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Network Structure Preserving Model Reduction with Weak A Priori Structural Information

E. Yeung[†], J. Gonçalves[*], H. Sandberg[°], S. Warnick[†]

† Information and Decision Algorithms Laboratories
Brigham Young University, Provo, UT
http://idealabs.byu.edu

∗ Control Group
Department of Engineering
University of Cambridge, Cambridge, UK

○ Automatic Control Laboratory
KTH School of Electrical Engineering, Stockholm, SE

Abstract— **This paper extends a state projection method for structure preserving model reduction to situations where only a weaker notion of system structure is available. This weaker notion of structure, identifying the causal relationship between manifest variables of the system, is especially relevant is settings such as systems biology, where a clear partition of state variables into distinct subsystems may be unknown, or not even exist. The resulting technique, like similar approaches, does not provide theoretical performance guarantees, so an extensive computational study is conducted, and it is observed to work fairly well in practice. Moreover, conditions characterizing structurally minimal realizations and sufficient conditions characterizing edge loss resulting from the reduction process, are presented.**

I. INTRODUCTION

Simplified representations of dynamical systems play an important role in understanding, redesigning, and controlling their underlying dynamic phenomena. This can be especially true in situations where the underlying system is large and structured in a complex network of interconnections. Examples of such systems include biological systems, social networks, the internet, economic industrial organization, chemical reaction networks, power systems, transportation networks, epidemic transmission systems for infectious diseases, etc.

Reducing the complexity, while preserving the fidelity of a model, is the primary challenge for developing these simplified representations. The key question then becomes what measures of complexity and fidelity to use to meaningfully formulate the simplification problem.

For linear systems, the measure of system complexity has traditionally been system order. Since linearity already constrains the richness of behavior available to each differential

Please direct questions and comments to E. Yeung at `enochy@byu.net` or Sean Warnick at `sean.warnick@gmail.com`.

equation in the system, the number of differential equations necessary to describe a particular input-output relationship then becomes a meaningful measure of system complexity.

System order is the measure considered by realization theory, which characterizes state representations with input-output characteristics that exactly match a given transfer function. There we see that rational, proper transfer functions can always be exactly realized by systems of order greater or equal to a particular number, called the McMillan degree of the transfer function.

Model reduction, however, considers state representations with order less than the degree of a given transfer function [10]. In this case, the input-output characteristics of the state equations will only approximate the transfer function, and it is precisely the sense of approximation that traditionally defines the fidelity of the simplified representation. This notion of fidelity emphasizes the input-output *dynamics* of the system.

For large complex systems, though, the input-output dynamics alone may not be an adequate measure of fidelity. For these systems, *system structure* may be an equally important characteristic for understanding, redesigning, or controlling the system since, for example, only distributed interaction may be feasible. In this case, model reduction must not only approximate the dynamics of the system, but it also must reflect a priori knowledge of the system structure to the extent possible.

Unfortunately, traditional approaches to model reduction focus only on dynamic approximation and do not generally preserve interconnection structure. Nevertheless, recent work has considered this issue, and a number of structure-preserving methods have been suggested [18], [8], [16], [1], [13], [14], [15]. These methods, however, all assume that the structure of the complex system is known a priori in a very strong sense. In particular, a partition of the entire state space is assumed to be known, enabling a decomposition of the

system into the interconnection of a finite set of subsystems.

Assuming knowledge of such a partition may be reasonable in some cases, especially when considering engineered systems constructed as the explicit interconnection of physically and spatially separate "solid-state" subsystems. This could be the case, for example, when interconnecting distinct mechanical or electrical modules to form a composite system, or in a networked control system where the interconnection structure is fully known.

In other situations, however, assuming a priori knowledge of a partition over the entire state space of the complex system could be unreasonable. For example, when modeling a chemical reaction network of a biological system, one may not even be aware of many of the chemical species involved in intermediate reactions, much less their reaction pathways. As a result, it could be unreasonable to expect that one would know how to meaningfully partition all of the states of the system into distinct subsystems. In these situations, a weaker notion of a priori structural information is necessary to formulate a meaningful structure-preserving model reduction problem.

A weaker notion of system structure has been developed in the context of the network reconstruction of biochemical systems [6]. This notion of structure is characterized by a factorization of a system's transfer function, called the *dynamical structure function*, and it characterizes the causal relationship between manifest variables (i.e. system inputs and outputs) without imposing any particular structural form on the rest of the system.

This paper formulates the structure-preserving model reduction problem assuming only weaker a priori structural information, as characterized by the dynamical structure function of the system. A state projection technique similar to that in [15] is then applied, and it is found to deliver reduced models with good dynamic fidelity. Nevertheless, unlike situations where strong a priori structural information is available, the technique is shown to not necessarily preserve structure in the weaker sense, and conditions for structure preservation are then provided.

