A Robust Control-Design Method Using Bode's Ideal Transfer Function

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***Abstract*— We propose a method for designing loop-shaping controllers using Bode's ideal transfer function. Bode's ideal transfer function is introduced using fractional calculus. The ideal loop transfer function is approximated using the first generation CRONE approximation, and then implemented by means of H_∞-optimization followed by closed-loop controller order reduction of the resulting controller. The design method is confirmed to be powerful and robust by simulating on a flexible transmission system.**

I. Introduction

Fractional calculus based on generalized integral-differential operators is known in mathematical analysis since the nineteenth century. In the last decades, there has been many fractional calculus applications also in control engineering problems. The range of applications covers both controller design and synthesis [1]–[3] and system modeling and identification [4]. Design of fractional-order controllers using Bode's ideal transfer function [5] is one of the applications [6]. Such controllers provide theoretically infinite gain margin. Most of these research studies result only in fractional-order controllers [7]. Although several methods for finite-dimensional approximation of fractional operators are available in the literature [4], [8], the arising finite-dimensional filters are of high-order. Therefore, they are not suitable for real-time implementation on control boards with limited computational capabilities and small memory units.

In this paper, we propose a systematic approach for designing low-order controllers using Bode's ideal loop shape on a given frequency interval and closed-loop controller reduction [9]. Like classical loop-shaping methods, the proposed control-design method uses specifications like phase-margin and cross-over frequency, and translates them into a reference model consisting of an ideal open-loop system and then designs the shaping filter using an H_∞-optimization method. At last, the method uses a closed-loop controller order reduction to obtain a low-order realizable controller. The advantage of this control-design method is that using a simple systematic approach and with a small number of tuning parameters, we get a low-order highly robust controller. Unfortunately, in some cases there are complex design specifications that cannot be met using Bode's ideal transfer function. Also the class of admissible plants is restricted to stable system.

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As a numerical example, we design a loop-shaping controller for a flexible transmission system. Controlling flexible transmission systems with low damping ratio in the presence of large load variation is a difficult task and a fixed high performance controller designed for one load condition may lead to instability for another load condition. The system we consider is well studied as a benchmark problem on robust digital control in [10]. Since there are already several good controllers [3], [11]–[14] for the considered system, the purpose here is not to obtain a better controller but to illustrate the method on a challenging well-known system. We design a highly robust controller for this benchmark problem that can satisfy *most* of the design specifications [10]. The controller is interesting to consider since its construction is rather straightforward and combines elements from both classical and modern control theory.

The paper is organized as follows. In Section II, Bode's ideal transfer function and its basic properties are described. Section III deals with the control-design method using the H_∞-optimization procedure and the closed-loop controller order reduction method. The numerical results regarding the flexible transmission system and the control design requirements are given in Section IV. Finally, Section V gives a general discussion on the results and the conclusion.

II. Bode's ideal transfer function

First, we need to introduce the notion of fractional derivative and integration.

A. Fractional Derivative

The integral-differential operator, denoted by ${}_aD_t^\alpha$, is a notation for taking both fractional derivative and fractional integral in a single expression

$$
{}_aD_t^\alpha = \begin{cases} d^\alpha/dt^\alpha, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases}
$$

There are several different definitions for fractional derivative and integral [15]. One of the most commonly used definitions is the Caputo definition

$$
{}_0D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m = \lfloor \alpha \rfloor, \\ \frac{d^m}{dt^m} f(t), & \alpha = m \in \mathbb{N}. \end{cases}
$$

The Laplace transform of the Caputo derivative is

$$
\mathcal{L}\{{}_0D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(t)|_{t=0}.
$$

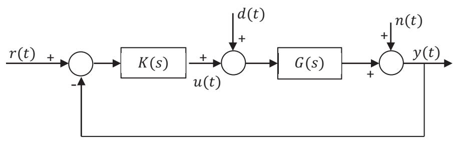

Fig. 1. The feedback interconnection of the controller $K(s)$ and the plant $G(s)$.

It has been shown that a system with transfer function $F(s^\alpha)$ is asymptotically stable if the following condition is satisfied

$$|\angle \text{pole}(F(s^\alpha))| > \alpha\pi/2,$$

for $0 < \alpha < 2$, where pole(.) denotes the poles of the transfer function and $\angle$ represents the angel of a complex number [16]. For $\alpha = 1$, this criteria becomes equivalent to the well known stability criteria of "all poles should be located in the left-half of the complex-plane (LHP)".

