Recursive Network Estimation for a Model with Binary-Valued States

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***Abstract*—This paper studies how to estimate the weighted adjacency matrix of a network out of the state sequence of a model with binary-valued states, by using a recursive algorithm. In the considered system, agents display and exchange these binary-valued states generated from intrinsic quantizers. It is shown that stability of the model and identifiability of the system parameters can be guaranteed under continuous random noise. Under standard Gaussian noise, the problem of estimating the real-valued weighted adjacency matrix and the unknown quantization threshold is transformed to an optimization problem via a maximum likelihood approach. It is further verified that the unique solution of the optimization problem is the true parameter vector. A recursive algorithm for the estimation problem is then proposed based on stochastic approximation techniques. Its strong consistency is established and convergence rate analyzed. Numerical simulations are provided to illustrate developed results.**

***Index Terms*—network estimation, binary-valued states, stochastic approximation, quantized identification, identifiability**

I. Introduction

Network estimation, or network inference, i.e., inferring underlying relationships between entities from data, is of great significance in multiple scientific disciplines. For example, traffic engineers can design new links to avoid network congestion, by estimating traffic volume between all pairs of nodes in a network from traffic flow, known as network tomography [1]. Theoretical modeling and empirical verification of gene regulatory networks can enhance our understanding of diseases and development [2]. Inferring social structures such as friendship and influence can help to analyze and predict collective behaviors in complex social networks [3].

A problem needs to be further studied in network estimation literature is to recursively estimate underlying networks based on quantized data. Quantized data is ubiquitous across domains, for example, active/inactive states of a gene [2] and ordinal rating of an individual [4]. However, existing researches focus more on reconstructing networks based on real-valued data [5]–[7]. In addition, recursive algorithms are of great importance for identification of networked systems [8]. They can be used for online tasks, such as adaptive control and decision-making, and thus have attracted much interest in the control community. More attention has been paid, however, on batch algorithms for network estimation [5]. Therefore, it is crucial to propose and investigate online network estimation algorithms for complex networked dynamics based on quantized time-series data [2], [5], [7].

A related field in system identification is quantized identification, i.e., estimating parameters based on quantized data [9]–[11]. But many of these methods cannot be applied directly to the aforementioned network estimation problems, since they require the design of either input signals [12] or quantizers [10], [13]. In contrast, it may be difficult to impose sufficient input signals when estimating a complex networked system. Moreover, quantized data could be generated from unknown components of a system, instead of some artificial quantized sensors.

This paper studies a recursive network estimation problem based on binary-valued data, which is a special but crucial case of general quantized data. In particular, we focus on the case where weighted adjacency matrices of underlying networks are real-valued, representing interaction strength between agents. In the considered system, agents display and exchange binary-valued states, which are generated from intrinsic quantizers.

Dynamics with binary-valued states can be encountered in a variety of domains. Here we present two motivating examples.

A. Motivating Examples

Example 1. (Neuronal Dynamics Under Random Noise) To model neuron behaviors, [14] introduces the following model of neuronal dynamics. Although higher-order dynamics are also studied in [14], [15], we focus on the first-order system in this paper. The model evolves over a network with an agent (neuron) set $\mathcal{V} = \{1, \dots, n\}$. Let S_k be the state vector at time $k \geq 0$. Agent $i \in \mathcal{V}$ has state $S_{k,i} \in \{0, 1\}$, which can be detected via experimental methods. The state of agent i updates in the following way

$$S_{k+1,i} = f\bigg(\sum_{j=1}^{n} a_{ij} S_{k,j} + D_{k,i} - c_i\bigg), \tag{1}$$

where the function f represents the McCulloch-Pitts neural model such that $f(x) = 1$ if $x \geq 0$ and $f(x) = 0$ otherwise, for real number x. Parameter $a_{ij} = \gamma_i t_{ij}/n$ represents the total interaction strength between agents i and j, given by the strength of synaptic connection γ_i and link strength t_{ij} between agents i and j. The term $D_{k,i}$ is the noise of agent i, assumed to be independent and identically distributed (i.i.d.)

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standard Gaussian, and it comes from underlying biochemical processes [16]. Finally, c_i is the individual threshold of agent i.

Since the state space of System (1) is $\{0,1\}^n$, it is a Boolean network (BN). BNs have been extensively studied in many disciplines [2], [17], due to their effectiveness in describing genetic regulatory networks, neuronal networks, and so on. There have been numerous related researches, such as stability [18] and filtering [19] of BNs, as well as stabilization [20] and observer design [21] of Boolean control networks.

Identification of BNs is a significant topic, because underlying relationships between agents, either logical or parametric ones, can be used for prediction and decision-making. For (1), the problem of interest is how to estimate the interaction strength matrix $A = (a_{ij})$ out of temporal data $\{S_k\}$, i.e., recovering the underlying connections between neurons from their behaviors. It should be noted that although a_{ij} consists of two unknown parameters γ_i and t_{ij}, there is no way to estimate the exact values of them. This is because infinite pairs of (γ_i, t_{ij}) can define the same a_{ij}.

Example 2. (Binary Choice Models of Social Interactions) Social interactions shape behaviors of individuals. Many interactive decisions are binary (e.g., voting), and these actions can be observed directly or obtained from questionnaires and online data. As a result, researchers have proposed lots of mathematical models, in order to analyze binary choices in social interactions [4]. A dynamical binary-choice model, whose update rule is similar to Example 1, is introduced in Section 4.1 of [22] and writes as follows,

$$P\{S_{k+1,i} = 1 | S_k\} = F(h_i + \textstyle\sum_{j=1}^n w_{ij} S_{k,j}), \qquad (2)$$

where $S_{k,i} \in \{0,1\}$ is the state of agent $i \in \mathcal{V} = \{1, \dots, n\}$ at time $k \geq 0$, $S_k \in \{0,1\}^n$ is the state vector, $F(\cdot)$ is a cumulative distribution function (cdf), and h_i and w_{ij} are parameters. As discussed in [22], update rule (2) can be seen as a consequence of agent i maximizing a random utility function $V_{k,i}(s, S_k)$ at time $k+1$, where $s \in \{0,1\}$ and

$$V_{k,i}(s, S_k) := h_i' s - \sum_{j=1}^n w_{ij}'(s - S_{k,j})^2 + \varepsilon_{k,i}(s).$$

Here h_i' is a private preference, w_{ij}' is the conformity strength of j on i, and $\{\varepsilon_{k,i}(1), k \geq 0\}$, $\{\varepsilon_{k,i}(0), k \geq 0\}$, $1 \leq i \leq n$, are mutually independent i.i.d. random sequences. When $V_{k,i}(1, S_k) \geq V_{k,i}(0, S_k)$, agent i chooses action 1 at time $k+1$ to maximize its utility. This process has update rule (2) with $h_i = h_i' - \sum_{j=1}^n w_{ij}$, $w_{ij} = 2w_{ij}'$, and $F(\cdot)$ being the cdf of $\varepsilon_{k,i}(0) - \varepsilon_{k,i}(1)$. It is of interest whether we can recover conformity relationships between agents based on the observed binary actions.

B. Related Work

For identification of BNs, most researches focus on estimating logical functions of deterministic BNs [2], [17], [23]. The problem considered in this paper, however, is to estimate the adjacency matrix, which constitutes the weights of Boolean threshold functions in a probabilistic BN. Identification of BNs with perturbation is studied in [24] without performance analysis. Identifiability and estimation of probabilistic BNs are studied rigorously in [25], [26]. [25] considers admissible Boolean threshold functions, but these functions do not fit into our case. In [26], authors investigate identification of logical Boolean functions, but here we focus on a specific model with threshold functions. Furthermore, [25], [26] assume that observation data comprises independent samples instead of time series.