The outline of the paper is as follows. The next section compares and contrasts the strong and weak notions of system structure introduced here. Section III then formulates the structure preserving model reduction problem, and Section IV describes a procedure to approximate its solution. Section V illustrates the performance of a state-projection reduction technique through an extensive computational study, and Section VI characterizes situations when the technique will fail to preserve structure. Section VII then concludes the paper.

II. BACKGROUND: CHARACTERIZING STRUCTURE

In this work we assume that the transfer function, $G(s)$, of a specified LTI system is known. Note that this representation of the system completely characterizes its input-output dynamics, but describes nothing of its internal structure.

This section then compares and contrasts different ways of characterizing partial structure information of the system.
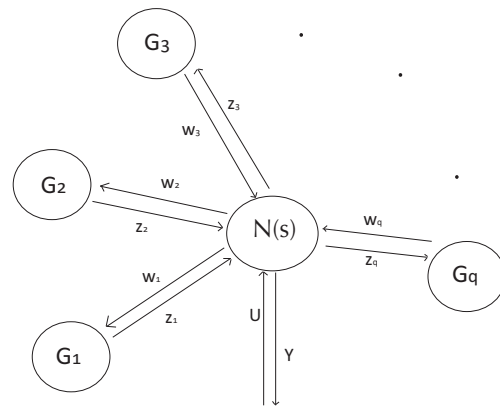

Fig. 1. Interconnection structure of a system with strong partial structure information. Notice that nodes represent systems, and the hub-spoke pattern emphasizes the existence of a global partition on all of the system states.

First, note that *complete structural information* is characterized by the state-space equations describing how the system actually computes its outputs given the input and initial conditions; we call this the *complete* or *computational structure* of the system. We next discuss *strong partial structure information* vs. *weak partial structure information* of the system and show that, although they may be described by graphical duals, the type of a priori information requirements they impose are very different.

A. Strong Partial Structure Information

One way to encode partial structure information about a system is to decompose the composite system into $q + 1$ distinct subsystems. One of these subsystems, $N(s)$, is a special module that characterizes the system structure and interconnects the other q subsystems; the others, $G_i(s)$, $i = 1, 2, ..., q$, are completely distinct and decoupled and do not interact except through $N(s)$. In this setting, replacing any subsystem $G_i(s)$ with another system, $\hat{G}_i(s)$, preserves the composite system structure, as encoded in $N(s)$, as long as the dimensions of the inputs and outputs of $\hat{G}_i(s)$ conform to the ports made available by removing $G_i(s)$. The mathematical representation of the composite system then becomes the lower fractional transformation of the system $N(s)$ and a block diagonal system $G(s)$, $\mathcal{F}_l(N, G)$, with $G_i(s)$ on the ith block of G.

The strong sense of structural understanding required by this definition of structure is a global partition on all the states of the complex, composite system resulting in the diagonalization of G. This may be the case, for example, in mechanical or electrical engineered systems where 1) complexity is achieved through the interconnection of a number of submodules, each with its own input-output characteristics, and 2) the interaction between subsystems is completely known (characterized by $N(s)$ and diagonal G), even if the internal structure (i.e. a state-space representation) of each subsystem is not specified. Other names for this strong type of partial structural information include *subsystem structure* or the *solid-state structure* of the system.

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In this setting, it is natural to describe the structure of the composite system graphically by associating each subsystem with a node of a graph. The input and output signals of these subsystems are then represented as edges of the graph, interconnecting the nodes commensurate with the interconnection information of $N(s)$. If only the input-output characteristics of $N(s)$ are known, then the graph takes on a hub-spoke pattern, with $N(s)$ as the hub and each subsystem $G_i(s)$ as a satellite system interconnected only to $N(s)$ (see Fig. 1). This pattern illustrates the strong information requirements emphasized by the diagonal structure of G.

B. Weak Partial Structure Information

A different way to encode partial structure information about a system is to characterize the open-loop causal dependence between manifest variables of the system. Note that unlike the transfer function, which communicates no structural information about the system, this characterization fixes the structure among measured outputs, and among system inputs and measured outputs, without specifying anything about the structure among the rest of the system states. In particular, unlike strong partial structure information, this notion of structure does not require knowledge or existence of a global partition of all system states in order to be meaningful.

