LMI-Based Model Reduction with Structural Constraints

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Model Reduction for Complex Dynamical Systems

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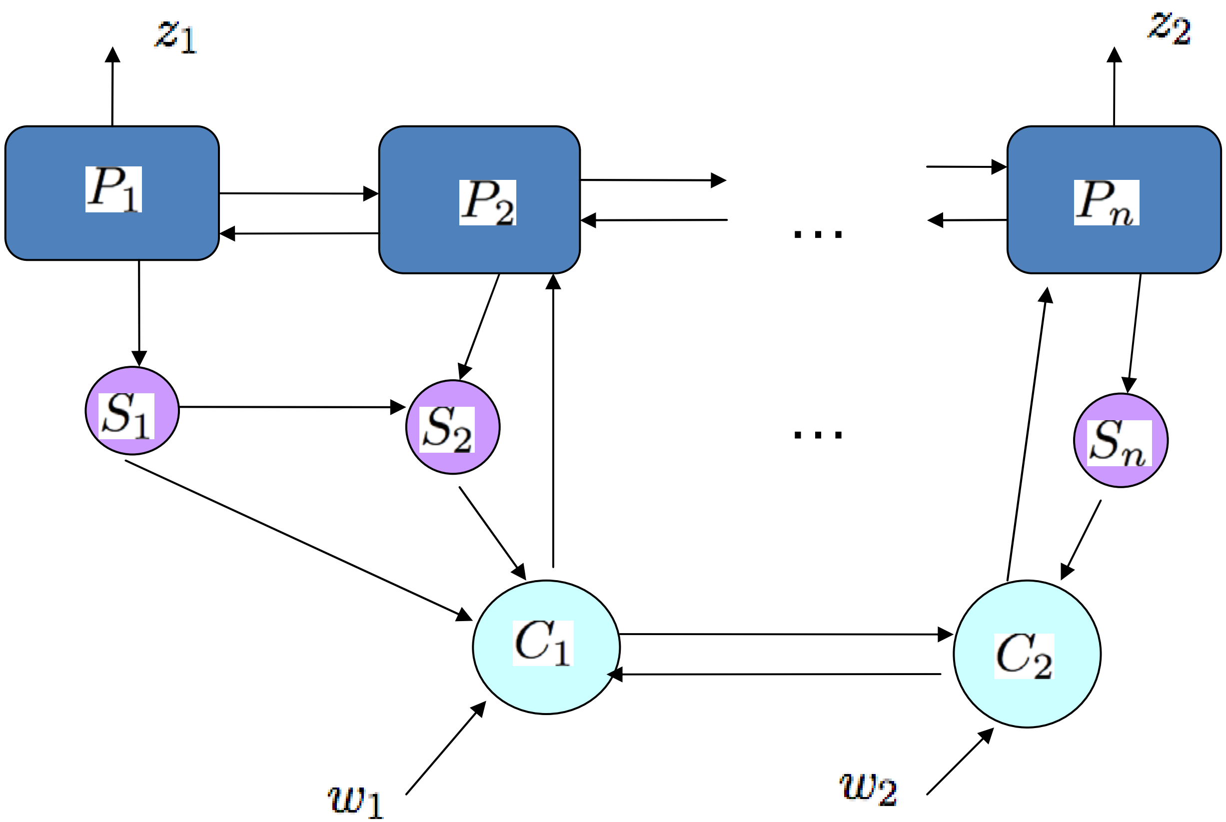
Outline

- Structured model reduction
- Structured (generalized) Gramians
- Model structures we can preserve:
  - Series/cascade and parallel systems
  - Feedback of passive systems
  - Port-Hamiltonian systems
- Example
  - Model reduction of boiler-header system

Structured Model Reduction

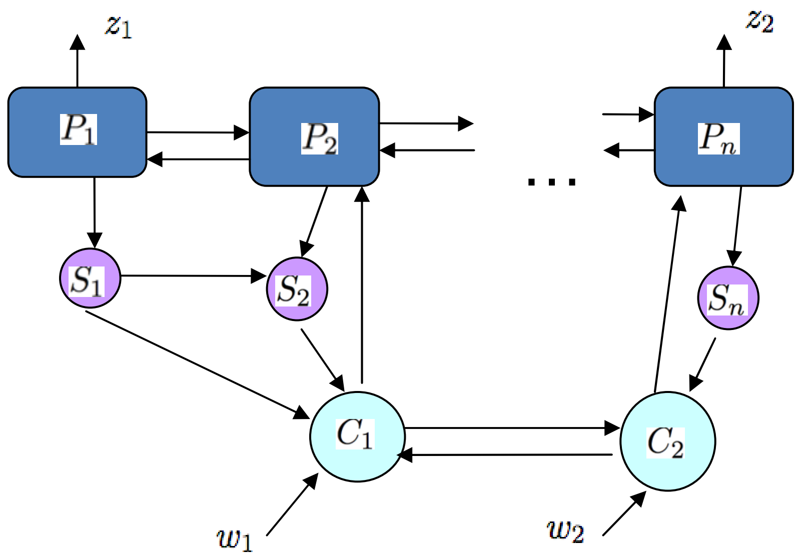


- Example: Distributed controller C_1,C_2 in a networked control system
- **Problem:** Find (local) approximations of C_1,C_2 such that (global) mapping $(w_1,w_2) \mapsto (z_1,z_2)$ is well preserved