There are several methods for approximating a stable fractional transfer function with a rational integer-order filter. One of the most famous methods is the CRONE approximation [15] which we are going to use in this paper.

B. CRONE Approximation for Fractional Filters

In this subsection, we introduce the *first generation* CRONE approximation for fractional filters. This method helps us to approximate a stable fractional order transfer function with a rational integer-order filter in a given frequency band.

Assume that we want to approximate $L(s) = 1/s^\alpha$ for some $\alpha > 0$ with an integer-order filter of degree N in the frequency interval $[\omega_l, \omega_h]$. The CRONE method results in

$$L(s) = \frac{1}{s^\alpha} \approx \tilde{L}(s) = C_0 \prod_{n=1}^{N} \frac{1 + s/\omega_{zn}}{1 + s/\omega_{pn}},$$

where

$$\omega_{p1} = \omega_l \sqrt{\eta}, \quad \omega_{zn} = \omega_{pn}\xi, \quad \omega_{p,n+1} = \omega_{zn}\eta,$$

and

$$\xi = (\omega_h/\omega_l)^{\alpha/N}, \; \eta = (\omega_h/\omega_l)^{(1-\alpha)/N}, \; C_0 = (1/\omega_l)^\alpha .$$

If we want the approximation error $|\tilde{L}(j\omega) - L(j\omega)|$ to be less than some real number $E > 0$ in the frequency range of interest $[\omega_l, \omega_h]$, the order of approximation N should satisfy

$$N = \left\lceil \frac{\log(\omega_h/\omega_l)}{\frac{E}{10}\left(\frac{1}{1+\alpha} + \frac{1}{2-\alpha}\right)} \right\rceil,$$

for $1 < \alpha < 2$ [15]. A good property of the this approximation method is that $\tilde{L}(s)$ is always stable.

C. Ideal Transfer Function

Consider the feedback interconnection of the controller $K(s)$ and the plant $G(s)$ in Fig. 1. Bode, in his study on design of feedback amplifiers [5], has suggested an ideal shape of the open-loop transfer function $GK(s)$ of the form

$$L(s) = (\omega_c/s)^\gamma \tag{1}$$

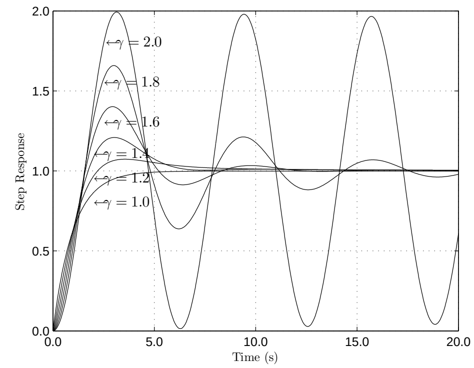

Fig. 2. Time response characteristics of the closed-loop system $T(s)$ for $\omega_c = 1$ and different γ.

for some $\gamma \in \mathbb{R}$, where ω_c is the gain cross-over frequency, that is, $|L(j\omega_c)| = 1$. The parameter γ determines both the slope of the magnitude curve on a log-log scale and the phase margin of the system, and may assume integer as well non-integer values. In fact, the transfer function $L(s)$ is a fractional-order transfer function for non-integer $\gamma \in \mathbb{R}$. The amplitude curve is a straight line of constant slope -20γdB/dec, and the phase curve is a horizontal line at $-\gamma\pi/2$rad. The Nyquist curve consists, simply, of a straight line through the origin. Let us now consider the unit feedback system with Bode's ideal transfer function $L(s)$ inserted in the forward path. This choice of $L(s)$ gives a closed-loop system with the desirable property of being insensitive to gain changes (Gain Margin $= \infty$). The variations of the gain change the cross-over frequency ω_c but the phase margin of the system remains $\pi(1 - \gamma/2)$rad, independent of the gain.