Results on identification of binary choice models in the field of econometric can be found in [27], which establishes sufficient conditions for identifiability. Many estimation methods and their asymptotic properties have, however, been studied under static games and situations where the network size tends to infinity [28], [29].

Probit models are a class of binary choice models relevant to our problem. Such a model has a binary outcome that is one if an unobserved continuous response exceeds a threshold and is zero otherwise. In the classic probit model [30], the unobserved response is the sum of a linear combination of multiple inputs and a Gaussian noise. Generalizations of the classic model include cases where the unobserved response depends on previous binary choices and the noise sequence is correlated and not necessarily Gaussian [31], [32]. To address identifiability issues, researchers have introduced various identification constraints [33]; for instance, the threshold and the noise variance can be assumed to be zero and one, respectively. A common estimation approach for the classic probit model follows the maximum likelihood framework [30]. Other estimation methods for dynamic probit models have been proposed in the literature. For example, [32] obtains asymptotic properties of the smoothed maximum score estimator, and [31] investigates the conditional likelihood estimator and the generalized method of moments. In this paper, we study how to estimate networks by using online algorithms rather than batch methods.

There is a growing interest on network inference for social dynamical systems, but not many researches focus on estimating networks out of quantized data [7]. The authors of [34] study a network estimation problem for a discuss-then-vote model, in which individuals display a discrete voting choice at the end of each discussion. But agents exchange continuous states during the discussion, and multiple episodes of the model are necessary for estimation.

In the literature of quantized identification, there are multiple methods not relying on the design of inputs or quantizers. For example, the maximum likelihood method is used in [10], [35]–[37]. Authors in [10] propose an online algorithm, based on the expectation-maximization (EM) algorithm and quasi-Newton method, to estimate autoregressive moving average models with quantized observations. But to achieve the best performance, quantizers need to be known and adaptive. To optimize likelihood functions, [35] and [36] use the EM algorithm, and [37] utilizes a variational approximation approach. Additionally, [11] applies a Bayesian framework. The authors of [38] propose an algorithm based on a recursive prediction error method, but both quantizers and the range of parameters are assumed to be known. This paper applies the maximum likelihood method to solve the network estimation problem

with unknown quantization thresholds, and online estimation is of interest.

C. Contributions

We study a recursive network estimation problem based on binary-valued data in this paper, motivated by aforementioned examples. More specifically, we propose an algorithm to recursively estimate the weighted adjacency matrix of a network, as well as unknown quantization thresholds, out of the state sequence of a model with binary-valued states.

Our contributions are summarized as follows.

1. Stability of the model is studied as a Markov chain under a general condition (Theorems 1 and 2). In addition, we investigate properties of an auxiliary Markov chain, and show that the stationary distribution of the auxiliary chain can be uniquely determined by the transition probability matrix of the original model (Lemma 1 and Theorem 3).

2. Identifiability of system parameters, consisting of a real-valued weighted adjacency matrix and quantization thresholds, is analyzed under the noise with non-zero density on the real line, including the case of standard Gaussian noise (Theorem 4).

3. The network estimation problem is studied via a maximum likelihood approach, under independent standard Gaussian noise. We define an objective function based on log-likelihood functions, and verify that the objective function is strictly concave and attains its unique maximum at the true parameter vector (Theorem 5).

4. By using stochastic approximation techniques, we propose a recursive estimation algorithm and prove that it is strongly consistent (Theorem 6). Furthermore, its convergence rate is estimated (Theorem 7).

The differences of this paper from the conference version [39] are that we add motivating examples, provide detailed proofs of results, further analyze the convergence rate of the algorithm, and conduct more numerical simulations.

D. Outline and Notation

The remainder of this paper is organized as follows. In Section II, the network estimation problem is formulated. Section III-A studies stability of the model, and Section III-B provides conditions such that the system parameters are identifiable. The network estimation problem is cast into an optimization problem in Section IV-A. We propose a recursive estimation algorithm in Section IV-B to solve the problem. Section IV-C investigates asymptotic properties of the estimation algorithm, including strong consistency and convergence rate. Section V presents numerical simulations illustrating theoretical results, and Section VI concludes the paper. To keep the paper fluent, some proofs are postponed to appendices.

Notation: By boldfaced lower-case or Greek letters we denote column vectors, and by upper-case letters we denote matrices and random vectors. We use $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\|\cdot\|$ to represent the set of real numbers, the n-dimensional Euclidean space, the set of $n\times m$ real matrices, and the Euclidean norm of a vector, respectively. Let $\mathbf{0}_n$, $\mathbf{1}_n$, and $\boldsymbol{e}_i$ be the n-dimensional all-zero vector, the n-dimensional all-one vector, and the unit vector with i-th entry being one. By a_i and $\boldsymbol{a}_{i:j}$ we denote the i-th entry of vector $\boldsymbol{a}$ and its sub-vector $(a_i, a_{i+1}, \ldots, a_j)^T$. For a matrix $A \in \mathbb{R}^{n\times m}$, a_{ij}, A_i, and A^T are used to represent its (i,j)-th entry, i-th row, and transpose. Define $\mathrm{vec}(A) := (a_{11}\ a_{12}\ \cdots\ a_{1n}\ a_{21}\ \cdots\ a_{2m}\ \cdots\ a_{nm})^T$. Denote the absolute value of $x \in \mathbb{R}$ by $|x|$, $|\boldsymbol{a}| := (|a_1|, \ldots, |a_n|)^T$, and $|A| := (|a_{ij}|)$. A matrix $A \in \mathbb{R}^{n\times n}$ is called stochastic if $A\mathbf{1}_n = \mathbf{1}_n$ and all its entries are nonnegative, and A is called absolutely stochastic if $|A|\mathbf{1}_n = \mathbf{1}_n$.

Let $E\{X_k\}$, $X_{k,i}$, and $X_{k,i:j}$ be the expectation, the i-th entry, and the sub-vector $(X_{k,i}, X_{k,i+1}, \ldots, X_{k,j})^T$ of a random vector X_k, $k \geq 0$. For $a, b \in \mathbb{R}$, denote $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Define $\mathcal{S}^n$ as the Descartes product $\mathcal{S}^n = \times_{i=1}^n \mathcal{S}_i$, where $\mathcal{S}_i = \{0, 1\}$, $1 \leq i \leq n$. $\mathbb{I}_{[\text{property}]}$ is the indicator function equal to 1, if the property in the bracket holds, and equal to 0 otherwise. The gradient and the Hessian of $f(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ are denoted by $\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$ and $\nabla^2_{\boldsymbol{x}} f(\boldsymbol{x})$, respectively. For two sequences $\{\boldsymbol{a}_k\}$ and $\{b_k\}$ with $\boldsymbol{a}_k \in \mathbb{R}^n$ and $b_k \neq 0$, $k \geq 1$, $\boldsymbol{a}_k = O(b_k)$ means that $\|\boldsymbol{a}_k/b_k\| \leq C$ for all $k \geq 1$ and some positive number C, and $\boldsymbol{a}_k = o(b_k)$ means that $\lim_{k\to\infty} \|\boldsymbol{a}_k/b_k\| = 0$. Let $[\cdot]$ be the rounding function such that $[k + 0.5] = k + 1$ for all integer k and $[x]$ is the nearest integer of x for all other $x \in \mathbb{R}$, and let $[\boldsymbol{x}] := ([x_1],\ \ldots,\ [x_n])^T$ for $\boldsymbol{x} \in \mathbb{R}^n$, i.e., the entry-wise rounding of $\boldsymbol{x}$.

