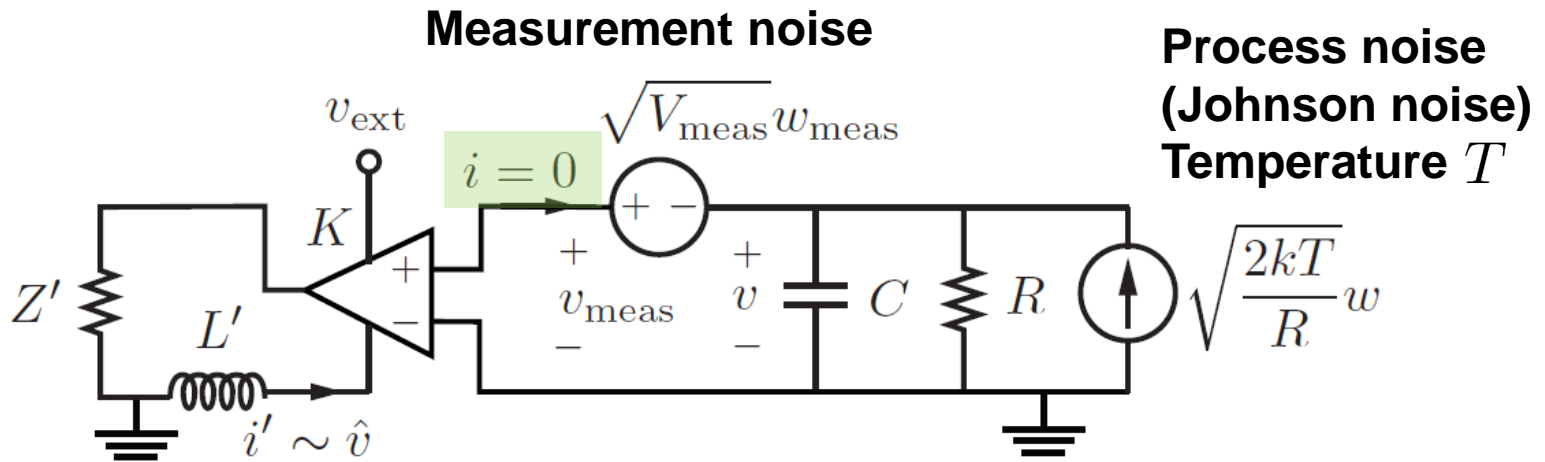
[S. Deffner and C. Jarzynski, Phys Rev X 041003, 2013]

Motivating Example: Optimal Estimation of Noisy Voltage



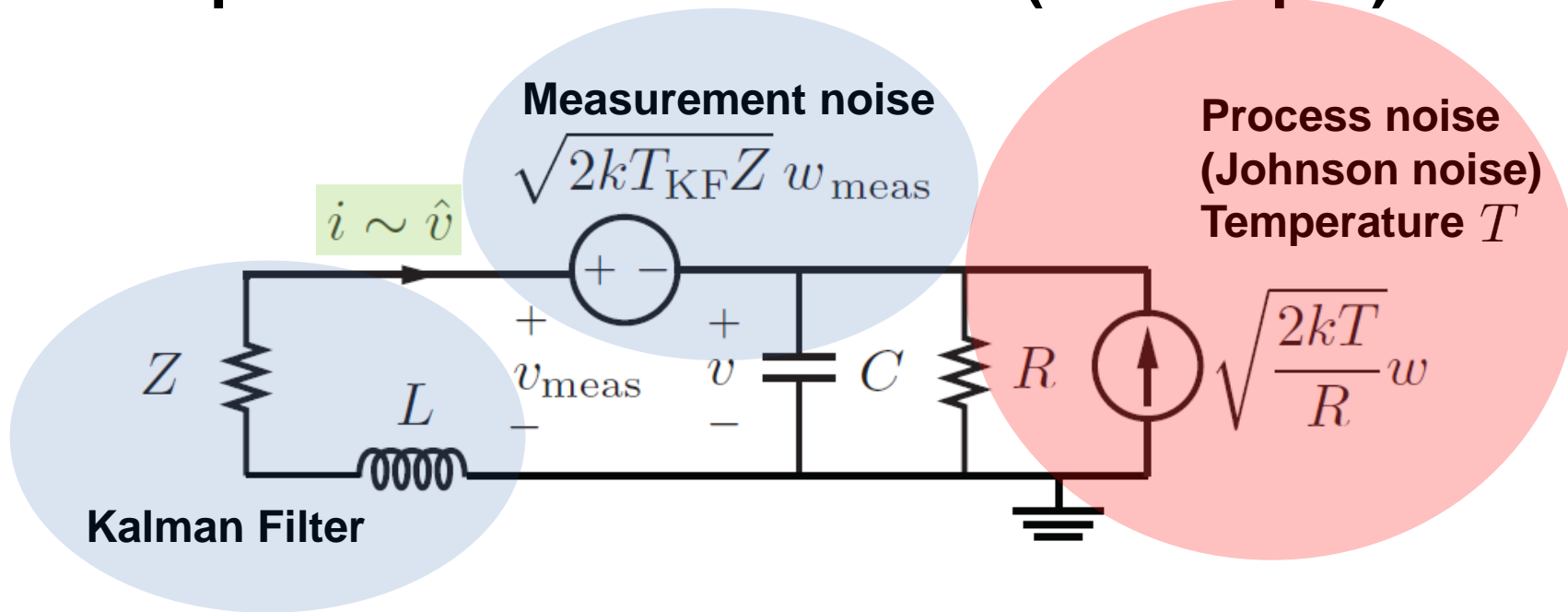
- What is a simple physical implementation of (analog) Kalman-Bucy filter?
- Is there a lower bound on required external power supply to filter?

