On Optimal System Design for Feedback Control over Noisy Channels

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***Abstract*— We study a closed-loop multivariable control system with sensor feedback transmitted over a discrete noisy channel. For this problem, we propose a joint design of the state measurement quantization, protection against channel errors, and control. The proposed algorithm leads to a practically feasible design of time-varying non-uniform encoding and control. Numerical results demonstrate the performance obtained by employing the proposed iterative optimization algorithm.**

I. INTRODUCTION

In recent years, the demand for sharing resources efficiently in large networked systems has been continuously increasing. However, in many situations, there is a challenging conflict between the amount of transmitted data and the response time. Limits imposed on available signaling bandwidth from communication channels can severely restrict the closed-loop performance and even destabilize the system. Networked control based on limited sensor and actuator information has therefore attracted considerable attention during the past decade. Up till now, results on control with limited information have often been derived based on rather simple system models. Generalizations to more complex scenarios, e.g., systems with process noise, measurement noise, and transmission errors, are challenging research topics. Some recent work in feedback control over noisy channels includes [1]–[3]. Further, the research interests have mainly been devoted to the stability properties of various control strategies. The counterpart of the optimal encoder design is rarely exploited in the literature. In most of the early work, the quantizers are typically considered as fixed system components, e.g., [3]–[5]. A closely related problem to the problem of the encoder design is the estimation of a Markov source. Some new results can be found in e.g., [6], [7]. In particular, [6] has studied the sequential vector quantization scheme, while [7] was interested in the design of the optimal finite memory encoder–decoders.

The main contribution of the present paper is a practical synthesis technique for joint optimization of the quantization and error protection for state observations over a bandlimited and noisy channel. The control system in Fig. 1 is considered. Suboptimal strategies for various encoder information settings are studied, as indicated by the dashed lines in the figure. In particular we extend our previous work [8] as follows: our new results are valid for multi-dimensional linear systems ([8] investigates the scalar case); the system model now includes measurement noise; we explicitly investigate the influence of side-information (SI) available at the encoder; and, we present new results on the validity of the certainty equivalence (CE) principle in control over noisy channels.

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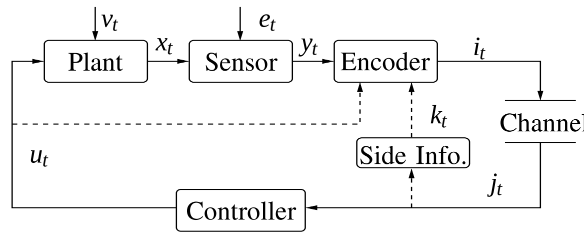

Fig. 1. The system description.

The paper is organized as follows. In Sec. II we define the control system with encoder, controller, and communication channel. The problem statement, which concerns a linear quadratic (LQ) objective over a finite horizon, is presented in Sec. III. The joint encoder–controller design and the training procedure are described in Sec. IV and Sec. V. Finally, we present the numerical experiments in Sec. VI and the conclusions in the last section.

II. PRELIMINARIES

Consider the control system with a communication channel depicted in Fig. 1. Sensor data are encoded and transmitted over an unreliable communication channel. Control commands are then derived based on the received data. (In [9] we motivated this scenario based on applications that involve closed-loop control using measurements from distributed wireless sensors.) In this section, we describe this system in detail. Let $\mathbf{x}_a^b = \{x_a, \ldots, x_b\}$ denote the evolution of a discrete-time multidimensional signal x_t from $t = a$ to $t = b$. The linear plant is governed by the following equations

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + v_t, \\ y_t &= Cx_t + e_t, \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, y_t \in \mathbb{R}^l$ are the state, the control, and the measurement, respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, are known. The variable $v_t \in \mathbb{R}^n$ is referred to as the process noise and $e_t \in \mathbb{R}^l$ is the measurement noise. The noise signals are white and mutually independent. We also assume that the pdf's of the initial state and the noises are known.