For a homogeneous and finite-state Markov chain $\{X_k\}$ in a state space $\mathcal{X}$, the transition probability from x to y is $P(x, y) := P\{X_1 = y | X_0 = x\}$, and the k-step transition probability from x to y is $P^k(x, y) := P\{X_k = y | X_0 = x\}$ for all $x, y \in \mathcal{X}$. We say that y is reachable from x, if there exists $k \geq 1$ such that $P^k(x, y) > 0$. The Markov chain is said to be irreducible, if y is reachable from x for all $x, y \in \mathcal{X}$. The greatest common divisor of set $\{k \geq 1 : P^k(x, x) > 0\}$ is called the period of x, denoted by $d(x)$. The Markov chain is aperiodic if $d(x) = 1$ for all $x \in \mathcal{X}$. A probability distribution π on $\mathcal{X}$, which is a row vector, is referred to as a stationary distribution of $\{X_k\}$, if $\pi(y) = \sum_{x\in\mathcal{X}} \pi(x)P(x, y)$ for all $y \in \mathcal{X}$.

II. Problem Formulation

A. Problem

In the sequel, suppose that the network size $n \geq 2$. The considered model with binary-valued states is as follows:

$$\begin{aligned} Y_{k+1} &= AS_k + D_k, \\ S_k &= \mathcal{Q}(Y_k, \boldsymbol{c}), \end{aligned} \tag{3}$$

where $k \geq 0$, $Y_k = (Y_{k,1}, \ldots, Y_{k,n})^T$, $D_k = (D_{k,1}, \ldots, D_{k,n})^T$, $S_k = (S_{k,1}, \ldots, S_{k,n})^T$ are the unobserved response, the noise, and the state vector at time k, respectively. $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix, and $\boldsymbol{c} = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ is the unknown quantization threshold vector. $\mathcal{Q}(Y_k, \boldsymbol{c}) := (\mathbb{I}_{[Y_{k,1}\geq c_1]}, \ldots, \mathbb{I}_{[Y_{k,n}\geq c_n]})^T$ is the intrinsic quantizer. See Fig. 1 for an illustration of this system.

For the weighted adjacency matrix A, we do not assume that it is primitive (corresponding to the strong connectivity of the network) or row stochastic. Negative weights, representing

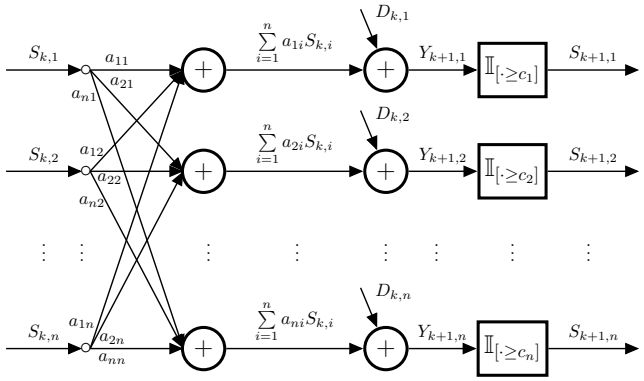

Fig. 1. The focus of this paper is to recursively estimate the weighted adjacency matrix $A = (a_{ij})$ and the quantization threshold $\boldsymbol{c}$ out of state sequence $\{S_k\}$.

antagonistic relationships, are also permitted. Instead of assuming that the entries of A are integers, this paper considers a more general case that A is real-valued in order to characterize the influence strength between agents [7].

In this paper we study the following two problems:

Problem 1. Provide conditions such that real-valued parameters A and $\boldsymbol{c}$ are identifiable from the binary-valued state sequence $\{S_k\}$.

Problem 2. For identifiable parameters A and $\boldsymbol{c}$, design a recursive estimation algorithm using the binary-valued state sequence $\{S_k\}$ and analyze the convergence of the algorithm.

B. Motivating Examples Revisited

We briefly revisit the motivating examples to show that they fit into System (3). System (1), the model of Example 1, is actually the same as System (3). System (2), the model of Example 2, is equivalent to System (3) with $c_i = -h_i = -h_i' + \sum_{j=1}^{n} w_{ij}'$, $a_{ij} = w_{ij} = 2w_{ij}'$, $1 \leq i, j \leq n$, and $D_{k,i} = \varepsilon_{k,i}(1) - \varepsilon_{k,i}(0)$, $1 \leq i \leq n$, $k \geq 0$.

III. Model Analysis

In this section, we study stability of the system and identifiability of the system parameters, to ensure that the network estimation problem is well-posed. To estimate the network as in Section IV, we need to ensure that the system provides sufficient diversity, which is related to its stability. Also, identifiability of parameters is necessary for consistent estimation. Hence, in Section III-A, Theorems 1 and 2 study stability of the system, and Lemma 1 and Theorem 3 show the relationship between the stationary distribution of an auxiliary Markov chain and the transition probability matrix of the original system. In Section III-B, Theorem 4 investigates a general condition for identifiability of parameters.

A. Stability

As in System (3), $\{S_k, k \geq 0\}$ is a Markov chain with finite states. The existence of stationary distributions is an essential property of Markov chains [40]. We study it under the following assumption.

Assumption 1. *(Noise)*
The sequences of system noise $\{D_{k,i}, k \geq 0\}$, $1 \leq i \leq n$, satisfy that
(i) they are i.i.d., mutually independent, and independent of S_0;
(ii) their distributions have density functions positive on $\mathbb{R}$.

Denote the transition probability matrix of $\{S_k\}$ by $P(\cdot, \cdot)$, and we have that

Theorem 1. *(Stability)*
Suppose that Assumption 1 holds, then Markov chain $\{S_k\}$ is irreducible and aperiodic. Moreover, $P(\boldsymbol{u}, \boldsymbol{s}) > 0$ holds for all $\boldsymbol{u}, \boldsymbol{s} \in \mathcal{S}^n$. Hence, $\{S_k\}$ converges in distribution, from any initial condition, to a unique stationary distribution π on $\mathcal{S}^n$ with $\pi(\boldsymbol{s}) > 0$ for all $\boldsymbol{s} \in \mathcal{S}^n$.

Proof. The conclusion follows from directly computing the transition probabilities of $\{S_k\}$, which is similar to the proof of Theorem 1 in the conference version [39]. □

Remark 1. *Theorem 1 provides a sufficient condition for the irreducible and aperiodic properties of $\{S_k\}$, and Assumption 1 is strong enough so that we do not need any assumption for the weighted adjacency matrix A. The behaviors of System* (3) *and related models have been extensively studied in different disciplines, e.g., [14], [15], [22], [32], [41]. We present this theorem to show that the observation sequence can exhibit sufficient diversity, as long as noise can surpass the influence of neighbors on an agent. The diversity is necessary for an accurate estimate of A.*

Define $\tilde{S}_k := (S_k^T \ S_{k-1}^T)^T$, $k \geq 1$. Note that $\{\tilde{S}_k\}$ taking values in $\mathcal{S}^{2n}$ is also a Markov chain. This auxiliary chain is critical for our estimation. For $k \geq 1$ and $\boldsymbol{s}^{k-1}, \boldsymbol{s}^k, \boldsymbol{s}^{k+1} \in \mathcal{S}^n$, it holds that

$$\begin{aligned} &P\{\tilde{S}_{k+1} = ((\boldsymbol{s}^{k+1})^T(\boldsymbol{s}^k)^T)^T | \tilde{S}_k = ((\boldsymbol{s}^k)^T(\boldsymbol{s}^{k-1})^T)^T\} \\ &= P\{S_{k+1} = \boldsymbol{s}^{k+1} | S_k = \boldsymbol{s}^k\}. \end{aligned} \tag{4}$$

So $\{\tilde{S}_k\}$ is aperiodic. For any two states $(\boldsymbol{s}^T \ \boldsymbol{u}^T)^T$, $(\boldsymbol{x}^T \ \boldsymbol{y}^T)^T \in \mathcal{S}^{2n}$ with $\boldsymbol{s}, \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^n$, since $\{S_k\}$ is irreducible, there exists $k \geq 1$ such that $P^k(\boldsymbol{x}, \boldsymbol{u}) > 0$. Moreover, from Theorem 1, $P(\boldsymbol{u}, \boldsymbol{s}) > 0$ holds. Hence it follows from (4) that

$$P\{\tilde{S}_{k+2} = (\boldsymbol{s}^T\boldsymbol{u}^T)^T | \tilde{S}_1 = (\boldsymbol{x}^T\boldsymbol{y}^T)^T\} > 0,$$

which implies that $\{\tilde{S}_k\}$ is also irreducible. Therefore, we have the following result.