The mathematical representation of the weak partial structure of a system employs a pair of matrix functions, similar to transfer functions, called the *dynamical structure function* of the system. In fact, it can be shown that the dynamical structure function of a system, $(Q(s), P(s))$, is a factorization of its transfer function, $G(s)$, given by $G(s) = (I - Q(s))^{-1}P(s)$. Note that the $G(s)$ here is the transfer function of the entire, interconnected system, whereas in the previous section G referred to a structured component of the complete system; the notational convention used in various sections of this paper follows that of each separate development in the literature, and any confusion should be clear from context.

To see how the dynamical structure function is derived, consider the system given by:

$$
\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\
y &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{D} \end{bmatrix} u
\end{aligned} \quad (1)
$$

where $\begin{bmatrix} z_1' & z_2' \end{bmatrix}' \in \mathbb{R}^n$, is the full state vector with $z_1 \in \mathbb{R}^p$, $y \in \mathbb{R}^p$ $(p < n)$ are the measured outputs, and $u \in \mathbb{R}^m$ is the control input. Without significant loss of generality, we assume $\begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$ has full row rank and $\bar{D} = 0$ (these assumptions simplify the exposition but do not restrict the results). We first consider the change of basis on the state variables yielding:

$$
\begin{aligned}
\begin{bmatrix} \dot{y} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
y &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}.
\end{aligned} \quad (2)
$$

Taking Laplace Transforms of the signals in (2), we find

$$
\begin{bmatrix} sY \\ sX \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \quad (3)
$$

Solving for X, gives

$$
X = (sI - A_{22})^{-1} A_{21} Y + (sI - A_{22})^{-1} B_2 U
$$

Substituting into the first equation of (3) then yields

$$
sY = WY + VU
$$

where $W = A_{11} + A_{12}(sI - A_{22})^{-1} A_{21}$ and $V = A_{12}(sI - A_{22})^{-1} B_2 + B_1$. Let D be a matrix with the diagonal term of W, i.e. $D = \text{diag}(W_{11}, W_{22}, ..., W_{pp})$. Then,

$$
(sI - D) Y = (W - D) Y + VU
$$

Note that $W - D$ is a matrix with zeros on its diagonal. We then have

$$
Y = QY + PU \quad (4)
$$

where

$$
Q = (sI - D)^{-1} (W - D) \quad (5)
$$

and

$$
P = (sI - D)^{-1} V \quad (6)
$$

The matrix Q is a matrix of transfer functions from Y_i to Y_j, $i \neq j$, or relating each measured signal to all other measured signals (note that Q is zero on the diagonal). Likewise, P is a matrix of transfer functions from each input to each output without depending on any additional measured state Y_i. Together, the pair $(Q(s), P(s))$ is the dynamical structure function for the system (1).

Note that when discussing the structure of a system, it is often convenient to distinguish between the *dynamical structure*, given by (Q, P), and its *network structure*, characterized by the Boolean structure of (Q, P). The Boolean structure of a matrix function Q is simply a matrix $\mathcal{B}(Q)$ with elements $\mathcal{B}(Q)_{ij} = 0$ if and only if $Q_{ij} = 0$, otherwise $\mathcal{B}(Q)_{ij} = 1$. The network structure of a system is thus given by $\mathcal{B}(Q, P)$.

This weak sense of structure may be particularly useful for describing the relationships between measured variables, for example, in biochemical or social systems. In these more "fluid" systems, not only might it be unreasonable to assume knowledge of an explicit non-trivial partition on all system states into distinct subsystems, but such a partition may not even exist. The dynamical structure of a system, however, always exists, regardless of whether the stronger sense of structure is meaningful or not.

In this setting, it is natural to describe the structure of the complex system graphically by associating each measured output and input with a node of the graph. Edges are then identified with each non-zero entry in (Q, P). In this way we observe that weak partial structure employs the dual graphical structure from that used by strong partial structure, with nodes representing signals and edges representing systems. For this reason, other names for this weak type of

partial structure include *signal structure*, *manifest structure*, or the *fluid structure* of the system. Note, however, that the internal states of systems represented by these edges are not necessarily distinct, thus truly weakening the structural information necessary to employ this representation.

III. PROBLEM FORMULATION

Realization problems transition to model reduction problems when the order of the representation is reduced beyond some threshold: systems with order greater than this threshold can exactly produce the desired behavior, while those with order less than the threshold must approximate the desired behavior. The context for model reduction is thus characterized by the minimal threshold defining this transition.

The McMillan degree of a transfer function is the minimal order necessary to realize it, and thus it is the relevant threshold for typical model reduction. Generating a particular structure, however, may demand a higher order than the McMillan degree, and so we distinguish between dynamic and structural minimality as follows.