We consider an encoder that causally utilizes the encoder information. By the *encoder information*, we mean the set of variables whose values are known to the encoder. The *encoder* is then a mapping from the set of the encoder information to a discrete set of symbols. We take each symbol to be represented by an integer index. At time t, the index is $i_t \in \mathscr{I}_L = \{0,1,\ldots,L-1\}$, where $L = 2^R$ with R denoting the *rate* of the transmission, in bits per state measurement. In particular, we are interested in the class of encoder mappings described by the function

$$i_t = f_t(y_t, \mathbf{k}_0^{t-1}). \tag{2}$$

That is, given the side-information $\mathbf{k}_0^{t-1}$, the encoder maps the present measurement y_t to an index i_t. The side-information k_t represents the feedback to the encoder about the symbol j_t, received at the controller. The mapping from j_t to k_t will be detailed later. We will also consider the encoders of the form

$$i_t = \bar{f}_t(\bar{y}_t, \mathbf{k}_0^{t-1}), \tag{3}$$

where we let $\bar{y}_t$ depend on both y_t and $\mathbf{u}_0^{t-1}$, as will be specified later.

Let the *discrete memoryless channel* have the input i_t and the output $j_t \in \mathscr{I}_L$, with one channel use defined by

$$j_t = \kappa_t(i_t), \tag{4}$$

where $\kappa_t : \mathscr{I}_L \to \mathscr{I}_L$ is a random mapping.

At the receiver side, we consider a controller that causally utilizes the controller information $\{\mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$. The *controller* is written as

$$u_t = g_t(\mathbf{j}_0^t, \mathbf{u}_0^{t-1}). \tag{5}$$

Finally, we denote the conditional mean estimator of the state, based on the received indices, as $\hat{x}_{s|t} = E\{x_s|\mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$ for $s \leq t$; moreover, let $\hat{x}_t = \hat{x}_{t|t} = E\{x_t|\mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$ and $\tilde{x}_t = x_t - E\{x_t|\mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$.

III. PROBLEM STATEMENT

In this section, the memoryless binary channel and the performance measure are specialized together with a discussion on the encoder side-information.

A. Binary Channel and Performance Measure

We consider a memoryless binary channel in this paper. Let $c(i_t) \in \{0,1\}^R$ be a binary codeword of length R representing the encoder output, $i_t \in \mathscr{I}_L$. The mapping between i_t and $c(i_t)$ is referred to as the *index assignment* [10]. In a similar way, $c(j_t)$ denotes the received binary codeword, where $j_t \in \mathscr{I}_L$ is the received index.

Our goal is to solve an optimal encoder–controller problem for the linear system (1). The performance measure for this integrated communication and control problem is the following LQ cost function, with a finite horizon $T > 0$,

$$E\{J_T\} = E\left\{\sum_{t=1}^{T} W_t(x_t, u_{t-1})\right\}, \; W_t = x_t' V_t x_t + u_{t-1}' P_{t-1} u_{t-1}, \tag{6}$$

where $(\cdot)'$ denotes matrix transpose. The matrices V_t's and P_t's are symmetric positive definite.

B. Encoder Side-Information

In general, the main reason for using memory-based encoder–controller is to increase the resolution of the quantized observation. For memory-based schemes, the system performance relies heavily on the encoder's knowledge about the controller memory state, and the controller's believe in the encoder memory state. In the presence of a noisy channel, care has to be taken in specifying how to "synchronize" the states of the encoder and controller.

We use the term *encoder side-information* (SI) to specify the potential feedback to the encoder about $\mathbf{j}_0^{t-1}$. Consequently, *no* SI is the extreme case when there is no feedback at all about the j_t's, and *full* SI denotes the situation that the encoder knows exactly the previously received symbols $\mathbf{j}_0^{t-1}$. This is the case when the channel is noiseless, so that $j_t = i_t$, or when there is an error-free side-information channel from the output of the forward channel to the encoder. Note that full SI can also be obtained if the encoder knows the previous control signals, $\mathbf{u}_0^{t-1}$, and the controller function is an invertible mapping, since then j_t can be deduced from $\mathbf{u}_0^t$. In this paper, we consider a particular class of side-information, namely,

$$k_t = \eta_t(j_t) \in \mathscr{K}_K = \{0,1,\ldots,K-1\}, \quad K \leq L. \tag{7}$$

Accordingly, $k_t = j_t$ and $K = L$ when full SI is available, while $K = 1$ when there is no SI at the encoder. Between the extremes, there is a variety of cases with incomplete SI, for which $1 < K < L$. One example is the case with no side-information channel and a non-invertible controller mapping, e.g., if u_t takes on only $K < L$ distinct values. We assume the general case, $1 \leq K \leq L$, in the paper. Note that in Fig. 1 we illustrate the mapping from j_t to k_t as an explicit side-information channel, even though this information can be obtained by other means, e.g., by inverting the controller mapping as previously discussed.