Theorem 2. *(Stability of the auxiliary chain)*
Suppose that Assumption 1 holds, then Markov chain $\{\tilde{S}_k\}$ is irreducible and aperiodic. Hence, it converges in distribution, from any initial condition, to a unique stationary distribution $\tilde{\pi}$ on $\mathcal{S}^{2n}$ with $\tilde{\pi}(\tilde{\boldsymbol{s}}) > 0$ for all $\tilde{\boldsymbol{s}} \in \mathcal{S}^{2n}$.

The next lemma illustrates the relationship between $\{S_k\}$ and the stationary distribution of $\{\tilde{S}_k\}$.

Lemma 1. *Suppose that Assumption 1 holds, and $\tilde{S}$ is subject to the stationary distribution of $\{\tilde{S}_k\}$. Then*

$$P\{\tilde{S}_{1:n} = \tilde{\boldsymbol{s}}_{1:n} | \tilde{S}_{n+1:2n} = \tilde{\boldsymbol{s}}_{n+1:2n}\} = P(\tilde{\boldsymbol{s}}_{n+1:2n}, \tilde{\boldsymbol{s}}_{1:n}),$$

for all $\tilde{\boldsymbol{s}} \in \mathcal{S}^{2n}$.

Proof. See Appendix A. □

Remark 2. *This lemma indicates that the conditional probability of the event* $\{\tilde{S}_{1:n} = \tilde{\boldsymbol{s}}_{1:n}\}$ *given* $\{\tilde{S}_{n+1:2n} = \tilde{\boldsymbol{s}}_{n+1:2n}\}$ *is the same as the transition probability of* $\{S_k\}$ *from* $\tilde{\boldsymbol{s}}_{n+1:2n}$ *to* $\tilde{\boldsymbol{s}}_{1:n}$*, for all* $\tilde{\boldsymbol{s}} \in \mathcal{S}^{2n}$*. This accords with the definition of* $\{\tilde{S}_k\}$.

Denote the system parameter vector by

$$\theta := \mathrm{vec}\big\{(A \ \boldsymbol{c})\big\}$$

with $(A \ \boldsymbol{c}) \in \mathbb{R}^{n\times(n+1)}$, the transition probability matrix of $\{S_k\}$ and $\{\tilde{S}_k\}$ by P_θ and $\tilde{P}_\theta$, respectively, and the stationary distribution of $\{\tilde{S}_k\}$ by $\tilde{\pi}_\theta$, emphasizing the dependency of these terms on the parameter vector θ. The following theorem shows that if $\{\tilde{S}_k\}$ has a unique stationary distribution, then it is uniquely determined by the transition probability matrix of $\{S_k\}$.

Theorem 3. *Suppose that Assumption 1 holds, then* $\tilde{\pi}_\theta = \tilde{\pi}_{\bar{\theta}}$ *implies* $P_\theta = P_{\bar{\theta}}$ *for all* $\theta, \bar{\theta} \in \mathbb{R}^{n(n+1)}$.

Proof. It suffices to study the structure of $\tilde{P}_\theta$. From the definition of $\{\tilde{S}_k\}$, we know that for $(\boldsymbol{s}^T \ \boldsymbol{u}^T)^T, (\boldsymbol{x}^T \ \boldsymbol{y}^T)^T \in \mathcal{S}^{2n}$ with $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{y} \in \mathcal{S}^n$, the transition probability from $(\boldsymbol{x}^T \ \boldsymbol{y}^T)^T$ to $(\boldsymbol{s}^T \ \boldsymbol{u}^T)^T$

$$\tilde{P}_\theta((\boldsymbol{x}^T\boldsymbol{y}^T)^T, (\boldsymbol{s}^T\boldsymbol{u}^T)^T)$$

is positive if and only if $\boldsymbol{u} = \boldsymbol{x}$. Moreover, for the column of $(\boldsymbol{s}^T \ \boldsymbol{u}^T)^T$ in $\tilde{P}_\theta$, denoted by $\tilde{P}_\theta(\cdot, (\boldsymbol{s}^T \ \boldsymbol{u}^T)^T)$, all of its positive entries are identical to $P_\theta(\boldsymbol{u}, \boldsymbol{s})$.

Note that under Assumption 1, both $\tilde{\pi}_\theta$ and $\tilde{\pi}_{\bar{\theta}}$ are unique. Hence from the definition of stationary distribution, it follows that

$$\tilde{\pi}_\theta \tilde{P}_\theta(\cdot, (\boldsymbol{s}^T\boldsymbol{u}^T)^T) = \tilde{\pi}_\theta((\boldsymbol{s}^T\boldsymbol{u}^T)^T),$$
$$\tilde{\pi}_{\bar{\theta}} \tilde{P}_{\bar{\theta}}(\cdot, (\boldsymbol{s}^T\boldsymbol{u}^T)^T) = \tilde{\pi}_{\bar{\theta}}((\boldsymbol{s}^T\boldsymbol{u}^T)^T).$$

By assumption, $\tilde{\pi}_\theta = \tilde{\pi}_{\bar{\theta}}$, so the above obtained property of $\tilde{P}_\theta$ implies $P_\theta(\boldsymbol{u}, \boldsymbol{s}) = P_{\bar{\theta}}(\boldsymbol{u}, \boldsymbol{s})$. The conclusion follows from the arbitrariness of $\boldsymbol{s}$ and $\boldsymbol{u}$. □

It should be noted that infinitely many aperiodic and irreducible transition probability matrices can define the same stationary distribution. But the above theorem indicates that the stationary distribution of the auxiliary chain is uniquely determined by the transition probability matrix of $\{S_k\}$. This shows the importance of the auxiliary chain and paves the way for the identifiability issue discussed in next subsection.

B. Identifiability

We have shown in the preceding subsection that certain condition ensures diverse information for estimation. Before proposing the estimation algorithm, it is necessary to study identifiability of the parameters in System (3). This is because consistent estimators cannot exist if the parameters are not identifiable.

For System (3), since the only available data is $\{S_k\}$ with Markov property, we define the identifiability from a statistical perspective, following the idea of [42] (also that of [43]), that is, "whether the values of the parameters are uniquely determined by the probability distribution of the model". Hence the identifiability problem here is whether the parameter vector θ can be specified from the finite-dimensional distribution of $\{S_k\}$.

Here two issues need to be clarified.

1. It is known that a discrete-state Markov chain is determined by its initial distribution and transition probability matrix. But in our problem, the data comes from one sample path of the Markov chain, and there is no way to distinguish different initial conditions. Therefore, when discussing identifiability of parameters, it is reasonable to assume that systems with different parameters start with the same initial distribution independent of these parameters. The identifiability problem then reduces to whether the parameters can be uniquely determined by the transition probability matrix of $\{S_k\}$, since the transition probability matrix uniquely define the finite-dimensional distribution under a fixed initial condition.

2. The other issue is that we should fix the distribution of noise when studying identifiability of parameters, otherwise there could be multiple noise distributions generating the same state sequence.

Based on the above discussion, we define the identifiability of parameters as follows by recalling that $\theta = \mathrm{vec}\big\{(A \ \boldsymbol{c})\big\}$ and that P_θ denotes the transition probability matrix of $\{S_k\}$.