Implementation 1: Op-Amp



- Possible implementation using ideal op-amp
- What is the smallest power supply?
- No back action (perturbation)

Implementation 2: RL Circuit (This Paper)



- Small, yet nonzero, back action (perturbation)
- Implementation using passive components
- $T_{\text{KF}} < T$ (non-equilibrium thermodyn. system)

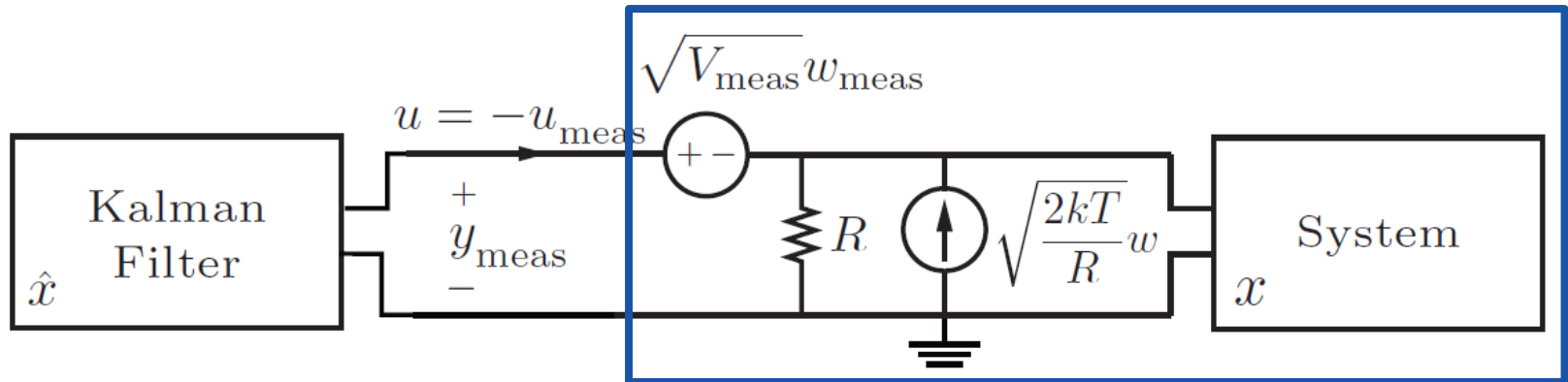


Contributions of Paper

- Identification of class of systems whose **Kalman-Bucy filters** are **implementable with passive components**. External power supply still needed, however
- Characterization of **trade-off power supply vs back action** for filter implementations (by application of 2nd law of thermodynamics)
- Connection made to **Landauer's principle** ($kT \ln 2$ Joules to erase one bit of information)

[R. Landauer, IBM J Res Dev, 1961]