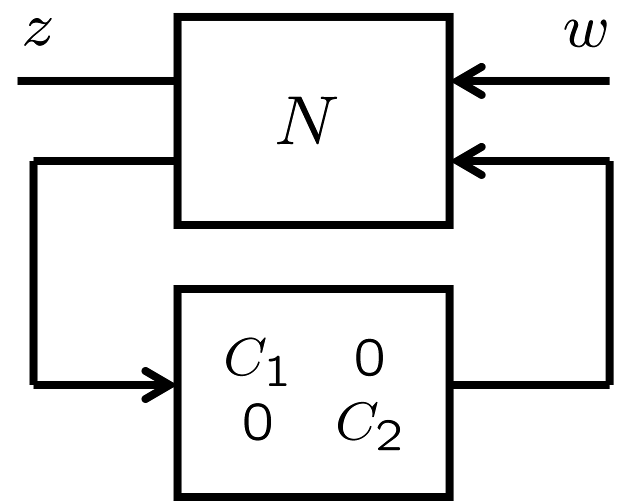
Related Work

- Controller reduction
  - Survey by Anderson and Liu in IEEE TAC, 1989
  - Closed-loop model reduction by Schelfout and De Moor in IEEE TAC, 1996
- Problem often encountered outside the control area
  - *Industrial processes*
  - Physical models
  - Chemical models
  - Biological models
- The reduction method should take *topological structure constraints* into account
- Hard problems! Do not expect optimal solutions
- But what can we do?

Equivalent Problems



$$z = \mathcal{F}_l(N, \{C_1, C_2\})w$$



$$\begin{bmatrix} \dot{x}_N \\ \dot{x}_{C_1} \\ \dot{x}_{C_2} \end{bmatrix} = \overbrace{\begin{bmatrix} A_N & A_{NC_1} & A_{NC_2} \\ A_{C_1N} & A_{C_1} & A_{C_1C_2} \\ A_{C_2N} & A_{C_2C_1} & A_{C_2} \end{bmatrix}}^{=A} \begin{bmatrix} x_N \\ x_{C_1} \\ x_{C_2} \end{bmatrix} + \overbrace{\begin{bmatrix} B_N \\ B_{C_1} \\ B_{C_2} \end{bmatrix}}^{=B} w$$

$$z = \underbrace{\begin{bmatrix} C_N & C_{C_1} & C_{C_2} \end{bmatrix}}_{=C} x$$

A Bad Heuristic

- Ignore surrounding subsystems and compute local Gramians

$$\begin{cases} A_{C_1}P_{C_1} + P_{C_1}A_{C_1}^T + \bar{B}_{C_1}\bar{B}_{C_1}^T = 0 \\ A_{C_1}^T Q_{C_1} + Q_{C_1}A_{C_1} + \bar{C}_{C_1}^T\bar{C}_{C_1} = 0 \end{cases} \quad \begin{cases} A_{C_2}P_{C_2} + P_{C_2}A_{C_2}^T + \bar{B}_{C_2}\bar{B}_{C_2}^T = 0 \\ A_{C_2}^T Q_{C_2} + Q_{C_2}A_{C_2} + \bar{C}_{C_2}^T\bar{C}_{C_2} = 0 \end{cases}$$

($\bar{B}_{C_1}, \bar{C}_{C_1}, \bar{B}_{C_2}, \bar{C}_{C_2}$ denote local input and output matrices)

- Balance the Gramians (P_{C_1},Q_{C_1}), (P_{C_2},Q_{C_2}), then truncate
- Well known from controller reduction this can give very bad/conservative *global* approximations

A Good Heuristic

- Compute Gramians $\begin{cases} AP + PA^T + BB^T = 0 \\ A^TQ + QA + C^TC = 0 \end{cases}$

$$P = \begin{bmatrix} P_N & * & * \\ * & P_{C_1} & * \\ * & * & P_{C_2} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_N & * & * \\ * & Q_{C_1} & * \\ * & * & Q_{C_2} \end{bmatrix}$$

- Balance the sub-Gramians (P_{C_1}, Q_{C_1}), (P_{C_2}, Q_{C_2})
- A *local* coordinate transformation in C_1, C_2 with a *global* objective
- Often works very well. No stability and error bounds, however… How can we get bounds?