Definition 1. *The parameters in System* (3) *are identifiable in* $\mathbb{R}^{n(n+1)}$*, if* $P_\theta = P_{\bar{\theta}}$ *implies* $\theta = \bar{\theta}$ *for all* $\theta, \bar{\theta} \in \mathbb{R}^{n(n+1)}$.

Remark 3. *There are also other types of identifiability definitions in various disciplines, for example, unique system representation of input-output relationships [2], [8]. Here we follow the idea of [42], [43] for global identifiability, with a justification for the current problem. These definitions, however, all demonstrate the same idea that system parameters can be uniquely determined by observations. Also note that it follows from Theorem 3 and Definition 1 that parameters can actually be uniquely determined by the stationary distribution of the auxiliary chain, under Assumption 1. In Section IV we show that under certain conditions the network estimation problem can be transformed to optimizing a function depending on the stationary distribution of* $\{\tilde{S}_k\}$*. This links the identifiability and consistent estimation of parameters together.*

Now we provide a general identifiability condition.

Theorem 4. *(Identifiability)*
Suppose that Assumption 1 holds, then the parameters in System (3) *are identifiable in* $\mathbb{R}^{n(n+1)}$.

Proof. The proof is similar to that of Theorem 3 in the conference version [39], since under the assumption, the cdf of the noise is strictly increasing and thus is invertible. □

In the following, we adopt the assumption of standard Gaussian noise, as in System (1).

Assumption 1′. *(Gaussian noise)*
The sequences of system noise $\{D_{k,i}, k \geq 0\}$*,* $1 \leq i \leq n$*, satisfy that*

(i) the same as (i) of Assumption 1;
(ii) their distributions are standard Gaussian.

Under this assumption the following identifiability result follows from Theorem 4.

Corollary 1. *Suppose that Assumption 1′ holds, then parameters in System* (3) *are identifiable in* $\mathbb{R}^{n(n+1)}$.

It is easy to observe that, when the variance of the Gaussian noise is not fixed, there exist a set of systems that define an identical state sequence:

Proposition 1. *Suppose that Assumption 1′ holds, then the following system has the same transition probability matrix as that of* $\{S_k\}$ *in System* (3)*,*

$$\begin{aligned} \bar{Y}_{k+1} &= \bar{A}S_k + \bar{D}_k, \\ S_k &= \mathcal{Q}(\bar{Y}_k, \bar{\boldsymbol{c}}), \end{aligned} \tag{5}$$

where $\bar{A} = BA$, $\bar{Y}_k = BY_k$, $\bar{D}_k = BD_k$, *and* $\mathcal{Q}(\bar{Y}_k, \bar{\boldsymbol{c}}) = (\mathbb{I}_{[\bar{Y}_{k,1} > \bar{c}_1]}, \dots, \mathbb{I}_{[\bar{Y}_{k,n} > \bar{c}_n]})^T$ *for* $k \geq 0$, *with* $B = \mathrm{diag}(b_1, \dots, b_n)$ *being a diagonal matrix with non-zero diagonal entries, and* $\bar{c}_i = b_i c_i$ *for* $1 \leq i \leq n$.

Remark 4. *If for all* $1 \leq i \leq n$, *there exists* j *such that* $a_{ij} \neq 0$, *then let* $b_i = (|A_i|\mathbf{1}_n)^{-1}$, $1 \leq i \leq n$. *In this way,* $\bar{A}$ *is absolutely stochastic, and* $\{\bar{D}_{k,i},\ 1 \leq i \leq n\}$ *become Gaussian random variables with different variances. In other words, if we assume that* A *is absolutely stochastic, then the variances of* $\bar{D}_{k,i}$ *are unconstrained. Hence the absolutely stochastic condition defines another identification constraint, in addition to those introduced in probit models [33]. Note that the actual estimated entity for each agent* i *is the ratio of* $|A_i|\mathbf{1}_n$ *and the standard deviation of its noise. This conclusion is similar to that in system identification with the presence of binary sensors (e.g. [35]).*

Motivating examples revisited: For System (1), Corollary 1 and discussions in Section II-B yield that c_i and a_{ij}, $1 \leq i, j \leq n$, are identifiable. As discussed in Example 1, there is no way to separate the effect of γ_i and t_{ij}. But we are still able to know the relative link strength between agent i and others from $\gamma_i t_{ij}$, $1 \leq i, j \leq n$. Parameters in the utility function in Example 2 are also identifiable under the assumption of Theorem 4, because h_i' and w_{ij}' can be computed from the transformation shown in Section II-B.

IV. Network Estimation

The previous section studies stability of the system and identifiability of parameters, laying a solid foundation for further investigation of network estimation. In this section, the network estimation problem is studied under the assumption of standard Gaussian noise. In Section IV-A, we cast the estimation problem into a concave optimization problem (Theorem 5), which is related to the stationary distribution of System (3). The optimization objective function, however, cannot be obtained directly, so an online algorithm is developed based on stochastic approximation (SA) techniques in Section IV-B. Finally, asymptotic properties of the proposed algorithm are studied in Section IV-C, including strong consistency (Theorem 6) and convergence rate (Theorem 7).

A. An Objective Function and Its Concavity

Recall that $\theta = \mathrm{vec}\big\{(A\ \boldsymbol{c})\big\} \in \mathbb{R}^{n(n+1)}$ is the parameter vector to be estimated, and further denote $\theta^{(i)} := (A_i\ c_i)^T \in \mathbb{R}^{n+1}$. To avoid ambiguity, we use $\theta^* := \mathrm{vec}\big\{(A^*\ \boldsymbol{c}^*)\big\} = (((\theta^*)^{(1)})^T, \dots, ((\theta^*)^{(n)})^T)^T$ to represent the true parameter vector. Given data $\{\boldsymbol{s}^k, 0 \leq k \leq T\}$, the log likelihood function is

$$\begin{aligned} & l(T;\theta) \\ &= \log P\{S_k = \boldsymbol{s}^k, 0 \leq k \leq T | \theta\} \\ &= \log P\{S_0 = \boldsymbol{s}^0\} + \sum_{k=1}^{T} \log P\{S_k = \boldsymbol{s}^k | S_{k-1} = \boldsymbol{s}^{k-1}, \theta\} \\ &= \log P\{S_0 = \boldsymbol{s}^0\} + \sum_{k=1}^{T}\sum_{i=1}^{n} \log g_i(\tilde{\boldsymbol{s}}^k | \theta^{(i)}), \end{aligned} \tag{6}$$

where $(\tilde{\boldsymbol{s}}^k)^T := ((\boldsymbol{s}^k)^T (\boldsymbol{s}^{k-1})^T)$, and

$$g_i(\tilde{\boldsymbol{s}}|\theta^{(i)}) := (1 - \Phi(c_i - A_i \tilde{\boldsymbol{s}}_{n+1:2n}))^{\tilde{s}_i} \Phi(c_i - A_i \tilde{\boldsymbol{s}}_{n+1:2n})^{1-\tilde{s}_i}, \tag{7}$$

for $\tilde{\boldsymbol{s}} \in \mathcal{S}^{2n}$ and $1 \leq i \leq n$. Here $\Phi(x)$ represents the cdf of the standard Gaussian variable.

For fixed θ, $g_i(\tilde{\boldsymbol{s}}|\theta^{(i)})$ is bounded since $\tilde{\boldsymbol{s}}$ takes values in $\mathcal{S}^{2n}$. Thus, by ergodic properties of Markov chains (Theorem 17.1.7 in [40]), the following equation holds for the chain $\{\tilde{S}_k\}$ and fixed θ a.s.:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T}\sum_{i=1}^{n} \log g_i(\tilde{S}_k|\theta^{(i)}) = E\Big\{ \sum_{i=1}^{n} \log g_i(\tilde{S}|\theta^{(i)}) \Big\},$$

where $\tilde{S}$ is subject to the stationary distribution of $\{\tilde{S}_k\}$.