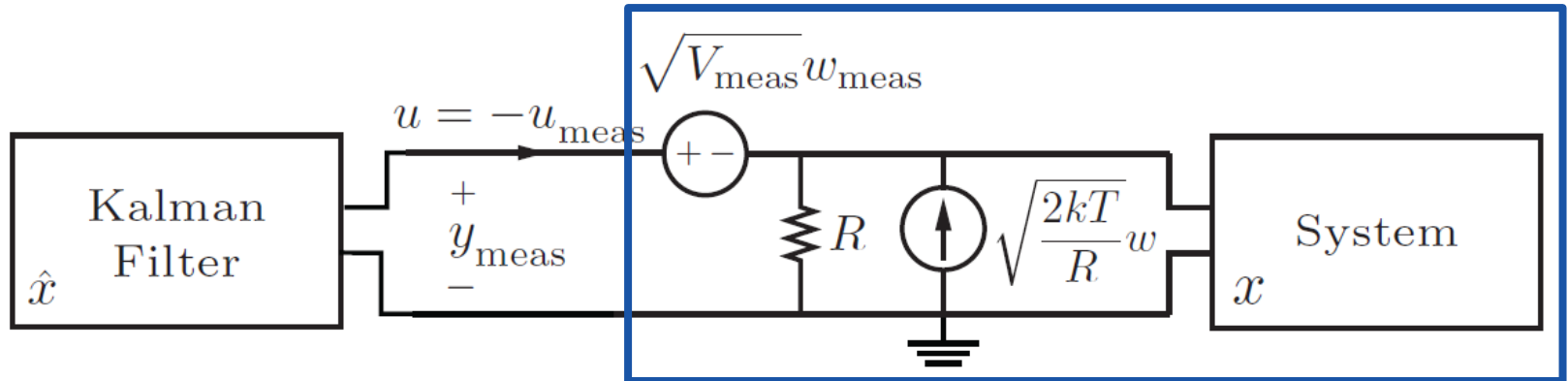
Considered Class of Systems



Linear port-Hamiltonian system (A. van der Schaft, *et al.*)

$$\begin{aligned} \dot{x} &= (J - GBB^T)Mx + Bu + B\sqrt{2kTG}w \\ y &= B^T Mx \\ y_{\text{meas}} &= B^T Mx + \sqrt{V_{\text{meas}}}w_{\text{meas}} \\ (J &= -J^T, \quad G := 1/R, \quad M = M^T) \end{aligned}$$

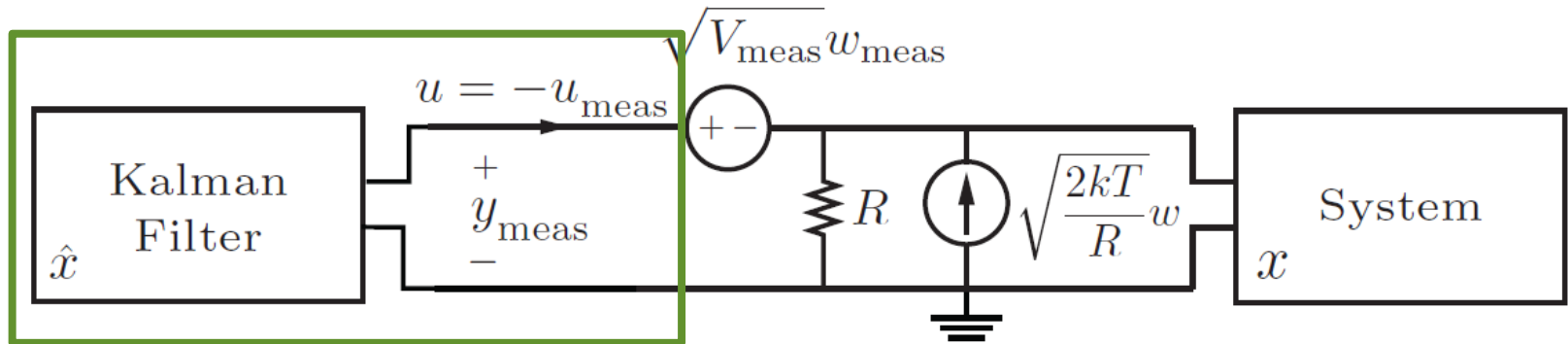
Considered Class of Systems



Signal-to-noise ratio (SNR):

$$\frac{(\text{process noise [V]})^2}{(\text{measurement noise [V]})^2} \equiv \frac{2kT}{GV_{\text{meas}}} =: \sigma$$

Kalman-Bucy Filter



$$\frac{d}{dt}\hat{x} = (J - GBB^T)M\hat{x} + Bu + K(y_{\text{meas}} - B^T M\hat{x})$$

Lemma: Kalman gain is
 $K = (\sqrt{1 + \sigma} - 1)GB \equiv g_K B$



Kalman-Bucy Filter is Port-Hamiltonian

Assumption: Admit linear back action current

$$\underbrace{u_{\text{meas}}}_{\text{current}} = -u = g \underbrace{B^T M \hat{x}}_{\text{voltage}}, \quad g = \text{gain} \in (0, \infty), \text{ free parameter}$$