[Vandendorpe and van Dooren, 2007; S. and Murray, 2009]

[S. and Murray, 2009], but see also [Beck et al., 1996]

Structured (Generalized) Gramians

- Compute generalized Gramians (LMIs):

$$\begin{aligned} \min \operatorname{trace} P: \quad & AP + PA^T + BB^T \leq 0 \\ \min \operatorname{trace} Q: \quad & A^TQ + QA + C^TC \leq 0 \end{aligned} \qquad (1a)$$

- Enforce structure constraints ("structured Gramians")

$$P = \begin{bmatrix} P_N & 0 & 0 \\ 0 & P_{C_1} & 0 \\ 0 & 0 & P_{C_2} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_N & 0 & 0 \\ 0 & Q_{C_1} & 0 \\ 0 & 0 & Q_{C_2} \end{bmatrix} \qquad (1b)$$

- *Local* approximations, *global* error bound:

$$\|z - \hat{z}\|_2 \leq 2 \left(\sum_{i=r_1+1}^{n_1} \sigma_{C_1,i} + \sum_{i=r_2+1}^{n_2} \sigma_{C_2,i} \right) \|w\|_2, \quad \sigma_{C_k,i} = \sqrt{\lambda_i(P_{C_k} Q_{C_k})}$$

Key problem: When are the LMIs (1a)-(1b) feasible?

Key Problem: When Can We Ensure Existence of Structured Gramians?

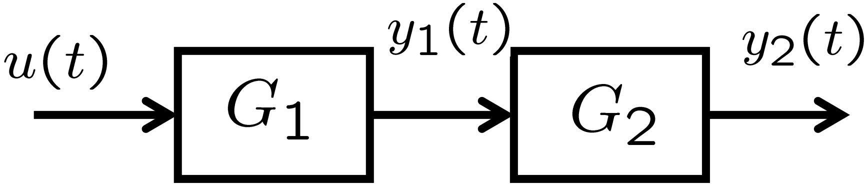
Existence can be proven for the following structures:

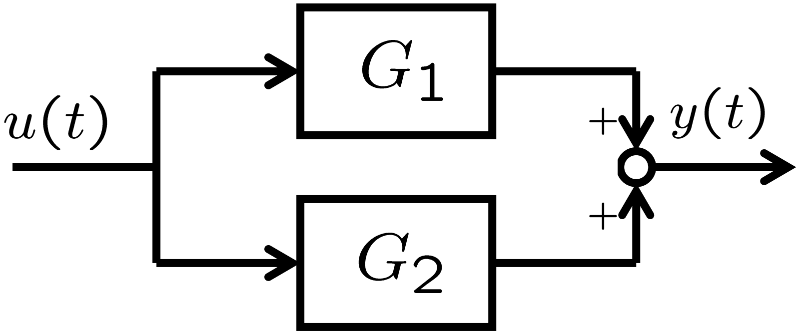
1. Series/cascade and parallel systems
2. Feedback of passive systems
3. Port-Hamiltonian systems
4. …

General feedback interconnections remain hard!

Existence is in fact equivalent to system being *strongly stable*, see [Beck, 2007].

1. Series/Cascade and Parallel Systems

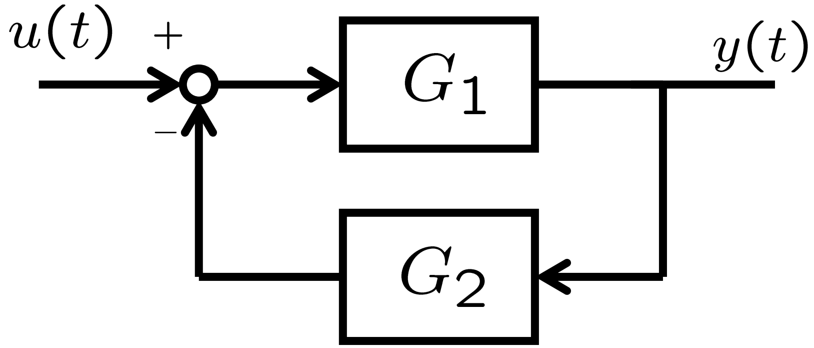




Assume $A = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}$, where A_1 and A_2 are stable. Then there exists a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} > 0$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} > 0$ satisfying $AP + PA^T + BB^T < 0$ and $A^TQ + QA + C^TC < 0$, respectively.

[Sturk et al., 2010]

2. Feedback of (Strictly) Passive Systems



The negative feedback interconnection of two strictly positive real systems with dimensions n_1, n_2 admits a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where $P_1, Q_1 \in R^{n_1 \times n_1}$ and $P_2, Q_2 \in R^{n_2 \times n_2}$.

[Sturk et al., 2010]

3. Port-Hamiltonian Systems

$$\dot{x} = (J - R)Hx + Bu, \qquad J = -J^T,\ R = R^T \geq 0$$
$$y = B^T H x, \qquad H = H^T > 0$$