IV. ENCODER–CONTROLLER DESIGN

This section presents the main results of the paper. Since the overall joint encoder–controller optimization problem is not tractable, we propose a method to optimize the encoder–controller pair iteratively. Similar to traditional quantizer design [10], the idea is to fix the encoder and update the controller, then fix the controller and update the encoder etc. At each updating, only one (time-)*component* (f_t or g_t for a certain t) of the encoder or controller is optimized. For the sake of clarity, we will refer to the sequence of components which define the full operation of the encoder as $\mathbf{f}_0^{T-1}$, and the notation f_t refers to the particular component at time t. Similar notation is used for the controller.

First in Sec. IV-A, we consider the problem of the optimal control for fixed encoding functions. Then, in Sec. IV-B, the optimal control equation is solved for a class of modified encoders. Thereafter, in Sec. IV-C, we address the problem of optimal encoding, assuming the controller is fixed.

A. Optimal Controller

In this section, we investigate the optimal controller mapping g_t, assuming the encoder $\mathbf{f}_0^{T-1}$ is fixed. This scenario fits well into the setting of stochastic optimal control [11], and we apply dynamic programming to derive the optimal control strategy. The observation available at the controller is the integer-valued received indices $\mathbf{j}_0^{t-1}$. The recursive derivation starts at the last stage $t = T$. Consider

$$\begin{aligned}\lambda_T =& E\{W_T|\mathbf{j}_0^{T-1},\mathbf{u}_0^{T-2}\} \qquad (8)\\ =& tr(V_T Q_{T-1}) + \int_{\mathbb{R}^n}[x'_{T-1}A'V_T A x_{T-1} + 2u'_{T-1}B'V_T A x_{T-1}\\ &+ u'_{T-1}(P_{T-1}+B'V_T B)u_{T-1}]p(x_{T-1}|\mathbf{j}_0^{T-1},\mathbf{u}_0^{T-2})dx_{T-1},\end{aligned}$$

where the term $Q_t = E\{v_t v_t'\}$ denotes the variance of the process noise, which is independent of the control u_{T-1}. The optimal u_{T-1} is the one minimizing λ_T, in particular,

$$u^*_{T-1} = -[P_{T-1}+B'V_T B]^+ B'V_T A\hat{x}_{T-1}, \qquad (9)$$

where $[\cdot]^+$ denotes matrix pseudoinverse and $(\cdot)^*$ indicates the optimal solution. Substituting u^*_{T-1} into λ_T, the optimal *cost-to-go* for the last stage $t = T$ is

$$\begin{aligned}\gamma^*_T =& \min_{u_{T-1}}\lambda_T = E\{x'_{T-1}I_1 x_{T-1} + \varpi_1|\mathbf{j}_0^{T-1},\mathbf{u}_0^{T-2}\},\\ I_1 =& A'V_T A - \pi_1,\\ \pi_1 =& A'V_T B[P_{T-1}+B'V_T B]^+ B'V_T A,\\ \varpi_1 =& tr(V_T Q_{T-1}) + E\{\tilde{x}'_{T-1}\pi_1\tilde{x}_{T-1}|\mathbf{j}_0^{T-1},\mathbf{u}_0^{T-2}\}.\end{aligned} \qquad (10)$$

When u^*_{T-1} is established, we can move to the second last stage $t = T-1$. The optimal u_{T-2} is the one minimizing γ_{T-1},

$$\begin{aligned}u^*_{T-2} =& \arg\min_{u_{T-2}}\{\gamma_{T-1}\}, \qquad (11)\\ \gamma_{T-1} =& \lambda_{T-1} + E\{\gamma^*_T|\mathbf{j}_0^{T-2},\mathbf{u}_0^{T-3}\},\\ \lambda_{T-1} =& E\{x'_{T-1}(V_{T-1}+I_1)x_{T-1} + u'_{T-2}P_{T-2}u_{T-2}|\mathbf{j}_0^{T-2},\mathbf{u}_0^{T-3}\}.\end{aligned}$$

Similarly, generalizing to any time t, the optimal u_{t-1} is the one minimizing γ_t, (especially, $\gamma_0 = E\{J_T\}$). Resembling the classical results, we present the following proposition.