Theorem: A realization of Kalman-Bucy filter is

$$\frac{d}{dt} \hat{x}_s = (J - Z B_s B_s^T) M \hat{x}_s + B_s y_{\text{meas}}$$

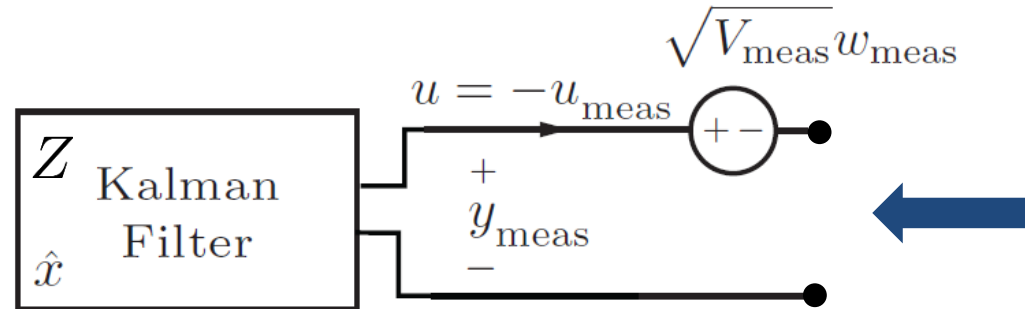
$$u_{\text{meas}} = B_s^T M \hat{x}_s$$

$$\hat{x}_s = \sqrt{g/g_K} \hat{x}$$

with effective resistance

$$Z = \frac{\sqrt{\sigma + 1}}{(\sqrt{\sigma + 1} - 1)g} + \frac{1}{(\sqrt{\sigma + 1} - 1)G} > 0$$

Temperature of Kalman-Bucy Filter



Fluctuation-dissipation theorem: $2kT_{\text{KF}}Z = V_{\text{meas}}$

Effective filter temperature:

$$T_{\text{KF}} = \frac{T}{\sqrt{1 + \sigma} + 1} \frac{g}{g + g_K + G} < \frac{1}{2}T$$

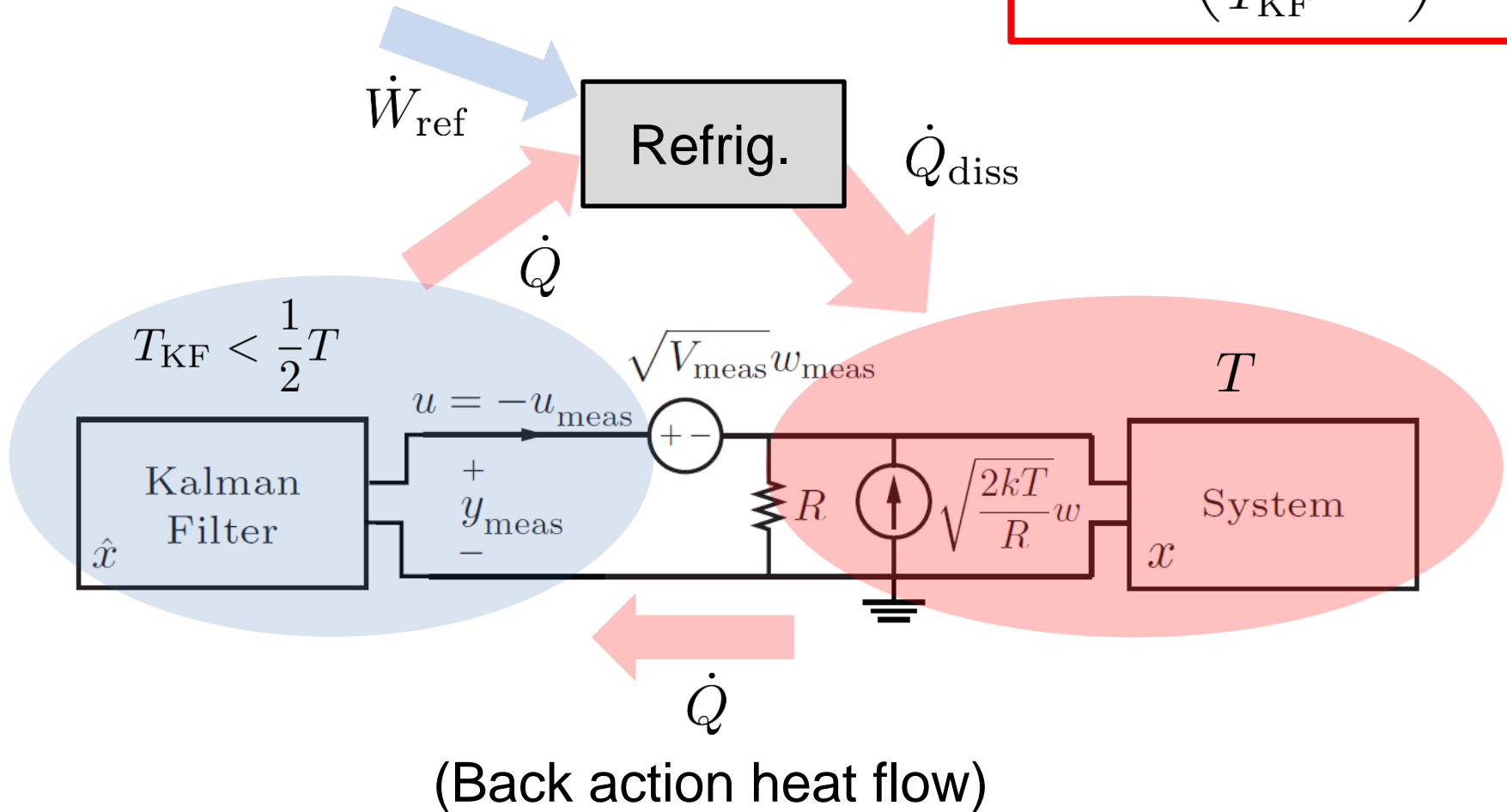
Insight:

As back action tends to zero, filter temperature tends to zero

Heat and Work Flows

2nd Law:

$$\dot{W}_{\text{ref}} \geq \left(\frac{T}{T_{\text{KF}}} - 1 \right) \dot{Q}$$



Power Supply Required by the 2nd Law

$$\dot{W}_{\text{ref}} \geq \kappa T (1 + \sigma - \sqrt{1 + \sigma}) \left(1 - \frac{\sqrt{1 + \sigma}}{1 + \sqrt{1 + \sigma}} \text{BA} \right)$$

No back action (BA \rightarrow 0)

$$\dot{W}_{\text{ref}} \geq \kappa T (1 + \sigma - \sqrt{1 + \sigma}) \approx \begin{cases} \frac{\kappa T}{2} \times \text{SNR} & (\text{low SNR}) \\ \kappa T \times \text{SNR} & (\text{high SNR}) \end{cases}$$

Max back action (BA \rightarrow 1)

$$\dot{W}_{\text{ref}} \geq \kappa T \frac{1 + \sigma - \sqrt{1 + \sigma}}{1 + \sqrt{1 + \sigma}} \approx \begin{cases} \frac{\kappa T}{4} \times \text{SNR} & (\text{low SNR}) \\ \kappa T \times \sqrt{\text{SNR}} & (\text{high SNR}) \end{cases}$$

Power Supply Required by the 2nd Law

$$\dot{W}_{\text{ref}} \geq \kappa T (1 + \sigma - \sqrt{1 + \sigma}) \left(1 - \frac{\sqrt{1 + \sigma}}{1 + \sqrt{1 + \sigma}} \text{BA} \right)$$