Definition 3.1: Given a structure (Q, P) of the transfer function G, with $G = (I - Q)^{-1}P$, then a realization of G with order n is *dynamically minimal* if every realization of G has order $\bar{n} \geq n$, and it is *structurally minimal* if it generates (Q, P), in the sense of satisfying (2)-(6), and if every other realization that generates (Q, P) has order $\bar{n} \geq n$. We call the order of a dynamically minimal realization the *degree* of the system, and the order of a structurally minimal realization the *structural degree* of the system.

It is easy to see from the definition that a dynamically minimal realization that generates the desired structure will also be structurally minimal. It may be less obvious that a realization may be structurally minimal without being dynamically minimal, that is, that certain structures of given transfer functions require non-dynamically minimal realizations.

Although dynamic minimality is fully characterized by the controllability and observability of a realization, a comparable characterization for structural minimality remains an open problem (see [19] for a more thorough discussion). A partial characterization, however, is available through the following conditions.

Theorem 3.1: Given the system characterized by (2) with associated dynamical structure function (Q, P) and transfer function $G(s) = (I - Q(s))^{-1}P(s)$, if the realization (2) is dynamically minimal, then it is also structurally minimal.

Proof: We prove by contradiction. Suppose that the realization (2) is dynamically minimal. Suppose, for contradiction, that the realization (2) is not structurally minimal. Then there exists a lower order realization $(\tilde{A}, \tilde{B}, [I0])$ with identical dynamical structure function (Q, P). But since $G(s) = (I - Q(s))^{-1}P(s)$, this realization is also a lower order realization of $G(s)$. This contradicts that the realization (2) was dynamically minimal. ∎

This first result draws a connection between dynamical minimality and structural minimality. It is important to note that the converse of Theorem 3.1 is not valid, i.e. not all structurally minimal realizations are dynamically minimal. To see why this is true, consider any system (2) with only two measured states and non-trivial network structure Q that is not dynamically minimal. Clearly, such a system is structurally minimal since removal of a state would abolish the notion of network structure, i.e. Q would not even be defined. Our next result is a necessary condition that further clarifies the subtle relationship between structural and dynamic minimality.

Theorem 3.2: Given the system characterized by (2) with associated dynamical structure function (Q, P), define the transfer function of the "hidden" system

$$H = A_{12}(sI - A_{22})^{-1} \begin{bmatrix} A_{21} & B_2 \end{bmatrix}. \qquad (7)$$

If the realization (2) is structurally minimal, then H is dynamically minimal.

Proof: G can be expressed as the lower fractional transform $\mathcal{F}_l(M, H)$, where the "measured" system M is given as

$$M = \begin{bmatrix} A_{11} & B_1 & I \\ \hline I & 0 & 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 \end{bmatrix}. \qquad (8)$$

Suppose that H is not minimal. Then there exist matrices $(\tilde{A}_{22}, [\tilde{A}_{21}\ \tilde{B}_2], \tilde{A}_{12})$ such that

$$A_{12}(sI - A_{22})^{-1} \begin{bmatrix} A_{21} & B_2 \end{bmatrix} = \tilde{A}_{12}(sI - \tilde{A}_{22})^{-1} \begin{bmatrix} \tilde{A}_{21} & \tilde{B}_2 \end{bmatrix}$$

and forming a dynamically minimal realization of H. The corresponding realization of G, given by

$$A = \begin{bmatrix} A_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ \tilde{B}_2 \end{bmatrix} \quad C = \begin{bmatrix} I & 0 \end{bmatrix}$$

has dynamical structure function (Q, P), but lower order than the realization (2), forming a contradiction. ∎

The definition of structural minimality (Definition 3.1) provides the context for structure preserving model reduction, with structure characterized in the weak sense of the dynamical structure function, and Lemma 3.2 demonstrates that any structure preserving model reduction problem will consider systems with dynamically minimal "hidden" component H. The problem we want to solve then becomes:

Problem: Given a system G with dynamical structure function (Q, P) and structural degree n, and a non-negative integer $\tilde{n} < n$, find an approximate system $\tilde{G}$ with dynamical structure function $(\tilde{Q}, \tilde{P})$ and structural degree $\tilde{n}$ such that

1) $\mathcal{B}(\tilde{Q}, \tilde{P}) = \mathcal{B}(Q, P)$, and
2) $\tilde{G}(s) = \arg \min \|G - \tilde{G}\|_\infty$.

This research does not solve this problem. Note that even without the structural requirements, the optimization problem given by the second point is well known to be non-convex and hard. Nevertheless, we demonstrate that a state projection reduction method known to preserve structure in the strong sense (when it exists) can be adapted to perform reasonably

well with only weak structural information. Moreover, although it does not necessarily preserve structure in the weak sense, conditions for when it fails to do so are provided. The next sections describe the method, empirically explore its performance, and analyze its ability to preserve structure as described by the Boolean structure of (Q, P).