Therefore, to solve the online estimation problem, we consider the function of θ

$$E\Big\{ \sum_{i=1}^{n} \log g_i(\tilde{S}|\theta^{(i)}) \Big\} \tag{8}$$

as an objective function. It has the following property:

Theorem 5. *(Strict concavity of* (8)*)*
Under Assumption 1′, the function (8) *of* θ *is strictly concave over* $\mathbb{R}^{n(n+1)}$, *and the true parameter vector* θ^* *is its unique maximum point.*

Proof. See Appendix B. □

Remark 5. *This theorem is the key to establish the consistent estimation of the weighted adjacency matrix* A^*, *since it shows that* $\theta^* = \mathrm{vec}\big\{(A^*\ \boldsymbol{c}^*)\big\}$ *can be obtained by optimizing* (8).

Therefore, our estimation task turns to seeking the unique maximum point of this function. Although $\tilde{S}$ cannot be directly obtained, the observations $\{\tilde{S}_k\}$ can be used. We introduce an SA algorithm in next subsection, and verify that the weighted adjacency matrix and the quantization thresholds can indeed be estimated out of the state sequence $\{S_k\}$.

B. Network Estimation Algorithm

We use an SA algorithm to deal with the estimation problem. For $1 \leq i \leq n$, denote

$$K_i(\theta^{(i)}, \tilde{\boldsymbol{s}}) := \nabla_{\theta^{(i)}} \log g_i(\tilde{\boldsymbol{s}}|\theta^{(i)}), \tag{9}$$

$$K(\theta, \tilde{\boldsymbol{s}}) := (K_1(\theta^{(1)}, \tilde{\boldsymbol{s}}), \ldots, K_n(\theta^{(n)}, \tilde{\boldsymbol{s}}))^T, \tag{10}$$

where $\theta = ((\theta^{(1)})^T, \ldots, (\theta^{(n)})^T)^T$, $\tilde{\boldsymbol{s}} \in \mathcal{S}^{2n}$, and $g_i(\cdot)$ is defined in (7).

The estimation algorithm is as follows:

$$\begin{aligned} \theta_{k+1} &= (\theta_k + a_k K(\theta_k, \tilde{S}_{k+1}))\mathbb{I}_{[\|\theta_k + a_k K(\theta_k, \tilde{S}_{k+1})\| \leq M_{\sigma_k}]}, \\ \sigma_k &= \sum_{i=1}^{k-1} \mathbb{I}_{[\|\theta_i + a_i K(\theta_i, \tilde{S}_{i+1})\| > M_{\sigma_i}]}, \end{aligned} \tag{11}$$

where $\theta_k = ((\theta_k^{(1)})^T, \ldots, (\theta_k^{(n)})^T)^T$ is the estimate of θ^* at time step k, a_k is the step size, $\{M_k\}$ is a sequence of positive numbers increasingly diverging to $+\infty$, and $\sigma_0 = \sigma_1 = 0$. The algorithm starts with a fixed initial vector θ_0.

Remark 6. *In Algorithm* (11)*, the truncation term (the indicator function) ensures the stability of the algorithm. This term is not necessary and can be removed if the algorithm without truncation is naturally bounded a.s.*

C. Asymptotic Properties

In this subsection we provide results on asymptotic properties of Algorithm (11), including strong consistency and convergence rate. First, we introduce the following step size condition, which is standard for SA algorithms.

Assumption 2. *Let a_k be the step size in Algorithm* (11)*, satisfying $a_k > 0$, $\sum_{k=1}^{\infty} a_k = \infty$, and $\sum_{k=1}^{\infty} a_k^2 < \infty$.*

Under Assumptions $1'$ and 2, we have the following strong consistency result, indicating that Algorithm (11) converges to the true parameter vector θ^*.

Theorem 6. *(Strong consistency)*
Suppose that Assumptions $1'$ and 2 hold, then the estimate θ_k in Algorithm (11) *converges to θ^* a.s., that is,*

$$P\Big\{ \lim_{k\to\infty} \theta_k = \theta^* \Big\} = 1,$$

from any fixed initial vector θ_0.

Proof. See Appendix C. □

Remark 7. *Theorem 6 establishes a theoretical guarantee for Algorithm* (11)*, showing that the weighted adjacency matrix and the quantization thresholds can be estimated under the independent standard Gaussian assumption.*

For convergence rate, we prove that by choosing an appropriate step size, our proposed algorithm can have a convergence rate arbitrarily close to $O(1/\sqrt{k})$ a.s. Three hyperparameters are given in the step size, which can be tuned to promote the performance of the algorithm in practice.

Assumption $2'$. *Let a_k be the step size in Algorithm* (11)*, satisfying $a_k = \frac{a}{k^{1-\beta}+\gamma}$ with $a, \gamma > 0$ and $\beta \in [0, 1/2)$.*

Theorem 7. *(Convergence rate)*
Suppose that Assumptions $1'$ and $2'$ hold.
If $\beta > 0$, then for θ_k in Algorithm (11)*,*

$$\|\theta_k - \theta^*\| = o(k^{-\delta}),\ \forall \delta \in \Big(0, \frac{1}{2} - \beta\Big),\ \textit{a.s.} \tag{12}$$

If $\beta = 0$, then

$$\|\theta_k - \theta^*\| = o(k^{-\delta}),\ \forall \delta \in (0, \delta'),\ \textit{a.s.,} \tag{13}$$

where $\delta' = (-a\lambda_{\max}(F)) \wedge \frac{1}{2}$, F is the Hessian of objective function (8) *at θ^*, and $\lambda_{\max}(F)$ is the largest eigenvalue of F.*

Proof. See Appendix D. □

Remark 8. *Theorem 7 further characterizes the performance of Algorithm* (11)*, whose convergence rate can be arbitrarily close to $O(1/\sqrt{k})$, the upper bound of SA algorithms [44]. In realistic scenarios one may encounter large-scale networks, so scalability of an algorithm is important. From the explicit expression of the gradient function* (9) *(given by* (18) *and* (19) *in Appendix B), the vector inner product $A_i S$ only needs to be computed once for agent i at each time with complexity $O(n)$, where n is the number of agents and $1 \leq i \leq n$. So to estimate $\theta^{(i)}$, a number of $O(n)$ operations are needed at each time step. Since there are n agents in the network, we know that the computational complexity of the proposed algorithm at each time is $O(n^2)$. Note that System* (3) *has $O(n^2)$ parameters to be estimated; this order is the same as the computational complexity. For large-scale networks, it is better to take sparsity into consideration, or learn low-dimensional features of a network such as communities.*

There are occasions where the entries of A are integers. For example, A is the adjacency matrix with $a_{ij} \in \{0, 1\}$. The consistency of Algorithm (11) means that under the assumptions of Theorem 6, the proposed algorithm converges to the true integer-valued parameters a.s. If there is prior knowledge about integer-valued parameters, a rounding operation can help to further improve the performance of the algorithm.

Corollary 2. *(Estimation of integer-valued parameters)*
Suppose that Assumptions $1'$ and 2 hold, and θ^* is an integer-valued vector. Then $[\theta_k] \to \theta^*$ a.s., where θ_k is given by Algorithm (11) *and $[\cdot]$ is the entry-wise rounding function defined in Section I-D. Moreover, this convergence happens in finite time, i.e., there exists a positive integer-valued random variable K such that $[\theta_k] = \theta^*$ for all $k > K$.*

Proof. From Theorem 6, there exists a positive integer-valued random variable K such that $\|\theta_k - \theta^*\| < 1/2$ for all $k > K$. Hence $[\theta_k] = \theta^*$ for all $k > K$. □

V. Numerical Simulations

In this section, we first demonstrate the asymptotic properties of the proposed algorithm, and then show how to deal with the case where parameters are integers. After that we compare our algorithm with other approaches. Finally, we investigate the performance of the algorithm under three kinds of unmodeled factors.