Heat conductivity of system

$$kGB^T MB$$

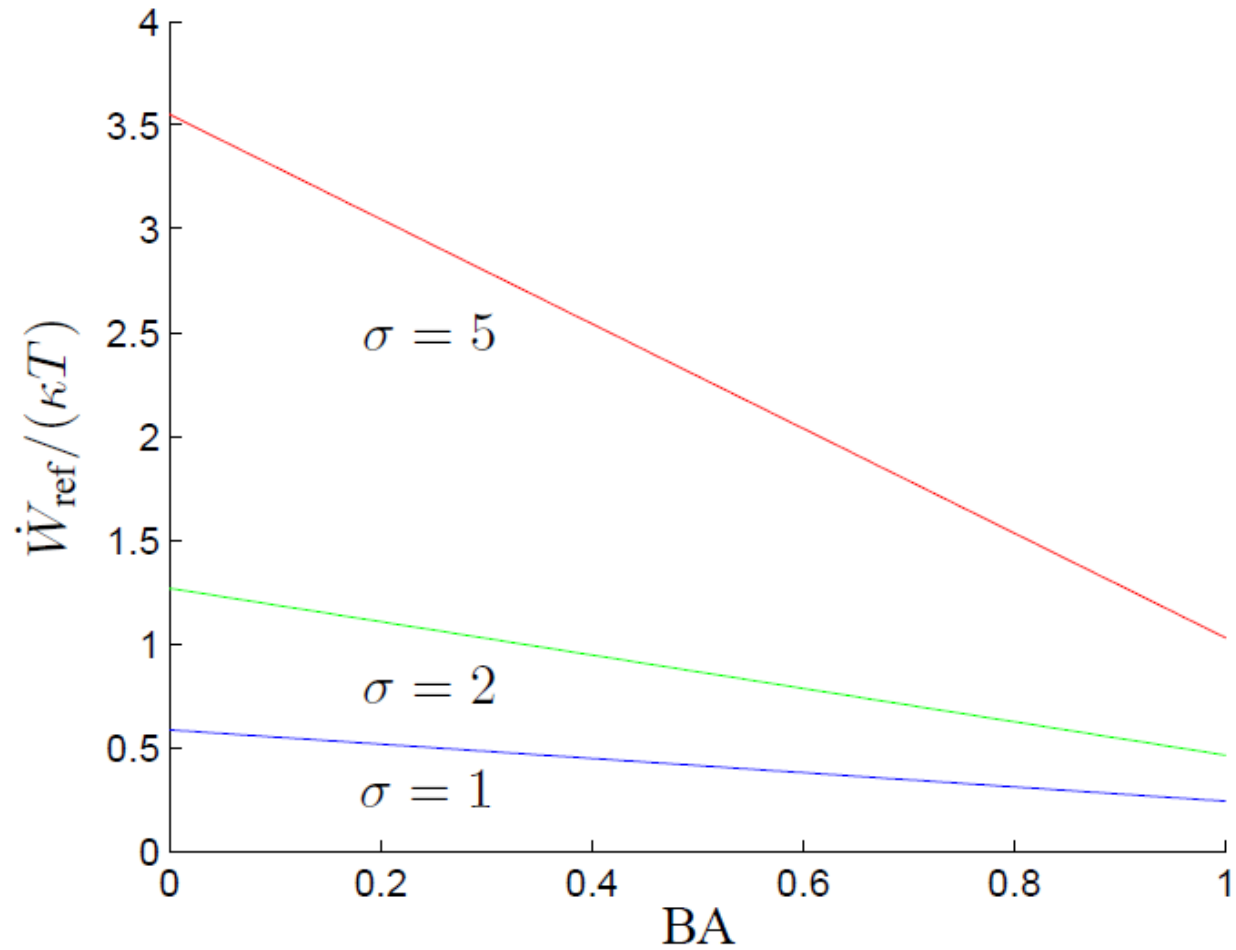
Normalized back action

$$\frac{g}{g + G}$$

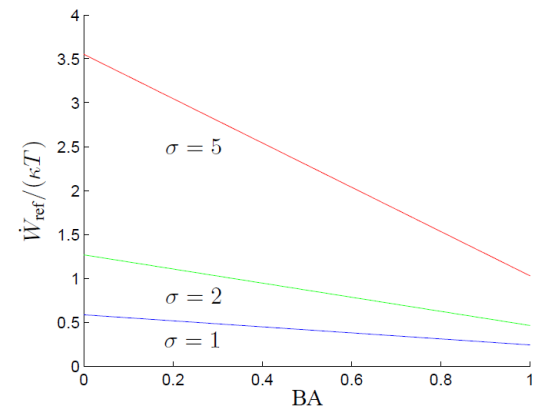
Max back action (BA \rightarrow 1)

$$\dot{W}_{\text{ref}} \geq \kappa T \frac{1 + \sigma - \sqrt{1 + \sigma}}{1 + \sqrt{1 + \sigma}} \approx \begin{cases} \frac{\kappa T}{4} \times \text{SNR} & (\text{low SNR}) \\ \kappa T \times \sqrt{\text{SNR}} & (\text{high SNR}) \end{cases}$$

Trade-Off Energy Supply/Back Action

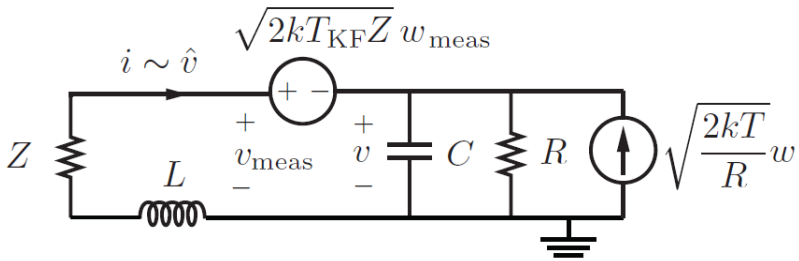


Observations



- **Main insight:** Large back action \Rightarrow lower power supply
- **Explanation:** Temperature difference $T - T_{\text{KF}}$ smaller \Rightarrow 2nd law less restrictive
- Trade-off more significant for high SNRs
- No back action costs a factor $\sqrt{\text{SNR}}$ more than max back action in high SNR regime