IV. STRUCTURED BALANCED TRUNCATION

Standard approaches to the unstructured version of the above problem, such as Hankel norm approximation or balanced truncation, are suboptimal in the H_∞ sense. Likewise, the approach taken here is also suboptimal, and it is an extension of balanced truncation adapted to accommodate system structure.

Typical balanced truncation is a state projection method that eliminates the least controllable and observable states in a system. It accomplishes this by first changing basis of the state space of a system realization so that the controllability and observability grammians of the system are equal and diagonal, and ordered (i.e. $X_c = Y_o = \Sigma \geq 0$, where Σ is diagonal with elements ordered from largest to smallest). The fact that such a transformation exists and is unique is well established, see for example [2], and the resulting diagonal elements of Σ are called the Hankel singular values of the system [9]. State projection is then used to truncate the states corresponding to the smallest Hankel singular values until the desired model order is realized.

Theoretical results that make balanced truncation useful include a guarantee that the reduced system will remain stable if the original system is stable, and the existence of both lower and upper a priori bounds on the H_∞ norm of the error between the reduced and complete system [11]. Nevertheless, system structure is not necessarily preserved in the reduced model [4], [3].

To preserve structure in the strong sense, balanced truncation was developed in [15], [8], [17], [12], [15]. A priori knowledge of system structure is reflected by the availability of a *structured realization* of the system, which is any realization of the system that conforms to the structure $\mathcal{F}_l(N, G)$ with G block diagonal. From this realization, the controllability gramian, X_c, and observability gramian, Y_o, are computed.

Note that even though the realization is structured, X_c and Y_o are generally full matrices with no apparent structure. If they were block diagonal, however–conformal with the subsystem structure of the system–then the transformation needed to balance them would likewise be block diagonal. Essentially, each subsystem could then be balanced independently without affecting the others.

This observation motivates the search for a block diagonal transformation that balances each subsystem (along the diagonal blocks of X_c and Y_o)–even though the off-diagonal blocks are non-zero with no particular relationship or structure. It can be shown that such a *structured transformation* always exists and is unique, generating a local change of basis within each subsystem such that the resulting controllability gramian and observability gramian

have a particular structure, with diagonal blocks that are 1) equal, 2) diagonal, 3) positive semidefinite, and 4) with entries ordered from largest to smallest within the block–while the off diagonal blocks have no particular relationship or structure.

If the off-diagonal blocks were zero, then the diagonal entries would, in fact, be the Hankel singular values of each subsystem, considered independently and "open loop" or disconnected from the rest of the system. When the off-diagonal blocks are non zero, however, these diagonal elements of the gramians do not correspond to the Hankel singular values of the subsystem. Nevertheless, they do carry a similar interpretation, that of identifying relatively more (or less) controllable and observable subsystem states–except as seen from the inputs and outputs of the *entire* interconnected system (i.e. external to the entire interconnected system) rather than from the inputs and outputs of the particular subsystem. As a result, these diagonal entries of the gramians are called the *structured Hankel singular values* of the interconnected system with strong a priori structure.

Subsystem structure preserving model reduction uses state projection to truncate states from the various subsystems that correspond to small structured Hankel singular values until the desired order is achieved for the complete, interconnected system. The method will preserve the subsystem structure of the system. Nevertheless, most of the other guarantees for typical balanced truncation are lost: the reduced system may be unstable, even when the the complete interconnected system is stable, and although an upper bound of the H_∞ norm of the error is provided, in [15] it is not an a priori bound that can be computed before the reduced system is constructed, although the recent work [14] does offer a priori bounds using an idea from frequency-based model reduction.

In this work we adapt this procedure to relax the need for strong, a priori partial structure information. The idea is that every system of the form (1) has a trivial subsystem structure found by changing basis as in (2) and considering the subsystem of measured states, y, and the subsystem of hidden states, x. The subsystems M and H, defined in (8) and (7), conform to this strong subsystem structure, with M as the "hub system" (denoted N in the structure preserving model reduction literature), and H as the only "satellite system" (denoted G in the structure preserving model reduction literature, where here it would only have a single block, G_1, since no other structure on H is assumed to necessarily exist).

Although this trivial two-block subsystem structure is not interesting in the strong structural sense, it is semi-universal, in that it applies to a very wide class of systems of the form (1), and it may still exhibit any conceivable weak structure among the system's manifest variables. Moreover, Lemma 3 from [5] demonstrates that dynamical structure is invariant to coordinate transformations on the hidden states, thus enabling the use of structured transformations without affecting manifest structure of the system.