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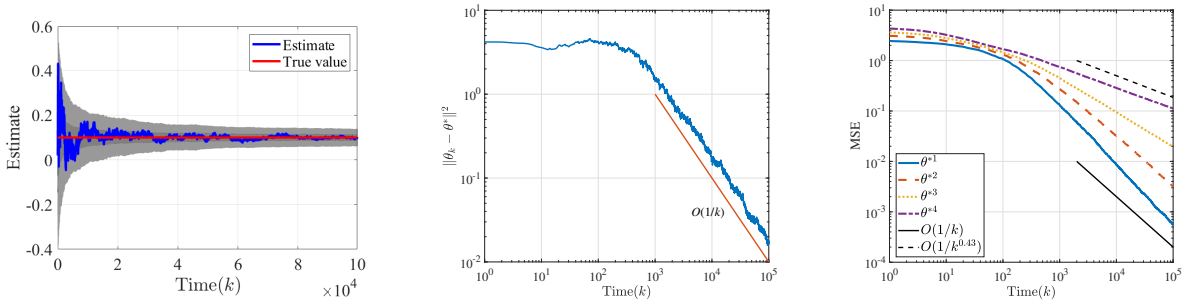


(a) Consistency of Algorithm (11).

(b) Convergence rate of Algorithm (11) with a nonzero β.

(c) Convergence rate of Algorithm (11) with $\beta = 0$ under parameter vectors θ^{*i}, $1 \le i \le 4$.

Fig. 2. Consistency and convergence rate of Algorithm (11).

A. Consistency and Convergence Rate

This subsection illustrates asymptotic properties of Algorithm (11). First, we consider System (1). We set $n = 10$ and $\gamma_i = 1/2$, $1 \le i \le n$, randomly generate $\boldsymbol{c}$ from uniform distribution on $(-1, 1)$, and generate $T = (t_{ij})$ according to the Hebbian learning rule [14]

$$T = \sum_{i=1}^{L} \boldsymbol{p}_i \boldsymbol{p}_i^T - LI_n,$$

where the entries of $\boldsymbol{p}_i$ are randomly chosen with equal probability from the set $\{1, -1\}$, and L is learning times. The explicit meaning of this learning rule is that the connection between two neurons increases if they activate at the same time. The term $-LI_n$ ensures that the diagonal entries of T are zero, i.e., no self-loops in the network. We can then obtain A according to Example 1.

In some references, e.g., [14], [22], system states are represented by ± 1. We note that the dynamics with states $\breve{S}_k \in \{1, -1\}$ is equivalent to System (3), by noticing the transformation $\breve{S}_k = 2S_k - \mathbf{1}_n$, $\breve{Y}_k = 2Y_k - A\mathbf{1}_n$, $\breve{D}_k = 2D_k$, and $\breve{\boldsymbol{c}} = 2\boldsymbol{c} - A\mathbf{1}_n$. The matrix A remains unchanged during the transformation. Note that the original model in [22] evolves over continuous time and has asynchronous updates. We can consider the embedded Markov chain, i.e., keeping only time steps when agents update, to get a discrete-time process. It can be verified that the stability results still hold for this case. When an agent updates, we can compute the estimate of the parameters corresponding to this agent by using the proposed algorithm. Knowing the update information is a common assumption for network estimation, e.g., [45].

For Algorithm (11), we set $\beta = 0.00001$, $a = 25$, and $\gamma = 500$, and run 200 times. Consistency and convergence rate are shown in Fig. 2. In Fig. 2(a), the blue line represents the estimates of one entry of A in one sample path, while the red line represents the true value. The dark (light) gray areas demonstrate error bands with one (three) standard deviation(s). This subfigure illustrates the consistency of Algorithm (11). It can be observed from Fig. 2(b) that the convergence rate is close to $O(1/\sqrt{k})$ as the first part of Theorem 7 shows. Using an averaging technique may speed up SA algorithms [44], but we leave the investigation to future work.

Illustration of the convergence rate result (13) in the case of $\beta = 0$ is shown in Fig. 2(c). Since (13) indicates that the convergence rate of Algorithm (11) may depend on system parameters, in this simulation, we consider System (3) with $n = 2$ and four parameter vectors, θ^{*i}, $1 \le i \le 4$, such that

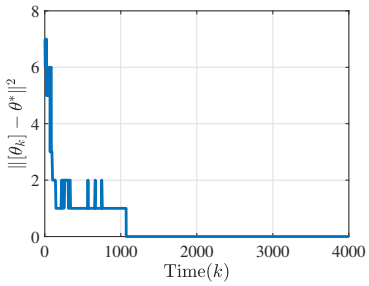
$$\theta^{*1} = (0.87\ 0.13\ 1\ 0.62\ 0.38\ 0.5)^T,$$

and $(\theta^{*i})_k = (\theta^{*1})_k$ for $2 \le i \le 4$ and $k = 1, 2, 4, 5, 6$ but $(\theta^{*2})_3 = 1.3$, $(\theta^{*3})_3 = 1.5$, and $(\theta^{*4})_3 = 1.72$, where $(\theta^{*i})_k$ is the k-th component of θ^{*i}. We run Algorithm (11) 100 times for each case, with $a = 8$ and $\gamma = 100$. The mean square error (MSE) is computed, defined by $\text{MSE}_k := \frac{1}{N}\sum_{i=1}^{N} \|\theta_k^{[i]} - \theta^*\|^2$ with $N = 100$ and $\theta_k^{[i]}$ being the i-th run's estimate at time k. It can be computed that $\lambda_{\max}(F)$ is approximately -0.07, -0.04, -0.02, and -0.01, respectively for each parameter vector. For θ^{*1} we observe the convergence rate is $O(1/\sqrt{k})$ as (13). But for System (3) with θ^{*i}, $i = 2, 3, 4$, the convergence rate slows down, indicating its dependency on parameters. The convergence rate for θ^{*4} is approximately 0.215. This rate is better than the lower bound 0.08 provided by (13), which is conservative. Note that large enough a can ensure the convergence rate to be $O(1/\sqrt{k})$ according to (13).

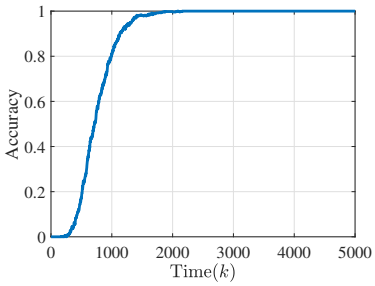
B. Estimating Integer-Valued Parameters

There are circumstances where one may know in advance that the parameters are integers. For example, A is a binary adjacency matrix. In this subsection we show that the proposed algorithm combined with a rounding operation can handle this issue, if it is known in advance that θ^* is an integer-valued vector, under the assumption of standard Gaussian noise. The theoretical result is given in Corollary 2.

We consider System (3) with $n = 2$ and an integer-valued parameter vector θ^*, and run the algorithm 100 times, under standard Gaussian noise. Fig. 3(a) shows the error of the algorithm with rounding operation in one run. It can be seen that the algorithm recovers the integer-valued parameters in finite time. Fig. 3(b) demonstrates the accuracy of the algorithm recovering the true parameters at each time, indicating the accuracy tends to one as $k \to \infty$. The accuracy is defined by $\frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{[[\theta_k^{[i]}]=\theta^*]}$, where $N = 100$ and $\theta_k^{[i]}$ is the estimate of θ^* at time k in the i-th run.