Proposition 1: Consider a fixed encoder $\mathbf{f}_0^{T-1}$. Given the linear plant (1) *and the memoryless channel* (4)*, a controller $u_t = g_t(\mathbf{j}_0^t,\mathbf{u}_0^{t-1})$ that minimizes the LQ cost* (6) *fulfills the following recursive equation*

$$\begin{aligned}u^*_{t-1} =& \arg\min_{u_{t-1}}\{\gamma_t\} \qquad (12)\\ \gamma_t =& \lambda_t + E\{\gamma^*_{t+1}|\mathbf{j}_0^{t-1},\mathbf{u}_0^{t-2}\}\\ \lambda_t =& E\{(Ax_{t-1}+Bu_{t-1}+v_{t-1})'V_t(Ax_{t-1}+Bu_{t-1}+v_{t-1})\\ &+ u'_{t-1}P_{t-1}u_{t-1}|\mathbf{j}_0^{t-2},\mathbf{u}_0^{t-3}\},\end{aligned}$$

for $t = 0,\ldots,T-1$ and $\gamma^*_{T+1} = 0$.

Unfortunately, the minimization problem (12) is hard to solve. One main obstruction lies in how $E\{\gamma^*_{t+1}|\mathbf{j}_0^{t-1},\mathbf{u}_0^{t-2}\}$ is affected by the past controls. Consider for example $t = T-1$; the quantity $E\{\tilde{x}'_{T-1}\pi_1\tilde{x}_{T-1}|\mathbf{j}_0^{T-1},\mathbf{u}_0^{T-2}\}$ in γ^*_T is then difficult to analyze, since the received index j_{T-1} itself is a function of u_{T-2} via encoding and transmission. Hence, obtaining an explicit solution to (12) is not feasible. Moreover, resorting to a numerical solution is computationally intensive and memory demanding.

B. Modified Encoder Information

As concluded in Sec. IV-A, an explicit solution to the optimal control problem (12) can in general be obtained only in few special cases. Here we study one example where we by changing the encoder information are able to arrive at an explicit solution for the optimal control. Namely, we extend the information available at the encoder to include also the values of all past controls (as illustrated in Fig. 1). Define

$$\bar{y}_t = y_t - \sum_{j=0}^{t-1} CA^{t-1-j}Bu_j = CA^t x_0 + \sum_{j=0}^{t-1} CA^{t-1-j}v_j + e_t, \quad (13)$$

that is, the part of y_t remaining after removing the effect of previous control signals. Consider the encoder

$$i_t = \bar{f}_t(\bar{y}_t,\mathbf{k}_0^{t-1}), \quad t = 0,\ldots,T-1,$$

(as defined in (3)), which uses the extended encoder information $\mathbf{u}_0^{t-1}$ to compute $\bar{y}_t$, and then produces an index i_t based on $\bar{y}_t$ and $\mathbf{k}_0^{t-1}$. According to the fact that $\mathbf{u}_0^{t-1}$ is completely determined by $\mathbf{j}_0^{t-1}$, when full SI is available, for any $f_t(\mathbf{y}_t,\mathbf{j}_0^{t-1})$, there is an equivalent $\bar{f}_t(\bar{\mathbf{y}}_t,\mathbf{j}_0^{t-1})$, and vice versa. For the encoder (3), we can verify that the estimation error $\tilde{x}_t$ is not a function of $\mathbf{u}_0^{t-1}$. This can be revealed by writing $\tilde{x}_t$ as,

$$A^t x_0 + \sum_{j=0}^{t-1}A^{t-1-j}v_j - E\left\{A^t x_0 + \sum_{j=0}^{t-1}A^{t-1-j}v_j|\mathbf{j}_0^t,\mathbf{u}_0^{t-1}\right\}.$$