As a result, the structure preserving model reduction method described above can be applied in situations where

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only a distinction between manifest and latent variables is known a priori, and the structure to be preserved is described by the dynamical structure function of the system. As there are few theoretical guarantees for this procedure–even when strong a priori structural information is available–the next section presents the results of an extensive computational study exploring the performance of the procedure for a variety of systems.

V. Results from a Computational Study

Since the state projection technique for reduction suggested in the previous section does not necessarily preserve stability of a system, nor provide bounds on the dynamic fidelity of the reduced model, nor guarantee structure preservation in the weak sense, we engaged a computational study to explore the performance of the technique. The results of the study are displayed in Table 1. The study generated approximately 50 stable random examples from 27 different classes of systems. Each of these systems were randomly chosen to be of order between 10 and 40 total states. The 27 classes were formed by considering three major parameters for each system: number of outputs, connectivity level of the network structure, and the number of states removed by truncation (i.e. "chop"). For each of these parameters, three categories were defined as follows:

1) Small number of outputs: $.05n \le p \le .20n$
2) Medium number of outputs: $.21n \le p \le .35n$
3) Large number of outputs: $.36n \le p \le .50n$
4) Sparse connectivity: $p \le$ number of edges in $Q \le p + \frac{p^2 - 2p}{3}$
5) Mid-level connectivity: $p + \frac{p^2 - 2p}{3} \le$ number of edges in $Q \le p + \frac{2(p^2 - 2p)}{3}$
6) Dense connectivity: $p + \frac{2(p^2 - 2p)}{3} \le$ number of edges in $Q \le (p^2 - p)$
7) Low chop: number of hidden states removed by truncation $\le \frac{n-p}{3}$
8) Medium chop: $\frac{n-p}{3} \le$ number of hidden states removed by truncation $\le \frac{2(n-p)}{3}$
9) High chop: $\frac{2(n-p)}{3} \le$ number of hidden states removed by truncation $\le (n - p)$

For each example, a number of performance characteristics were measured. First, the stability of the reduced system was checked. Then, the ability of the procedure to preserve structure in the weak sense was measured by counting the number of edges lost in the network structure of the reduced system. Finally, the dynamic fidelity was measured by the relative or "scaled" size of the error, $\|G - \tilde{G}\|_\infty / \|G\|_\infty$. The results of these experiments are presented in Table 1. The average value of each measure over the roughly 50 random experiments performed for each category are reported. From these results, we draw the following conclusions.

First, note that although instability is possible, it was rare to lose stability through truncation. Although this may have been affected by our use of diagonally dominate A matrices in the system realization when generating stable random systems, it is still encouraging to note that even without

stability guarantees the procedure generally delivers stable reduced order models.

Second, we observe that although weak structure was preserved most of the time, an average as high as 15% of the edges in Q can be lost by the procedure. This motivates the characterization of edge loss discussed in the next section.

Finally, the dynamic fidelity of the procedure appears to be excellent. Although one category reports an average error as high as 30% of the norm of G, generally the error is well below $1-3\%$. Note that ϵ refers to values smaller than 10^{-3}. The singular value plot and structured singular values for a typical example are shown in Figures 2 and 3 respectively. In this example, $n = 40$, $p = 3$, and 27 hidden states were truncated.

VI. Sufficient Conditions for Edge Loss

It is easy to understand how an edge may be lost through the reduction process. If, for example, the edge connecting two measured outputs y_1 and y_2 is realized through a hidden state x_n, so that $y_1 \longrightarrow x_n \longrightarrow y_2$, and that hidden state is eliminated in the truncation process, then we would expect the edge not to be present in the network structure of the reduced model. This observation lead us to the following characterization of edge loss.

Definition 6.1: A realization of the form (2) is *hidden balanced* if its controllability and observability gramians (with block structure conformal to the partition in (2)),

$$X_c = \begin{bmatrix} X_{c11} & X_{c12} \\ X_{c21} & X_{c22} \end{bmatrix}, \quad Y_o = \begin{bmatrix} Y_{o11} & Y_{o12} \\ Y_{o21} & Y_{o22} \end{bmatrix},$$

satisfy $X_{c22} = Y_{o22} = \Sigma$, were $\Sigma > 0$ is diagonal.