Next, we study the step-response of the closed-loop system consist of the fractional order transfer function $L(s)$ given in (1) with unitary feedback

$$T(s) = L(s)/(1 + L(s)) = 1/(1 + (s/\omega_c)^\gamma).$$

Thus, the step-response would be

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\omega_c^\gamma}{s(\omega_c^\gamma + s^\gamma)}\right\} = 1 - \sum_{n=0}^{\infty} \frac{[-(-\omega_c t)^\gamma]^n}{\Gamma(1 + \gamma n)},$$

which results in

$$y(\infty) = \lim_{t\to\infty} y(t) = 1, \quad y(0^+) = \lim_{t\to 0^+} y(t) = 0.$$

Fig. 2 shows the overshoot M_p, peak-time T_p, rise-time T_r, and settling-time T_s of the step-response of the fractional-order transfer function $T(s)$ for $\omega_c = 1$ and different γ. Using these results and Fig. 2, it is relatively easy to find suitable values ω_c and γ based on design specifications.

III. Control Design Method

The control-design method consists of two separate parts: (i) loop-shaping using H_∞-optimization, and (ii) closed-loop controller order reduction. We discuss these two parts in the subsequent subsections.

A. Control Design: Loop-Shaping Using H_∞-Optimization

In the design procedure using Bode's ideal transfer function loop shaping, first, we must find the optimal loop-gain $L(s)$ based on the design specifications and fix both γ and ω_c in (1). This can be done using the results of Subsection II-C (Fig. 2). The next step is to approximate this fractional order transfer function with a rational integer order transfer function $\tilde{L}(s)$ using the CRONE method introduced in Subsection II-B. Finally, we should make the loop-gain $GK(s)$ as close as possible to this approximation $\tilde{L}(s)$ in the frequency range of interest $[\omega_l, \omega_h]$. This part of the design procedure can be done using a weighted H_∞-norm minimization

$$\arg\min_{K(s)} \|W_o(s)[GK(s) - \tilde{L}(s)]W_i(s)\|_\infty, \tag{2}$$

where the search domain should be on the set of stable controllers $K(s)$. This optimization problem is easy to solve [17]. The weight functions $W_o(s)$ and $W_i(s)$ are selected based on the frequency range of interest $[\omega_l, \omega_h]$. This frequency range is usually dictated by the cross-over frequency ω_c and the open-loop characteristics of the plant-to-be-controlled. Furthermore, if the open-loop plant model varies under different working conditions, we may rewrite the H_∞-optimization in (2) as

$$\arg\min_{K(s)} \left\| \begin{array}{c} W_o(s)[G_1K(s) - \tilde{L}(s)]W_i(s) \\ \vdots \\ W_o(s)[G_nK(s) - \tilde{L}(s)]W_i(s) \end{array} \right\|_\infty, \tag{3}$$

where the transfer functions $G_\ell(s)$ for $\ell = 1, \ldots, n$ represent the plant-to-be-controlled under different working conditions. It should be noted that, if $\omega_c \in [\omega_l, \omega_h]$, both the controller and the open-loop plant are stable, and the minimization error in (2) is small enough, then the closed-loop system should be stable with $K(s)$. This is true because $GK(s)$ is close to $\tilde{L}(s)$, and using the properties of Bode's ideal transfer function and the CRONE approximation, we know that the Nyquist diagram of $\tilde{L}(s)$, and therefore, the Nyquist diagram of $GK(s)$ does not encircle -1.

B. Control Design: Controller Reduction Method

The controller $K(s)$, introduced in Subsection III-A is a high-order controller and it is not suitable for real-time implementation with low memory usage and low computational power. In this section, we use the model-order-reduction method introduced in [9] to a get a low-order controller with satisfactory closed-loop performance.

Consider the closed-loop interconnection of the system and the controller in Fig. 1 with the closed-loop state-space representation

$$\begin{bmatrix} \dot{x} \\ \dot{x}_k \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ x_k \end{bmatrix} + \tilde{B} \begin{bmatrix} r \\ d \\ n \end{bmatrix}, \quad \begin{bmatrix} y \\ u \end{bmatrix} = \tilde{C} \begin{bmatrix} x \\ x_k \end{bmatrix},$$

where

$$\tilde{A} = \begin{bmatrix} A & BC_k \\ -B_kC & A_k \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 0 & B & 0 \\ B_k & 0 & -B_k \end{bmatrix},$$

and

$$\tilde{C} = \begin{bmatrix} C & 0 \\ 0 & C_k \end{bmatrix}.$$

In the above state-space realization, the matrices $[A, B, C]$ and the matrices $[A_k, B_k, C_k]$ are respectively the realization matrices of the plant transfer function and the controller transfer function

$$G(s) \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \; K(s) \triangleq \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right].$$