Since $\mathbf{u}_0^{t-1}$ can be derived from $\mathbf{j}_0^{t-1}$; and, $\{x_0,\mathbf{v}_0^{T-1},\mathbf{e}_0^{T-1}\}$ are not affected by $\mathbf{u}_0^{T-1}$, one can show by the following induction that the indices $\mathbf{j}_0^t$ are not functions of $\mathbf{u}_0^{t-1}$. Start the induction at $t = 0$. The statement holds true at $t = 0$ since

$$i_0 = \bar{f}_0(\bar{y}_0) = \bar{f}_0(x_0+e_0), \quad j_0 = \kappa_0(i_0), \quad k_0 = \eta_0(j_0).$$

If the statement is valid for time t, then at time $t+1$,

$$i_{t+1} = \bar{f}_{t+1}(\bar{y}_{t+1},\mathbf{k}_0^t), j_{t+1} = \kappa_{t+1}(i_{t+1}), k_{t+1} = \eta_{t+1}(j_{t+1}),$$

which hence do not explicitly depend on $\mathbf{u}_0^t$. Therefore, $\{i_t,j_t,k_t\}_{t=0}^{T-1}$ depend only on $\{x_0,\ \mathbf{v}_0^{T-1},\ \mathbf{e}_0^{T-1}\}$ and potential channel errors, but not on $\mathbf{u}_0^{T-1}$.

The fact that the estimation error $\tilde{x}_t$ is not a function of $\mathbf{u}_0^{t-1}$ for the extended encoder information will significantly simplify the derivation of the optimal control. Consider equation (11). Since the covariance of the estimation error $\tilde{x}_{T-1}$ is independent of $\mathbf{u}_0^{T-2}$, the optimal u_{T-2} turns out to be

$$\begin{aligned}u^*_{T-2} =& \ell_{T-2}\hat{x}_{T-2},\\ \ell_{T-2} =& -[P_{T-2}+B'(V_{T-1}+I_1)B]^+B'(V_{T-1}+I_1)A,\end{aligned} \qquad (14)$$

and the optimal cost-to-go at $t = T-1$ is given by

$$\begin{aligned}\gamma^*_{T-1} =& E\{x'_{T-2}I_2 x_{T-2} + \varpi_2|\mathbf{j}_0^{T-2},\mathbf{u}_0^{T-3}\}, \qquad (15)\\ I_2 =& A'(V_{T-1}+I_1)A - \pi_2,\\ \pi_2 =& A'(V_{T-1}+I_1)B[P_{T-2}+B'(V_{T-1}+I_1)B]^+B'(V_{T-1}+I_1)A,\\ \varpi_2 =& \varpi_1 + tr((V_{T-1}+I_1)Q_{T-2}) + E\{\tilde{x}'_{T-2}\pi_2\tilde{x}_{T-2}|\mathbf{j}_0^{T-2},\mathbf{u}_0^{T-3}\},\end{aligned}$$

with $\{I_1, \pi_1, \varpi_1\}$ as in (10). Then, since $E\{\tilde{x}_t' \pi_{T-t} \tilde{x}_t | \mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$ does not depend on $\mathbf{u}_0^{t-1}$ at any time instance t, the optimal control of (5) at time t is obtained as

$$\begin{aligned} u_t^* &= \ell_t \hat{x}_t, \\ \ell_t &= -[P_t + B'(V_{t+1} + I_{T-t-1})B]^+ B'(V_{t+1} + I_{T-t-1})A. \end{aligned} \quad (16)$$

The resulting optimal cost-to-go is

$$\begin{aligned} \gamma_{t+1}^* &= E\{x_t' I_{T-t} x_t + \varpi_{T-t} | \mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}, \\ I_{T-t} &= A'(V_{t+1} + I_{T-t-1})A - \pi_{T-t}, \quad (17) \\ \pi_{T-t} &= A(V_{t+1} + I_{T-t-1})B[P_t + B'(V_{t+1} + I_{T-t-1})B]^+ \\ &\quad \times B'(V_{t+1} + I_{T-t-1})A, \\ \varpi_{T-t} &= \varpi_{T-t-1} + tr((V_{t+1} + I_{T-t-1})Q_t) + E\{\tilde{x}_t' \pi_{T-t} \tilde{x}_t | \mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}. \end{aligned}$$

The results in (16) and (17) illustrate that, for the fixed encoder (3), the optimal control strategy (12) admits an explicit solution. We summarize the above result in Proposition 2.