Example: Power Supply for RC Circuit



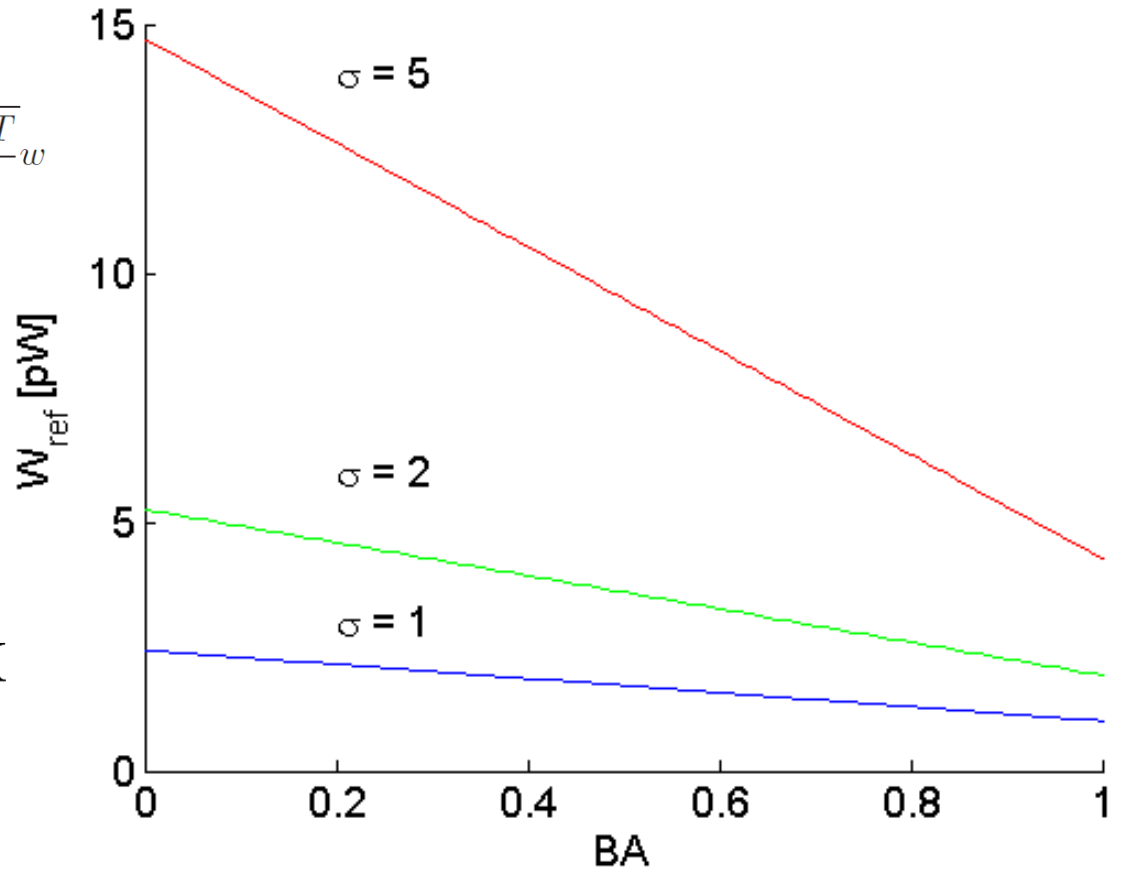
$$R = 1 \Omega$$

$$C = 1 \text{ nF}$$

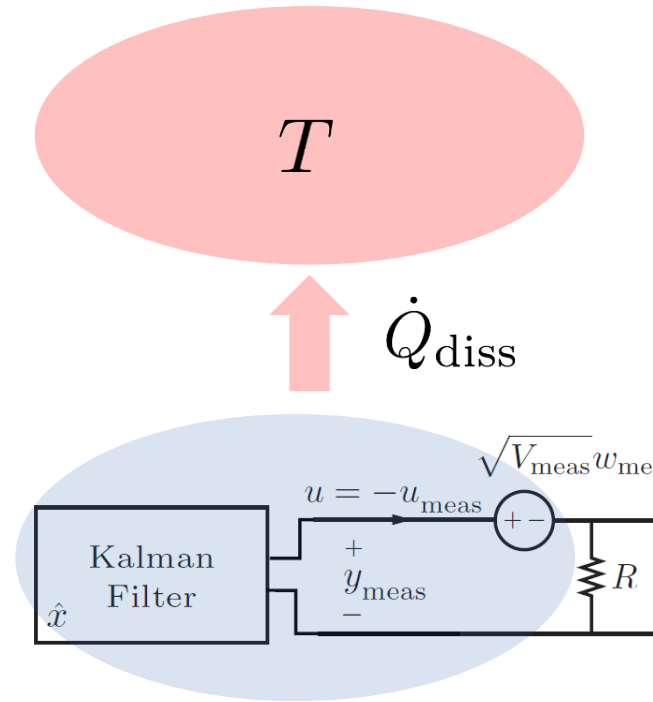
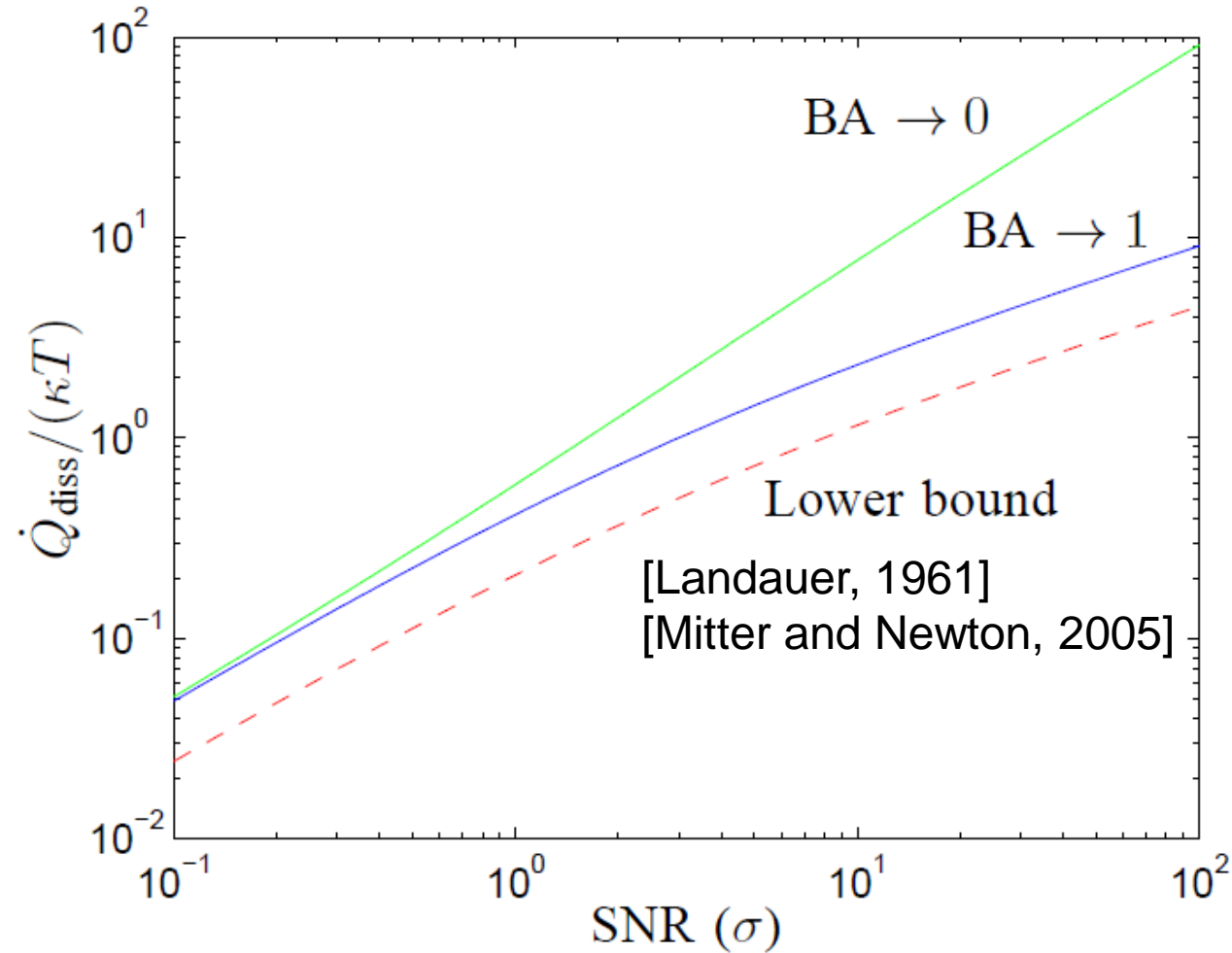
$$k = 1.38 \cdot 10^{-23} \text{ J/K}$$

$$\kappa = \frac{k}{RC} = 1.38 \cdot 10^{-14} \text{ W/K}$$

$$T = 300 \text{ K}$$



Filter Heat Dissipation (Lower Bound from Landauer's Principle)





Observations

- We have at least a factor 2 more dissipation than required by lower bound
- **Explanations:**
 - Landauer's principle holds for **infinitely slow erasure**. Here finite erasure rate, which costs more
 - Used information rate is a **lower bound** on **actual information rate** written to memory. Actual rate can be up to a factor 2 larger

[Sandberg *et al.*, Phys. Rev. E, 2014]



Summary

- Class of systems with “passive” Kalman-Bucy filters found. “Passive” but active cooling required (unless we own a cold heat bath...)
- A trade-off identified: Admit measurement perturbations for lower power supply
- Connection made to a basic principle from physics: Landauer’s principle
- **Open problem:** What is the best *attainable* lower bound on power supply?



Related References

- S.K. Mitter and N.J. Newton: “Information and entropy flow in the Kalman-Bucy filter” (*Journal of Statistical Physics*, 2005)
- J.-C. Delvenne and H. Sandberg: "Finite-Time Thermodynamics of Port-Hamiltonian Systems“ (*Physica D*, 2014)
- H. Sandberg, J.-C. Delvenne, N.J. Newton, S.K. Mitter: “Maximum work extraction and implementation costs for nonequilibrium Maxwell's demons” (*Physical Review E*, to appear, arXiv:1402.1010)