Now, we can compute reachability and observability Gramians $\tilde{P}$ and $\tilde{Q}$ using the following Lyapunov equations

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0, \qquad \tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{C}^T\tilde{C} = 0. \tag{4}$$

and extract the weighted reachability and observability Gramians for the controller $K(s)$ using

$$P = \begin{bmatrix} 0_{n_k \times n} & I_{n_k \times n_k} \end{bmatrix} \tilde{P} \begin{bmatrix} 0_{n_k \times n} & I_{n_k \times n_k} \end{bmatrix}^T, \tag{5}$$

$$Q = \begin{bmatrix} 0_{n_k \times n} & I_{n_k \times n_k} \end{bmatrix} \tilde{Q} \begin{bmatrix} 0_{n_k \times n} & I_{n_k \times n_k} \end{bmatrix}^T, \tag{6}$$

where n_k is the order of the controller $K(s)$ and n is the order of the plant. Based on these weighted Gramians, one can balance the coordinates of the controller and use the singular perturbation method to find the reduced-order controller [18]. The order of the reduced-controller r is obtained by incrementally increasing r until the design specifications are fulfilled. If the open-loop plant model varies under different working conditions, we can use this controller reduction method on the hardest plant-to-be-controlled with the design specifications. In this controller reduction method, we can choose to use a different set of inputs and outputs for the closed-loop system based on the design specifications.

IV. Numerical Example

First, we need to introduce the flexible transmission system. The discrete-time model of flexible transmission system under different load conditions is given in [10] but in this paper, we use the continuous-time model introduced in [19].

A. Flexible Transmission System

Fig. 3 shows the schematic diagram of the flexible transmission system. This system consists of three pulleys connected to each other by two elastic belts. The first pulley is driven by a DC motor whose position is controlled by local feedback. Since the dynamics of the electrical actuator is much faster than that of the mechanical parts of the systems, this part can be neglected in modeling and analysis of the system. The objective is to control the position of the third pulley which may be loaded by small disks. The system is characterized by two low-damped vibration modes, subject to large variations in the presence of different loads. Fig. 4 gives the magnitude Bode diagram of the system under three different loadings: no-load, half-load, and full-load. The transfer function for these three different cases can be found in (7)-(9) based on modeling done in [19]. $G_{NL}(s)$, $G_{HL}(s)$, and $G_{FL}(s)$ stand for the transfer functions of the flexible transmission system under no-load, half-load, and

$$G_{NL}(s) = \frac{11.56}{7.140\times10^{-5}s^4 + 1.225\times10^{-4}s^3 + 8.607\times10^{-2}s^2 + 7.395\times10^{-2}s + 11.56} \tag{7}$$

$$G_{HL}(s) = \frac{11.56}{2.159\times10^{-4}s^4 + 2.458\times10^{-4}s^3 + 2.017\times10^{-1}s^2 + 7.395\times10^{-2}s + 11.56} \tag{8}$$

$$G_{FL}(s) = \frac{11.56}{3.604\times10^{-4}s^4 + 3.690\times10^{-4}s^3 + 3.173\times10^{-1}s^2 + 7.395\times10^{-2}s + 11.56} \tag{9}$$

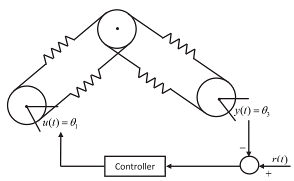

Fig. 3. The Schematic of the flexible transmission system.

full-load conditions, respectively. According to [19], the goal is to obtain a low-order controller that achieves the following specifications:

- S_1: Rise time of less than 1.0 sec for all loads,
- S_2: Overshoot of less than 10% for all loads,
- S_3: Disturbance attenuation in low frequency band from 0 to 0.4πrad/s for all loads; i.e., small output sensitivity function $S_o(s)$ for the frequency range $[0, 0.4\pi]$ where the output sensitivity function $S_o(s)$ is the transfer function between the output disturbance $n(t)$ and the plant output $y(t)$,
- S_4: A maximum value of less than 6dB of the output sensitivity function for all loads,
- S_5: A maximum value of less than 10dB of the transfer function $T_{nu}(s)$ between the output disturbance $n(t)$ and the controller output $u(t)$,
- S_6: Perfect rejection of constant disturbances (using integral action),
- S_7: Rejection of the step-output disturbance filtered by $1/A(s)$ within 1.2 sec (for 90% rejection of the measured peak value) for all loads where for each load $A(s)$ is the denominator of the transfer function of the corresponding load,
- S_8: Delay margin of at least 70ms.