Proposition 2: Consider a fixed encoder $\bar{\boldsymbol{f}}_0^{T-1}$. *Given the linear plant* (1) *and the memoryless channel* (4)*, the controller* $u_t = g_t(\boldsymbol{j}_0^t, \boldsymbol{u}_0^{t-1})$ *that minimizes the LQ cost* (6) *is given by*

$$u_t = \ell_t \hat{x}_t, \quad (18)$$

with ℓ_t *specified in* (16)*, and* $\hat{x}_t$ *is the conditional mean estimate of the state* x_t*, i.e.,* $\hat{x}_t = E\{x_t | \boldsymbol{j}_0^t, \boldsymbol{u}_0^{t-1}\}$.

Observe that, the optimal control strategy (18) can be decomposed into a separate decoder and a controller. Hence, the *separation property* holds, e.g., [11], [12]. Additionally, one can show that the derived optimal control strategy (16) is a *certainty equivalence* (CE) controller, as discussed next.

A CE controller is obtained by first computing the optimal deterministic control, in the absence of process noise and assuming perfect state observations. Thereafter, the perfect state observations are replaced with estimates of the partially observed states, c.f., (16). The CE controller does in general not provide optimum performance. In our case, we were able to show that the resulting CE controller specified in (16) is optimal for the fixed encoder $\bar{f}_t$ in (3). In many applications it is not reasonable to assume that the encoder has access to the control sequence for computing $\bar{y}_t$. However, in the case of the original encoder (2), we can still employ the CE controller, together with the optimal encoder (19) (detailed in the next sub-section), to implement a computationally feasible approximation to the optimal solution.

C. Optimal Encoder

In this section, we address the problem of optimizing the encoder component f_t, for a fixed controller $\mathbf{g}_0^{T-1}$ and the fixed encoder components $\mathbf{f}_0^{t-1}$ and $\mathbf{f}_{t+1}^{T-1}$. The optimal encoder needs to take the impact of the predicted future state evolutions into account. Hence, the following results are evident from the construction.

Proposition 3: Consider a fixed controller $\boldsymbol{g}_0^{T-1}$ *and the fixed encoder components* $\boldsymbol{f}_0^{t-1}$ *and* $\boldsymbol{f}_{t+1}^{T-1}$. *Given the linear plant* (1) *and the memoryless channel* (4)*, the encoder component* $f_t(y_t, \boldsymbol{k}_0^{t-1})$ *that minimizes the LQ cost* (6) *is given by*

$$i_t = \arg\min_{i \in \mathscr{I}_L} E\left\{ \sum_{s=t+1}^{T} W_s(x_s, u_{s-1}) \Big| y_t, \mathbf{k}_0^{t-1}, i_t = i \right\}. \quad (19)$$

Observe that, the optimization problem (19) requires the probability densities $p(x_t | y_t, \mathbf{k}_0^{t-1})$ and $p(\mathbf{j}_0^{t-1} | y_t, \mathbf{k}_0^{t-1})$ to be estimated, and then used in the prediction of the future states and controls. The plant (1), the memoryless channel (4), the fixed controller and the encoder components $\mathbf{g}_0^{T-1}$, $\mathbf{f}_0^{t-1}$, $\mathbf{f}_{t+1}^{T-1}$, and the design criterion (6), are all involved in the estimation and prediction procedures. We also present the analogous result for the encoder $\bar{f}_t$.

Proposition 4: Consider a fixed controller $\boldsymbol{g}_0^{T-1}$*, and the fixed encoder components* $\bar{\boldsymbol{f}}_0^{t-1}$ *and* $\bar{\boldsymbol{f}}_{t+1}^{T-1}$. *Given the linear plant* (1) *and the memoryless channel* (4)*, the encoder component* $\bar{f}_t(\bar{y}_t, \boldsymbol{k}_0^{t-1})$ *that minimizes the LQ cost* (6) *is given by*

$$i_t = \arg\min_{i \in \mathscr{I}_L} E\left\{ \sum_{s=t+1}^{T} W_s(x_s, u_{s-1}) \Big| \bar{y}_t, \mathbf{k}_0^{t-1}, i_t = i \right\}. \quad (20)$$

V. TRAINING ALGORITHM

In this section, we propose an encoder–controller design algorithm which is suitable for low data rate, accomplishing source compression, channel protection and control simultaneously. Since, there are only a finite number of admissible control signals (due to the finite number of symbols $E\{x_t | \mathbf{j}_0^t, \mathbf{u}_0^{t-1}\}$), these values can be pre-calculated and stored in a codebook at the controller.