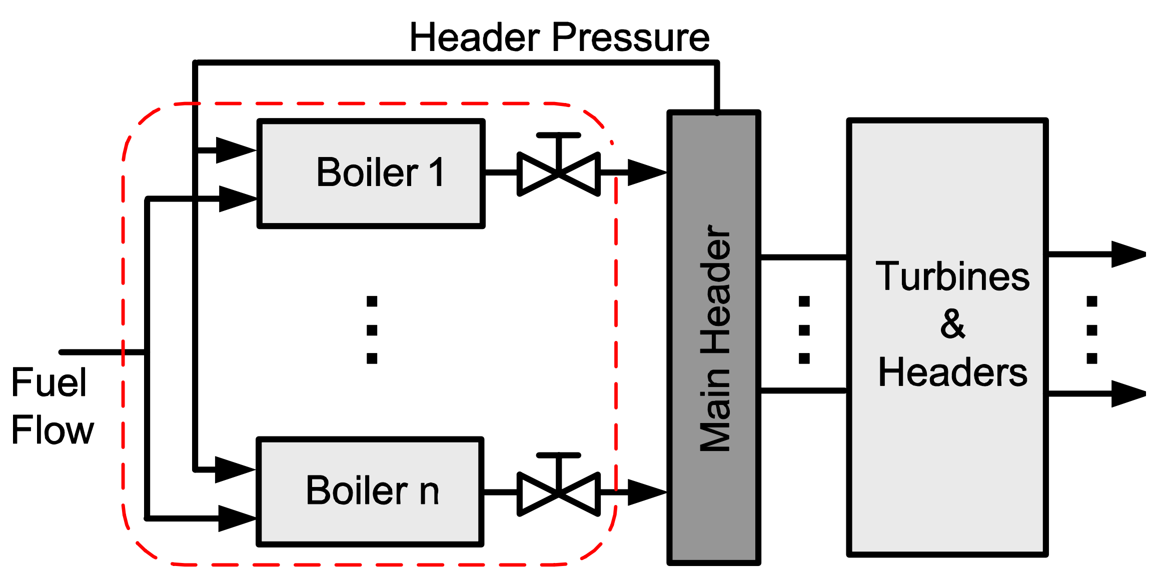
Suppose the Hamiltonian has the structure $H = H^T = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$, and that the input matrix B lies in the range of the dissipation matrix R (see note), i.e., $B = \mathcal{R}(R)$. Then the port-Hamiltonian system admits structured Gramians $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$.

Note: This is often ensured by the *fluctuation-dissipation theorem*.

[S., 2010]

Example of Structured Model Reduction: Boiler-Header System

- Multiple configurations of boilers
- MPC requires models for each configuration
- Combine identified boiler and header models to form the full system $\mathcal{F}_l(N, G)$



[Sturk et al., 2010]

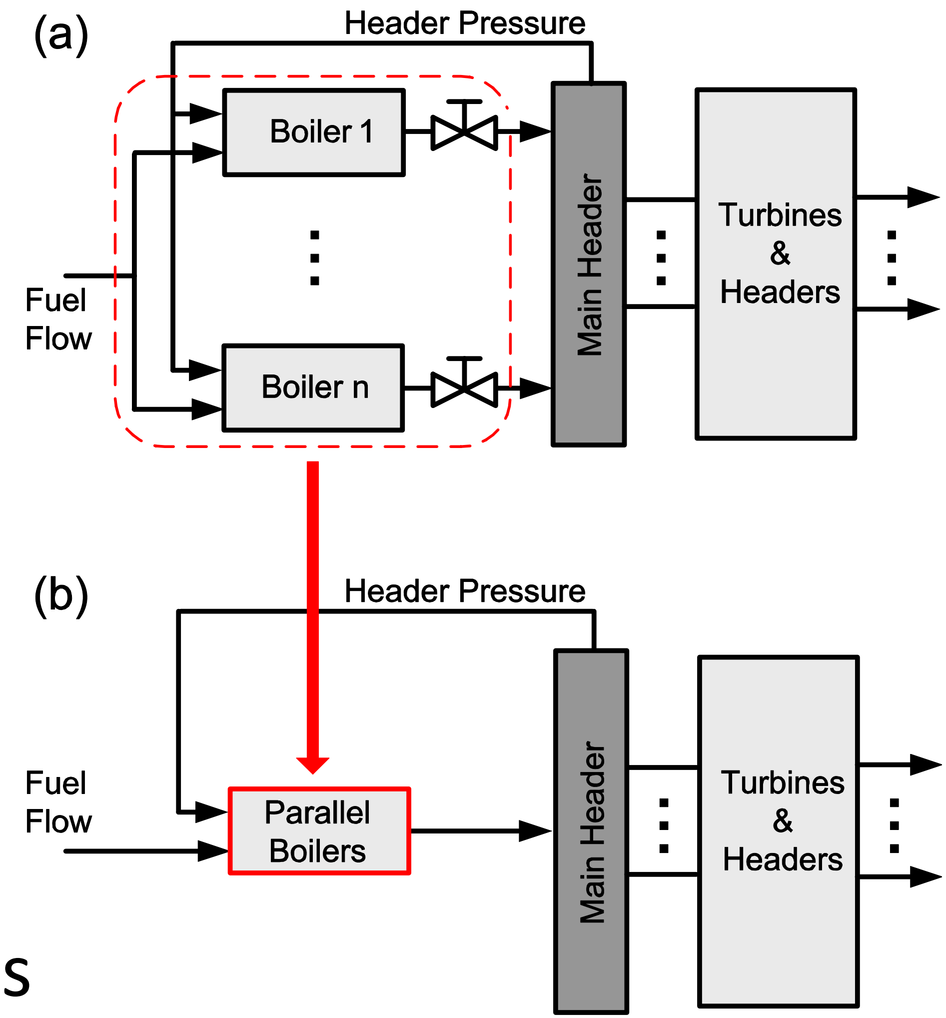
Problem Formulation

- Inputs and outputs

$$w = \begin{pmatrix} \mathsf{FF} \\ \mathsf{SD} \end{pmatrix}, \; z = \begin{pmatrix} \mathsf{HP} \\ \mathsf{SF} \end{pmatrix}$$