B. Control Design: Loop-Shaping Using H_∞-Optimization

As the first step, we have to pick good γ and ω_c for Bode's ideal transfer function to get good results from the design procedure. Based on the results of Subsection II-C, it can be seen through Fig. 2 that $\gamma = 1.15$ results in $M_p = 4.87\%$ and $\omega_c = 2.22$rad/s gives a rise-time about 0.74s. From Fig. 4, we know that there can be a resonance peak in the system at approximately 35rad/s, and we should choose ω_h larger than 35rad/s. We fix $\omega_h = 1\times10^3$rad/s. On the other hand, the lower we choose ω_l, the better step reference tracking and low frequency disturbance rejection the closed-loop system will have. But if we make ω_l too small, the controller becomes very high-order (because of the high approximation degree of the fractional order Bode's ideal transfer function). We pick $\omega_l = 1\times10^{-2}$rad/s.

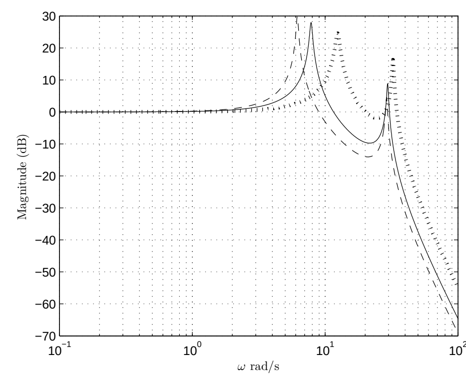

Fig. 4. Magnitude Bode diagram of the system for three different loads: full-load (dash), half-load (solid), no-load (dot).

Now, we can use CRONE method to find the approximation $\tilde{L}(s)$. We do this approximation with $N = 15$.

As the next step of the control-design procedure, we should set up an H_∞-optimization for loop-shaping. In our modeling of the flexible transmission system, we have three different transfer functions, thus the controller design problem becomes

$$\arg\min_{K(s)} \left\| \begin{bmatrix} G_{NL}K(s) - \tilde{L}(s) \\ G_{HL}K(s) - \tilde{L}(s) \\ G_{FL}K(s) - \tilde{L}(s) \end{bmatrix} W(s) \right\|_\infty . \tag{10}$$

We can simplify the two different weight functions $W_i(s)$ and $W_o(s)$ in (3) into one weight function $W(s)$, because the underlying systems are single-input single-output dynamical systems. Using a simple weight function

$$W(s) = \frac{100}{\omega_l} \frac{s}{(1+s/\omega_l)(1+s/\omega_h)},$$

the resulting controller $K(s)$ is of degree 30. Furthermore, this controller does not have an integrator, and therefore, it does not reject the step disturbances perfectly. To overcome this problem, we multiply the designed controller $K(s)$ by a simple lag-filter

$$\hat{K}(s) = K_{lag}(s)K(s) = \left(\frac{s+\omega_l}{s}\right)K(s).$$

The lag-filter $K_{lag}(s)$ helps the controller to satisfy S_6 and it does not change the loop-gain in the frequency range of interest $[\omega_l, \omega_h]$ much. The controller $\hat{K}(s)$ is of degree 31 which may not be easy to implement. In the next subsection, we use the method given in Subsection III-B to find reduced-order controller.

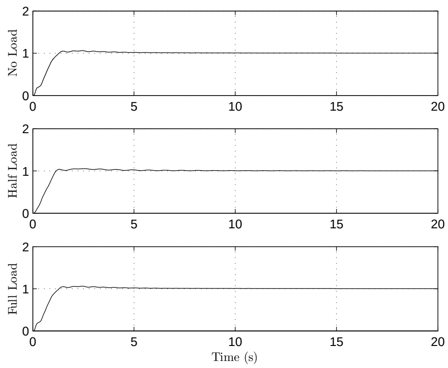

Fig. 5. The closed-loop step-response of the flexible transmission system under three different load condition with the designed controller $\hat{K}(s)$ (S_1 and S_2).