We summarize the results of Sec. IV in the following design algorithm. The CE controller in (16) is employed for the both encoders (2) and (3).

Encoder–Controller Design Algorithm

1) Initialize the encoder–controller mappings.
2) For each $t = 0, \ldots, T-1$,
   - Update the encoder component f_t using (19).
   - Update the controller g_t using (16)–(17).
3) If J_T has not converged, go to Step 2); otherwise stop.

The analogous training algorithm for the encoder $\bar{f}_t$ can be formulated by simply replacing f_t and (19) in Step 2) with $\bar{f}_t$ and (20).

Unfortunately, the design algorithm does not guarantee global optimality. The result converges to a local minimum, which has shown to work well in practice. The encoder–controller design presented above is still fairly computationally intensive. In our simulations, we resort to a sequential Monte Carlo approach to handle the nonlinear filtering problems.

VI. NUMERICAL EXAMPLES

Here we present numerical results to demonstrate the performance obtained using iterative encoder–controller design. We study a scalar system for simplicity. The system equations and the design criterion are,

$$x_{t+1} = a x_t + u_t + v_t, \quad y_t = x_t + e_t, \quad W_t = x_t^2 + \rho u_{t-1}^2.$$

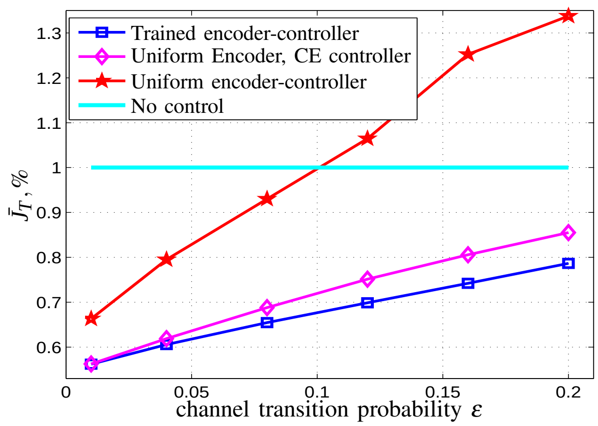

Fig. 2. A comparison among the proposed encoder–controller, the uniform encoder–controller and the uniform encoder with the CE controller.

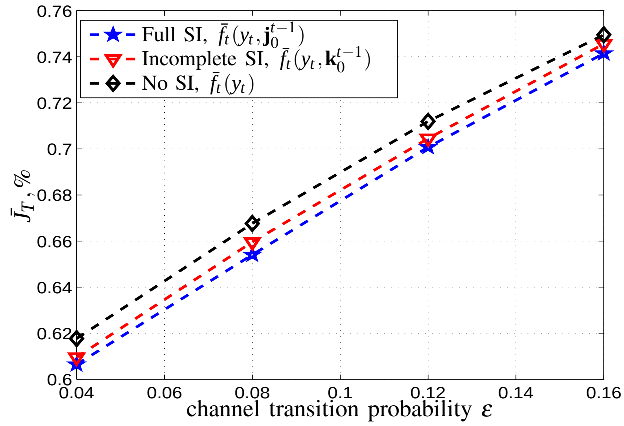

Fig. 3. Performance for various SI, k_t. Incomplete SI is generated by a noiseless feedback channel with the data rate $\frac{R}{2}$.

The system parameters are $a = 1.2$, $T = 2$, $R = 2$, and $\rho = 0.1$. The initial state x_0, the process noise v_t and the measurement noise e_t are all white Gaussian distributed, in particular, $p(x_0)$ is $\mathcal{N}(0,1)$; $p(v_t)$ and $p(e_t)$ are $\mathcal{N}(0,0.5)$. The encoded symbols are transmitted over a binary symmetric channel.