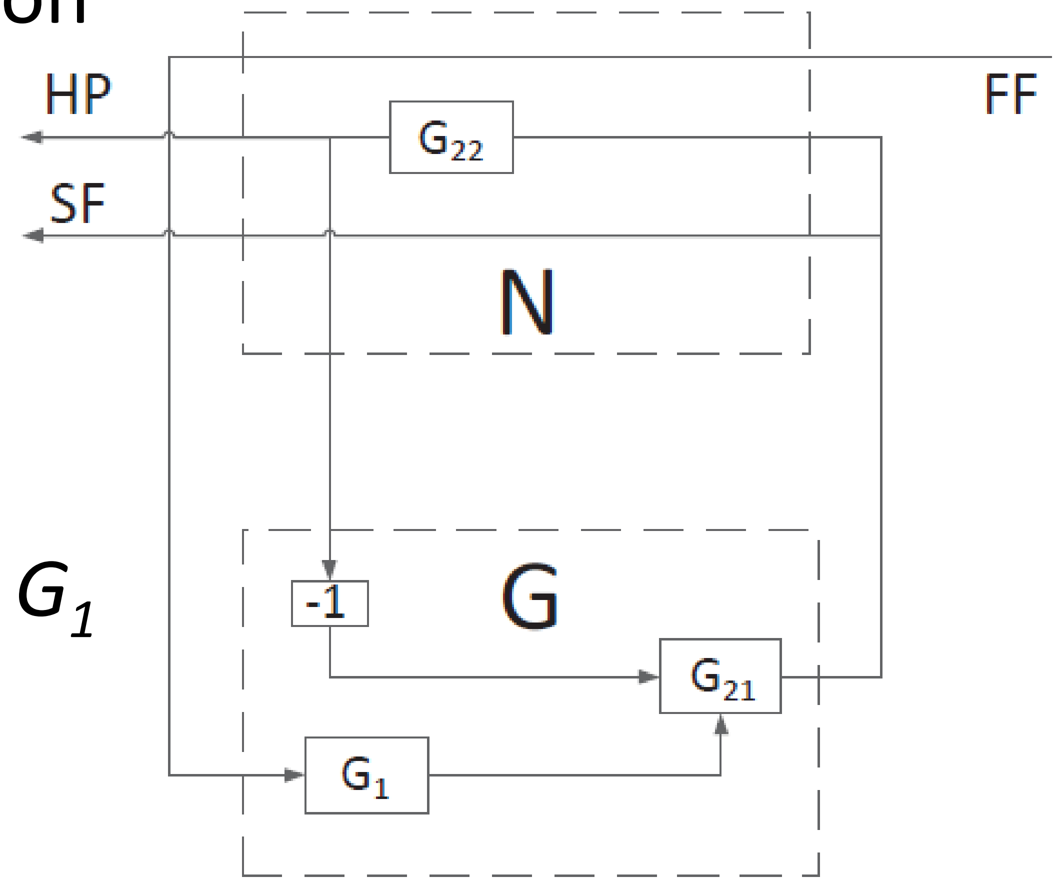
- Retain physical interpretation of subsystems and interconnection signals

- Header has 1 state, boilers have 2 states each

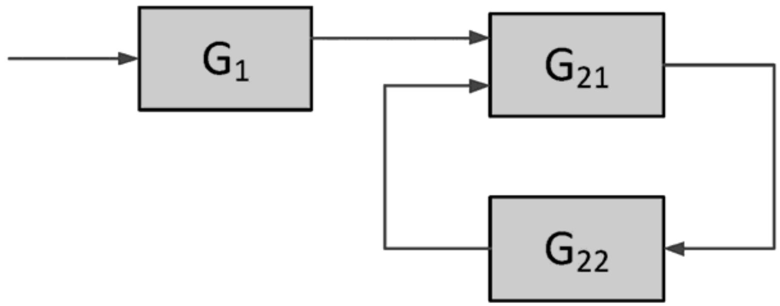


Model $\mathcal{F}_l(N, G)$

- *G* – Parallel connection of boilers
- *N* – Header
- Series connection of G_1 with the negative feedback loop of G_{21} and G_{22}