Theorem 6.1: Given a system G with network structure $\mathcal{B}(Q, P)$ and a hidden balanced realization given by

$$\begin{bmatrix} \dot{y} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} y \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u$$
$$y = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x_1 \\ x_2 \end{bmatrix}, \quad (9)$$

where x_1 and x_2 partition the hidden states of the system. Let $a^T = [a_2^T \ a_3^T]$ be the ith row of $[A_{12} \ A_{13}]$, $b = [b_2^T \ b_3^T]^T$ be the jth column of $[A_{21}^T \ A_{31}^T]^T$, and T_2 be the unitary transformation constructing the Schur form of A_{22}. Consider the system $\tilde{G}$ with network structure $\mathcal{B}(\tilde{Q}, \tilde{P})$ and hidden balanced realization given by

$$\begin{bmatrix} \dot{y} \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ x_1 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} y \\ x_1 \end{bmatrix}, \quad (10)$$

obtained by truncating x_2 from the system (9). Then an edge in the ij^{th} position is lost as a result of truncation, meaning $[\mathcal{B}(Q)]_{ij} = 1$ and $[\mathcal{B}(\tilde{Q})]_{ij} = 0$, if $[A_{11}]_{ij} = 0$, $[\mathcal{B}(Q)]_{ij} = 1$, and the matrix $T_2 a_2 b_2^T T_2^*$ is lower triangular with zero on the diagonals.

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Q Size	Connectivity	Chop	Unstable	n	p	States Chopped	Edges in Q	Edges Lost	Scaled Error
Small	Sparse	Low	4%	19.8	2.3	3.7	2.6	0	ϵ
Small	Sparse	Med	6%	22.4	2.66	10.2	3.3	0	.0609
Small	Sparse	High	0%	20.0	2.5	12.2	2.8	0	0.0119
Small	Mid	Low	0%	20.5	2.8	3.7	5.2	0	ϵ
Small	Mid	Med	0%	22.7	2.7	10.8	4.1	0	ϵ
Small	Mid	High	0%	21.6	3.1	12.9	5.9	.5	.0184
Small	Dense	Low	0%	29.1	4.2	4.9	14.6	0	ϵ
Small	Dense	Med	2%	27.1	4.1	12.6	13.6	0	ϵ
Small	Dense	High	4%	29.4	4.2	18.1	14.8	0	.0060
Med	Sparse	Low	0%	22.2	6.3	3.7	21.2	0.1	.0014
Med	Sparse	Med	0%	22.5	6.7	3.8	13.2	0	ϵ
Med	Sparse	High	0%	25.01	6.7	13.1	11.8	1.9	.0140
Med	Mid	Low	2%	25.7	3.7	4.1	11.1	.2	.0075
Med	Mid	Med	0%	22.5	6.4	9.0	23.3	0	.0106
Med	Mid	High	0%	23.3	6.6	11.6	23.4	.5	.0028
Med	Dense	Low	0%	25.9	7.8	3.6	60.2	0	.0303
Med	Dense	Med	0%	28.1	8.0	10.2	62.2	0	.0959
Med	Dense	High	0%	24.7	7.2	12.4	49.7	0	.3147
High	Sparse	Low	0%	22.1	10.0	2.9	23.8	0	ϵ
High	Sparse	Med	0%	23.5	10.0	7.3	19.5	0	.0081
High	Sparse	High	0%	22.72	10.0	8.28	25.92	0	.0646
High	Mid	Low	0%	20.2	9.0	2.6	49.6	0	ϵ
High	Mid	Med	0%	20.6	8.9	6.5	44.5	2.0	.0013
High	Mid	High	0%	25.9	11.8	9.8	83.54	.14	.3001
High	Dense	Low	0%	25.5	11.3	2.8	127.6	.04	.0519
High	Dense	Med	0%	21.8	9.4	7.3	93.2	.12	.2256
High	Dense	High	0%	24.4	11.0	9.1	120.9	.12	.1700

TABLE I

NUMERICAL RESULTS

Proof: We begin by noting that $[\mathcal{B}(\tilde{Q})]_{ij} = 0$ if and only if $A_{11} = 0$ and $[A_{12}(sI - A_{22})^{-1}A_{21}]_{ij} = 0$, since $\tilde{W} = A_{11} + [A_{12}(sI - A_{22})^{-1}A_{21}]$ and $\tilde{Q} = (sI - \tilde{D})^{-1}(\tilde{W} - \tilde{D})$, where $\tilde{D} = diag(\tilde{W}_{11}, \tilde{W}_{22}, ..., \tilde{W}_{pp})$.