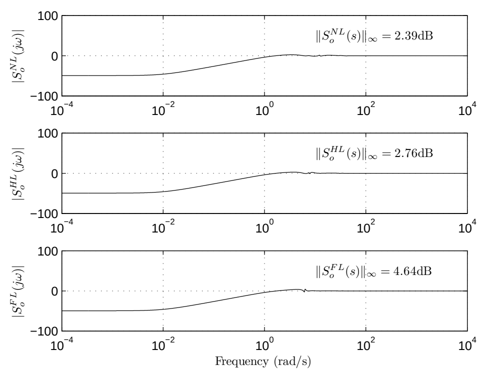

Fig. 6. Output sensitivity function $S_o(j\omega)$ versus frequency for three different plants with the designed controller $\hat{K}(s)$ (S_3 and S_4).

Fig. 5 shows the step responses of the closed-loop systems with $\hat{K}(s)$ as the controller. As it can be seen, the step-response satisfies the design specifications S_1 and S_2. Fig. 6 and Fig. 7 show the output sensitivity function $S_o(j\omega)$ and the transfer function $T_{nu}(j\omega)$ versus frequency. Both of these transfer functions satisfy the design criteria S_3, S_4, and S_5 under different load conditions.

C. Control Design: Controller Order Reduction

In this subsection, we use the method given in Subsection III-B to find reduced-order controller. Since we want to satisfy the controller design criteria S_1-S_8 after the the model reduction, we choose $\begin{bmatrix} r & d & u \end{bmatrix}^T$ as the input for the closed-loop system and $\begin{bmatrix} y & u \end{bmatrix}^T$ as the output of the closed-loop system (because each pair of these input-output transfer functions corresponds to at least one of the control design specifications). It is worth mentioning that we choose $G_{FL}(s)$ for the controller order reduction, because it is the hardest plant-to-be-controlled with the design specifications S_1-S_5. The reduced-order controller K_r takes the form (11), when we choose the order $r = 5$. As mentioned earlier in Subsection III-B, the order $r = 5$ is obtained by incrementally increasing r until the specifications are fulfilled. Fig. 8 shows the step-responses of the closed-loop

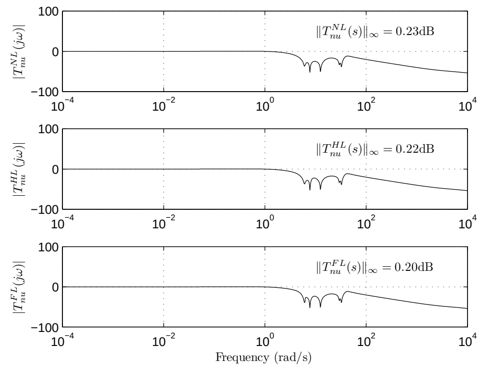

Fig. 7. Transfer function $T_{nu}(j\omega)$ versus frequency for three different plants with the designed controller $\hat{K}(s)$ (S_5).

TABLE I

DESIGN SPECIFICATION OF THE PROPOSED CONTROL-DESIGN METHOD.

	No Load	Half Load	Full Load
T_r(<= 1.2s)	0.99 s	0.96 s	0.86 s
M_p(<= 10%)	9.2 %	9.9 %	8.1 %
$\|S_o^{NL}(s)\|_\infty$(<= 6.0dB)	5.09 dB	5.90 dB	5.52 dB
$\|T_{nu}^{HL}(s)\|_\infty$(<= 10.0dB)	0.00 dB	0.00 dB	0.00 dB
Delay Margin(>= 70ms)	629 ms	604 ms	576 ms
T_s	2.9 s	4.7 s	7.1 s