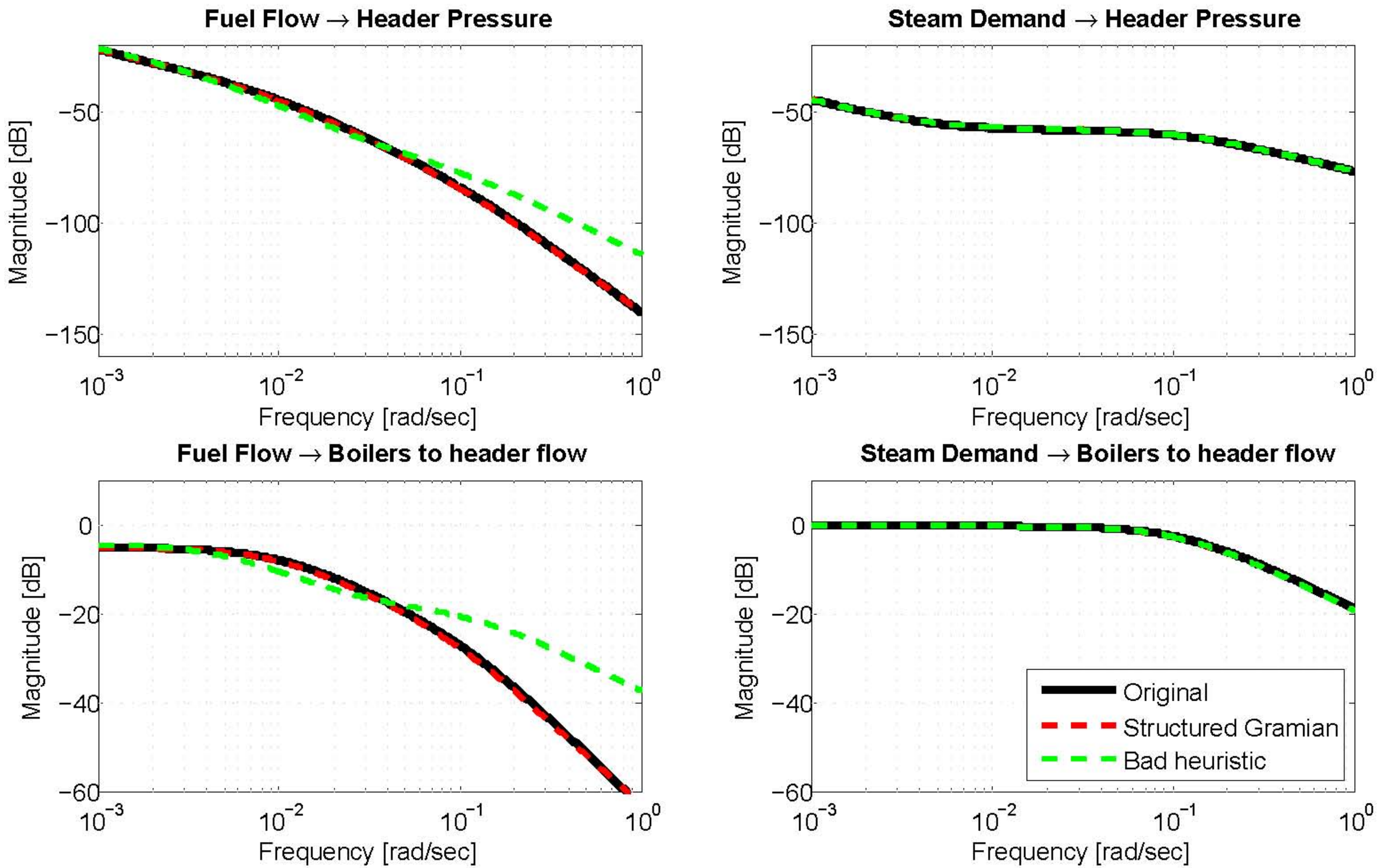


Existence of Structured Gramians (Combination of Cases 1. and 2.)