Although $(sI - A_{22})^{-1}$ may not have any particular structure in general, $(sI - T_2 A_{22} T_2^*)^{-1}$ will be upper triangular since $T_2 A_{22} T_2^*$ is the Schur form of A_{22}, and the inverse of an upper triangular matrix is upper triangular. Noting that $[A_{12}(sI - A_{22})^{-1}A_{21}] = [A_{12}T_2^*(sI - T_2 A_{22} T_2^*)^{-1}T_2 A_{21}]$, we see that the ij^{th} entry will be identically 0 if and only if $a_2^T T_2^*(sI - T_2 A_{22} T_2^*)^{-1}T_2 b_2 = 0$ for almost all $s \in \mathbb{C}$.

The triangular structure of $(sI - T_2 A_{22} T_2^*)^{-1}$ can now be exploited to force certain combinations of the entries of the vectors $a_2^T T_2^*$ and $T_2 b_2$ to be zero in order to keep the product $a_2^T T_2^*(sI - T_2 A_{22} T_2^*)^{-1}T_2 b_2 = 0$ for almost all s. These zeroing combinations are efficiently expressed in the requirement that the outer product $T_2 a_2 b_2^T T_2^*$ be lower triangular, with zeros on the diagonal. These conditions are only sufficient, however, since for particular cases there may be other ways to achieve $a_2^T T_2^*(sI - T_2 A_{22} T_2^*)^{-1}T_2 b_2 = 0$ for almost all s without zeroing out pairwise products of certain entries of a and b. ∎

This theorem characterizes sufficient alignment conditions for an edge to be lost through the truncation process discussed in this paper. Although as a practical matter it may be easy to compute $\mathcal{B}(\tilde{Q}, \tilde{P})$ and directly compare it with $\mathcal{B}(Q, P)$ to detect edge loss, these conditions help distinguish situations where truncation is guaranteed to simplify the dynamical structure of the reduced order model.

This loss of edges can actually be valuable in situations where a structured system is identified from noisy data. In these situations, network reconstruction algorithms will find fully connected structures fit the data as well or better than sparse structures. As a result, there is a tendency to over estimate structure, and many approaches actively work to compensate for this tendency by explicitly rewarding sparsity of the estimate [7].

The alignment conditions provided in Theroem 6.1 offer the hope for a new approach to this problem by using model reduction to both simplify the dynamic expression of a model and to purposefully simplify the structure estimate, eliminating only those edges that result from noise. Future work will explore these issues.

VII. CONCLUSION

This paper extended a state projection technique for structure preserving model reduction to situations where only a weak notion of system structure is available. Strong partial structure information and weak partial structure information were defined and compared, and characterizations for structurally minimal realizations, in the weak sense, were given. An extensive computational study of the reduction process was conducted, demonstrating that it typically performs well in spite of the lack of theoretical guarantees on stability or performance bounds. Nevertheless, the technique does not always strictly preserve structure in the weak sense, and sometimes a reduced model may lose edges compared with the network structure of the original system. Sufficient conditions for edge loss were then provided, demonstrating
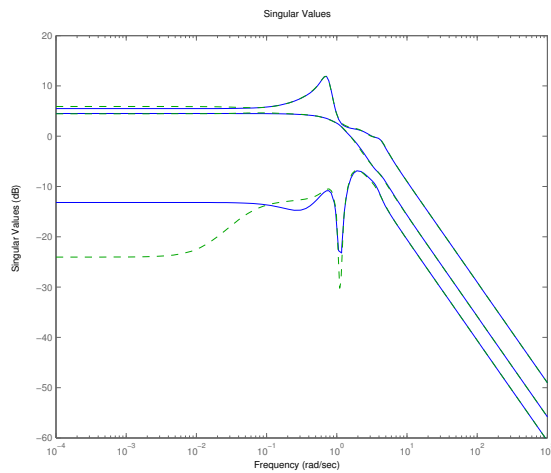
Fig. 2. σ-Plot of G(solid) and G_r(dashed)

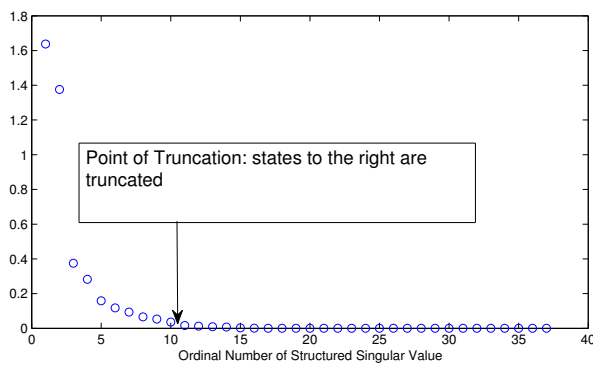Point of Truncation: states to the right are truncated

Fig. 3. Structured Hankel Values for the Hidden Balanced Realization

certain alignment properties that reveal how this approach to model reduction may contribute in the future to network reconstruction from noisy data.

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