system with the reduced controller under three different load conditions. Clearly, the design specifications S_1 and S_2 are satisfied for all loading situations. Fig. 9 and Fig. 10 show the output sensitivity function $S_o(j\omega)$ and the transfer function $T_{nu}(j\omega)$ versus frequency with the reduced controller for different load conditions which shows that the reduced-order controller satisfies the design criteria S_3, S_4, and S_5 too. The integral action must be preserved, since the singular perturbation method preserve the static gain; i.e., $|K_r(0)| = |\hat{K}(0)|$. Therefore, the reduced-order controller $K_r(s)$ also has integral action and fulfills S_6. The detailed result of the numerical example is given in Table I. The only specification that cannot be satisfied is S_7, because this specification cannot be translated to Bode's ideal transfer function. This specification cannot be satisfied even with the approximation of Bode's ideal transfer function $\tilde{L}(s)$ in (1) for any γ and ω_c, so this is an inherent limitation of the method.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, we have presented an strategy for designing low-order robust controllers using Bode's ideal transfer function loop shaping. The proposed method is based on the H_∞-minimization between the desired transfer function, produced by an approximation of fractional-order open-loop transfer function in the frequency range of interest, and the transfer function of the loop-gain with the robust controller. The design procedure is systematic and simple. The tuning parameters, for this control-design method, are: (i) γ (phase-margin) and ω_c (cross-over frequency) which can be determined from design specifications on rise-time,

$$K_r(s) = -1.38 \times 10^{-1} \frac{(s - 5.91 \times 10^4)(s^2 - 3.17 \times 10^{-1}s + 36.43)(s^2 + 1.31 \times 10^{-2}s + 73.78)}{s(s + 5.97 \times 10^4)(s + 4.638)(s^2 + 2.05s + 41.24)} \tag{11}$$

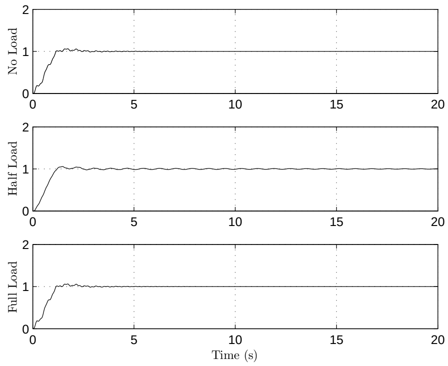

Fig. 8. The closed-loop step-response of the flexible transmission system under three different load condition with the reduced controller $K_r(s)$ (S_1 and S_2).

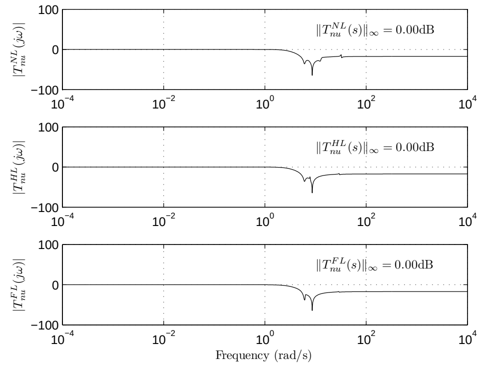

Fig. 10. Transfer function $T_{nu}(j\omega)$ versus frequency for three different plants with the reduced controller $K_r(s)$ (S_5).

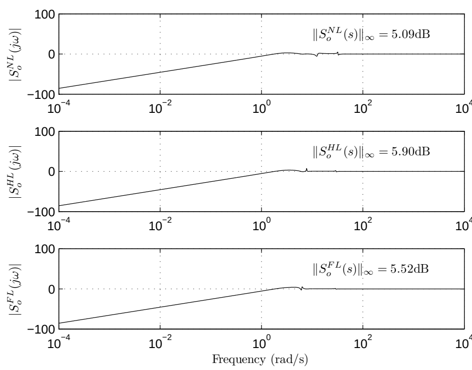

Fig. 9. Output sensitivity function $S_o(j\omega)$ versus frequency for three different plants with the reduced controller $K_r(s)$ (S_3 and S_4).

percentage-overshoot, and settling-time, (ii) ω_h and ω_l that are dictated by ω_c and open-loop characteristics of the plant-to-be-controlled, (iii) N (approximation degree in the CRONE method) which is characterized mainly by ω_h/ω_l, and (iv) $W(s)$ (weighting filter for H_∞-minimization) which is a simple filter only based on ω_h, ω_l. This control-design method uses classical loop-shaping specifications like phase-margin and the cross-over frequency, translates them into a reference model consisting of an ideal open-loop system and then designs the shaping filter using an H_∞-optimization method. Finally, we use a closed-loop controller order reduction to get a low-order realizable controller.

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