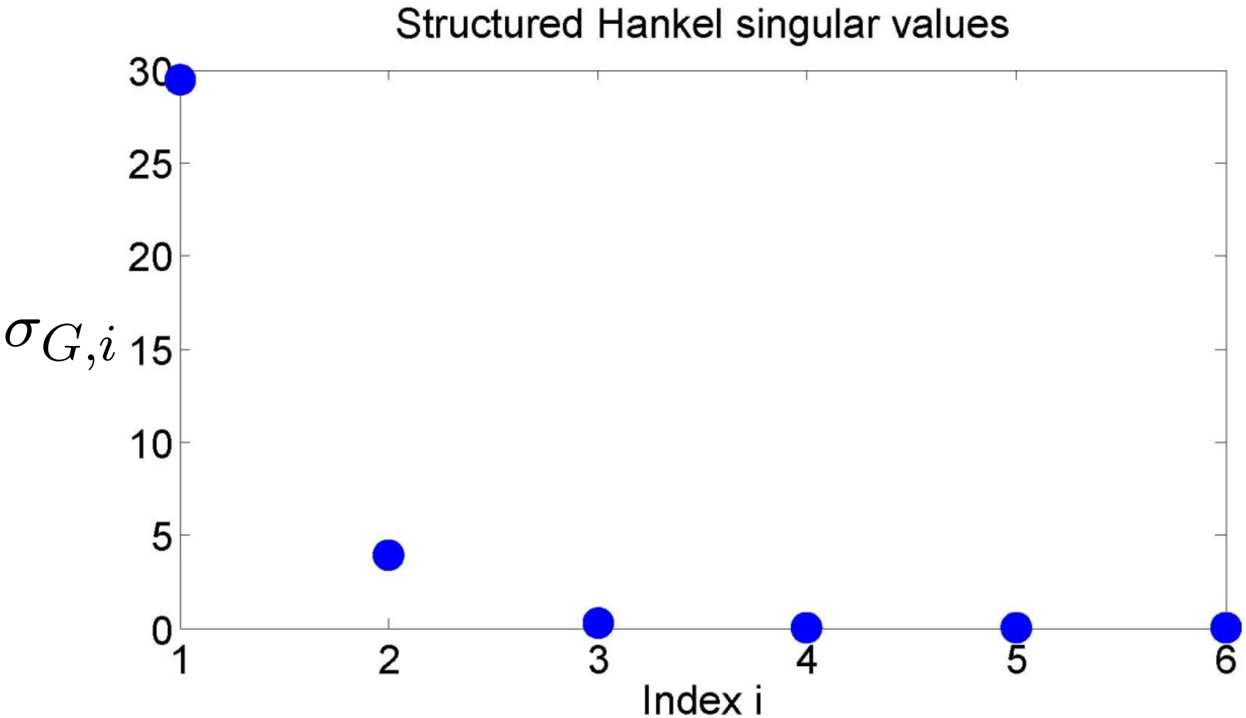
A stable LTI system of dimension n_1 connected in series with two strictly positive real systems of order n_{21} and n_{22} connected with negative feedback admits a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where $P_1, Q_1 \in R^{(n_1+n_{21})\times(n_1+n_{21})}$ and $P_2, Q_2 \in R^{n_{22}\times n_{22}}$.

Structured Gramians vs. Bad Heuristic



Bad heuristic does not capture the global dynamics!

Hankel Singular Values and Error Bound



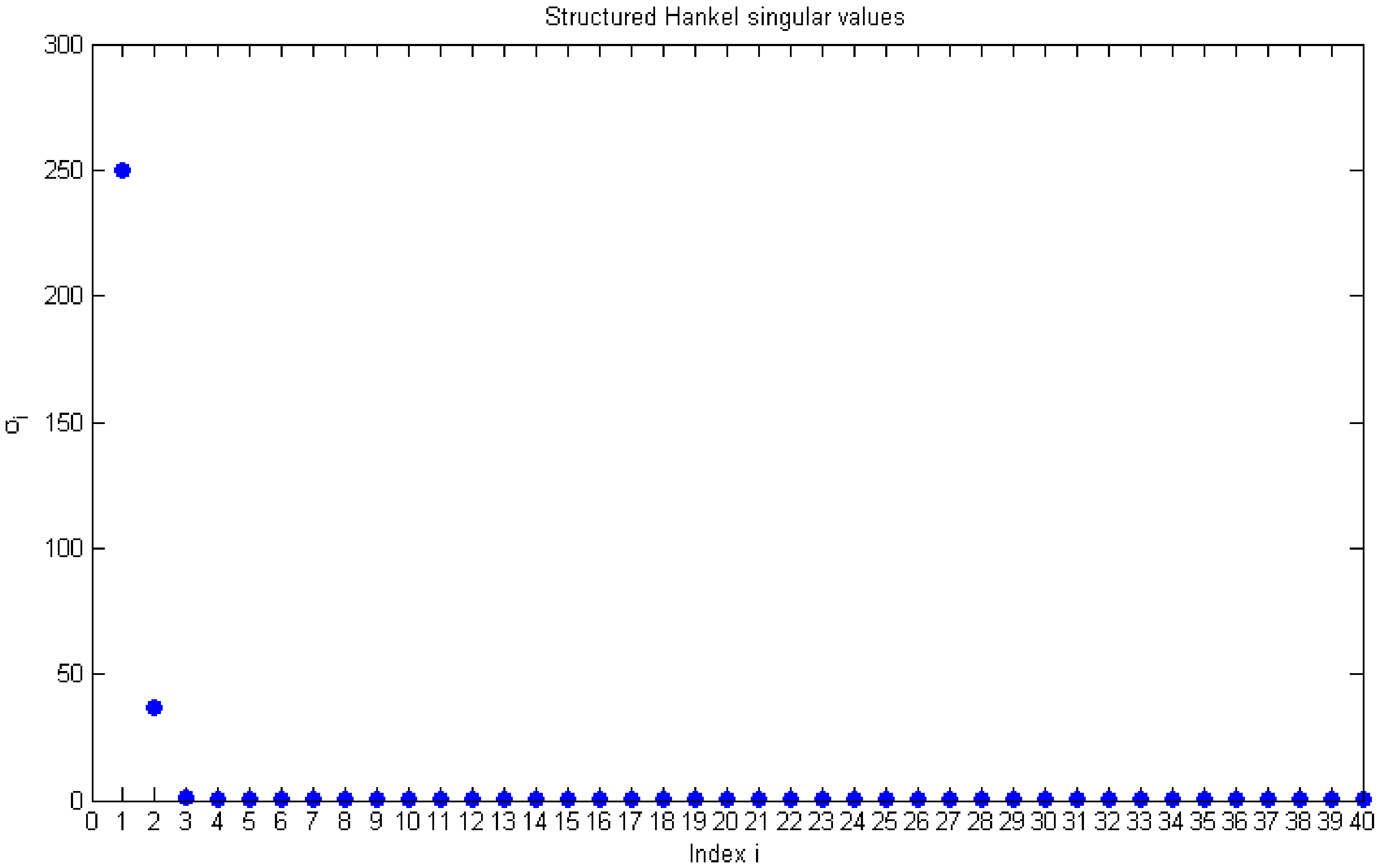
- Upper a priori error bound and a posteriori error:

$$\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\widehat{G})\|_\infty \leq 2\sum_{i=3}^{6} \sigma_{G,i} = 0.6274$$

$$\|\mathcal{F}_l(N,G) - \mathcal{F}_l(N,\widehat{G})\|_\infty = 0.3146$$

- Interpretation: Three parallel boilers have been lumped into one, while preserving global performance

Hankel Singular Values (20 Boilers)



Other Uses of Generalized Gramians

- Used to improve error bounds in balanced truncation (enforce multiplicity of Hankel singular values), see Hinrichsen and Pritchard, 1990
- To obtain *optimal* H_∞-model approximations, see Kavranoglu and Bettayeb, 1993
- Model reduction of uncertain systems, see Beck, Doyle and Glover, 1996
- …

- **Computational problems:** Computationally expensive to solve LMIs; $O(n^5)$-$O(n^6)$ operations.
- **Idea:** If we know there are structured solutions to LMIs, look for non-optimal solutions by solving many much smaller Lyapunov equations.

Key Problem: When Can We Ensure Existence of Structured Gramians?

Existence can be proven for the following structures:

- Series/cascade and parallel systems
- Feedback of passive systems
- Port-Hamiltonian systems

What to do when no structured Gramians can be found?

- Use the good heuristic without error bound
- Search for *extended structured Gramians*

Extended Gramians (in Discrete Time)

Lyapunov inequalities

$$P - APA^T - BB^T > 0, \quad P > 0, \tag{2}$$

$$Q - A^TQA - C^TC > 0, \quad Q > 0, \tag{3}$$

Extended Lyapunov inequalities

$$\begin{bmatrix} P & AF & B \\ F^TA^T & F + F^T - P & 0 \\ B^T & 0 & I \end{bmatrix} > 0, \tag{4}$$

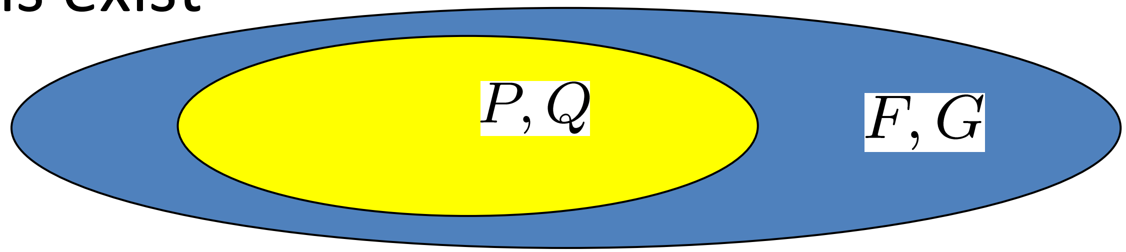
$$\begin{bmatrix} G + G^T - Q & GA & 0 \\ A^TG^T & Q & C^T \\ 0 & C & I \end{bmatrix} > 0, \tag{5}$$

Theorem The inequalities (2)–(3) have solutions $P = P^T, Q = Q^T$ if, and only if, the inequalities (4)–(5) have solutions $P = P^T, Q = Q^T$ and F, G.

[De Oliviera *et al*. 1999]

Extended Balanced Truncation

- Larger feasibility set $\Rightarrow$More likely structured extended Gramians exist



- Balance the extended Gramians and truncate *F*=*G*=Σ_e
- Extended Hankel singular values give error bound:

$$\Sigma_e = \text{diag}\{\sigma_{e,1}, \ldots, \sigma_{e,n}\}$$

$$\|z - \hat{z}\|_2 \leq \left(2 \sum_{i=r+1}^{n} \sigma_{e,i}\right) \|w\|_2$$

[S., ACC 2008, TAC 2010]

- Example in [S.,2010]: Structured extended balanced truncation works with 22% larger feedback gain.

Summary

- Avoid bad heuristic. Take surrounding into account
- Good heuristic often works well. Error bounds if structured generalized (or extended) Gramians exist
- Structured generalized Gramians do exist in many cases! (Series/cascade, parallel, passive, port-Hamiltonian systems)
- Method illustrated on boiler-header system