In Fig. 2, we show the system performance as a function of the channel transition probability ε. Performance $\bar{J}_T$ is obtained by normalizing J_T in (6) with the cost obtained when no control action is taken, c.f., the horizontal line in Fig. 2. In this experiment, the encoder f_t, with full SI, is employed. Three types of encoder–controller pairs are illustrated, namely, the trained encoder–controllers, time-varying uniform encoder–controllers, and time-varying uniform encoders with CE controllers. The trained encoder–controller pairs evidently outperform the other coding–control schemes. The improvement is essentially attributed to the CE controller.

The SI affects the system in several ways. For the encoder (3), the SI is involved in e.g., the probabilities $p(x_t|\bar{y}_t, \mathbf{k}_0^{t-1})$ and $p(\mathbf{j}_0^{t-1}|\bar{y}_t, \mathbf{k}_0^{t-1})$. Similarly, for the encoder (2), the SI affects $p(x_t|y_t, \mathbf{k}_0^{t-1})$ and $p(\mathbf{j}_0^{t-1}|y_t, \mathbf{k}_0^{t-1})$. In Fig. 3, we show a comparison of different degrees of the SI when the encoder (3) is employed. In particular, we explore no, incomplete and full SI scenarios. In the experiment, the incomplete SI is generated as follows. The least significant bit of j_t is discarded and the resulting index is fed back to the encoder over a noiseless link. The figure shows, full SI provides the best training result, while the incomplete SI scenario outperforms the no SI scenario. However, a similar experiment for the encoder f_t (2) shows, for this type of encoders, the improvement given by knowing the SI becomes insignificant. The main reason is the trained encoder–controllers have resulted in minor differences among the densities $p(x_t|y_t, \mathbf{k}_0^{t-1})$'s, for different $\mathbf{k}_0^{t-1}$'s.

VII. CONCLUSION

This paper has investigated the jointly optimization of the encoder and the controller in closed-loop control of a linear plant with low-rate feedback over a memoryless binary channel. After recognizing the difficulties in solving the general optimal control problem, we resort to a suboptimal strategy. Moreover, we showed that the closed-form solution of the optimal controller for fixed encoders is possible by an extension of the information set at the encoder. Thereafter, we introduced an iterative approach to optimize the encoder–controller pairs. We have performed various numerical investigations of the proposed optimization algorithm. Numerical results demonstrate the promising performance obtained by employing the proposed algorithm. Finally, we have also investigated the impact on system performance for different degrees of SI at the encoder.

REFERENCES

[1] A. Sahai, "The necessity and sufficiency of anytime capacity for control over a noisy communication link," in *Proc. of the 43rd IEEE Conference on Decision and Control*, December 2004.

[2] T. Simsek, R. Jain, and P. Varaiya, "Scalar estimation and control with noisy binary observation," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1598–1603, Sept. 2004.

[3] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1196–1201, Sept. 2004.

[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, July 2000.

[5] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Jour. Contr. Opt.*, Febrary 2003.

[6] V. S. Borkar, S. K. Mitter, and S. Tatikonda, "Optimal sequenctial vector quantization of Markov sources," *SIAM J. Control Optim.*, vol. vol.40, no. 1, pp. 135–148.

[7] A. Mahajan and D. Teneketzis, "Fixed delay optimal joint source–channel coding for finite–memory systems," in *Proc. of 2006 IEEE International Symposium on Information Theory*, July 2006, pp. 2319–2323.

[8] L. Bao, M. Skoglund, and K. H. Johansson, "Encoder-decoder design for feedback control over the binary symmetric channel," in *Proc. of 2006 IEEE International Symposium on Information Theory*, 2006, pp. 2481–2485.

[9] ——, "Encoder-decoder design for event-trigged feedback control," in *Proc. of 2006 American Control Conference*, 2006, pp. 4183–4188.

[10] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Transactions on Information Theory*, vol. vol.36, no. 4, pp. 799–809, 1990.

[11] M. Aoki, *Optimization of Stochastic Systems - Topics in Discrete-Time Systems*. Academic Press, 1967.

[12] H. S. Witsenhausen, "Separation of estimation and control for discrete time systems," in *Proc. of the IEEE*, vol. 59, no. 11, Nov. 1971, pp. 1557–1566.