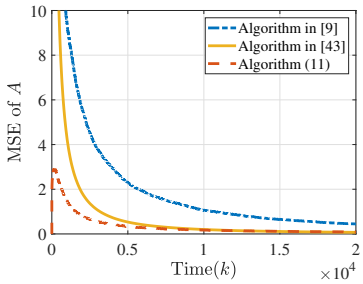


(a) Consistency of the algorithm.

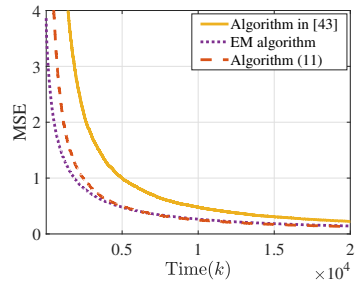


(b) The accuracy of recovering the true parameters at each time.

Fig. 3. Performance of Algorithm (11) combined with the rounding operation when estimating integer-valued parameters.



(a) Performance of the proposed algorithm and those in [9] and [46].



(b) Performance of the proposed algorithm, the EM algorithm, and that in [46].

Fig. 4. Performance comparison.

C. Algorithm Comparison

In this subsection, we apply several other methods to the network estimation problem for performance comparison.

First, we study the performance of two quantized identification algorithms [9], [46]. We note that many quantized identification algorithms require known quantization thresholds, except for the maximum likelihood approach and a method for identifying Wiener systems [47]. However, [47] needs input signals to be independent standard Gaussian, which cannot hold in our case. To implement the algorithms in [9] and [46], we consider System (3) as a collection of n subsystems, S_k as the quantized inputs of these subsystems, and $S_{k+1,i}$ as the output of the i-th subsystem. In addition, we assume $\boldsymbol{c}$ is known in advance. Behavior of the two algorithms shown in Fig. 4(a) indicates that they may be valid under more general situation, though they are designed for identifying FIR systems. The first example in Section V-A is used for illustration. We run each algorithm for 100 times and compute the MSE. All algorithms start with the same initial condition, but the compared algorithms have larger MSE at the beginning due to insufficient data. Since Algorithm (11) estimates both A and $\boldsymbol{c}$, we compare only the MSE of Algorithm (11) estimating A with the other two algorithms in Fig. 4(a).

The projection algorithm in [46] is recursive, with computational complexity $O(n^2)$ at each time, the same as Algorithm (11). From Fig. 4(a), we can see that the algorithm in [46] has slightly smaller MSE than Algorithm (11), but this could result from that it has more information, including the true value of $\boldsymbol{c}$ and the range of the parameters. The algorithm in [9] is a batch one with total computational complexity $O(n^2l^2)$, where $l \geq n$ is the number of considered quantized inputs. At each time step k, we run the algorithm using data $\{S_t, 0 \leq t \leq k\}$. The algorithm does not perform well, because it needs to estimate the probability of $\{A_i\boldsymbol{s}+D_{k,i} \geq c_i\}$ for every state $\boldsymbol{s} \in \mathcal{S}^n$ (considered as an input), $1 \leq i \leq n$. Although we can focus on estimating the above probability for a small number of states which ensure the full rank condition in [9], there could be only a few samples for each $\boldsymbol{s}$. This is because all $\boldsymbol{s} \in \mathcal{S}^n$ appear with positive probability in System (3), and $|\mathcal{S}^n| = 1024$ when $n = 10$. It is unlike the situation in quantized identification where one can impose a small number of inputs. This idea of studying the effect of each input is also introduced in [25], [26], but systems therein are assumed to have only K variables, where $K \ll n$. Thus algorithms in [25], [26] have less computational complexity and sample complexity.

Next, we examine methods estimating both A and $\boldsymbol{c}$. In fact, by considering the following equivalent form of System (3), we can regard $\boldsymbol{c}$ as a linear term during estimation,

$$Y_{k+1}^{\ddagger} = AS_k - \text{diag}(\boldsymbol{c}) + D_k,$$
$$S_k = \mathcal{Q}(Y_k^{\ddagger}, \mathbf{0}_n),$$

where $\text{diag}(\boldsymbol{c})$ is the diagonal matrix with entries of $\boldsymbol{c}$ on the diagonal. In this way, we can apply the algorithm in [46] by taking $(S_k^T \ -1)^T$ as the input. We also test the EM algorithm, since the unobserved response $Y_k^{\ddagger}$ can be seen as missing data. It turns out that the maximum likelihood estimator has the form of the solution to a least-squares problem, and thus for online estimation, we can rewrite the EM algorithm into the form of recursive least-squares [10], [35]. We run each algorithm for 100 times, and Fig. 4(b) shows the result. We find that in this case the algorithm in [46] has larger MSE than Algorithm (11). The EM algorithm has smaller MSE when the

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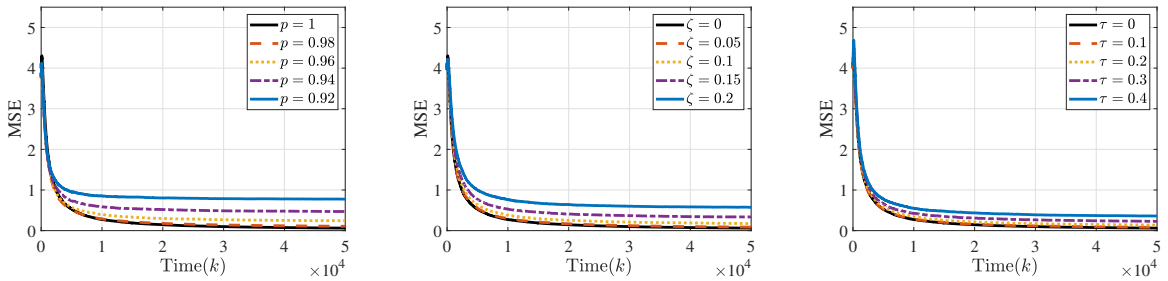

(a) The asynchronous update scenario. (b) The occasional noise scenario. (c) The random communication scenario.

Fig. 5. Performance of Algorithm (11) under unmodeled factors.

data size is small, but its computational complexity is $O(n^3)$ at each time step.

D. Performance Under Unmodeled Factors

We now study the influence of three possible unmodeled factors on the performance of our algorithm, and consider the first example in Section V-A.

The first unmodeled factor is that agents may update asynchronously. For simplicity, suppose that at each time agents update mutually independently and with probability $p \in (0, 1]$. When $p = 1$, this is System (3). For different p, we compute the MSE with $N = 100$ runs. The result is shown in Fig. 5(a), and it can be found that the proposed algorithm performs well when p is close to one. This indicates that estimating adjacency matrix without update information is a tricky task, since one cannot know when an update happens with the presence of random noise. But if update information is known, then the proposed algorithm still works, as discussed in Section V-A.

The second factor is that noise occurs less frequently. More specifically, for $\zeta \in [0, 1)$, the true dynamic of Y_k is

$$Y_{k+1} = \begin{cases} AS_k & \text{with probability } \zeta, \\ AS_k + D_{k,i} & \text{with probability } 1 - \zeta. \end{cases}$$

In other words, the agents are affected by noise occasionally. When ζ is large, noise behaves more like a step or pulse signal [8]. The MSEs with $N = 100$ for different ζ are illustrated in Fig. 5(b). The system is the same as (3) when $\zeta = 0$. As ζ grows larger, the error of the algorithm increases, but the MSE remains small for ζ around 0.1.

The final scenario considered here is that the network is time-varying because of environment randomness or communication outages. Let $\{u_{k,ij}, k \geq 0\}$, $1 \leq i, j \leq n$, be mutually independent sequences of i.i.d. Bernoulli random variables with mean $1 - \tau$. The true dynamic of Y_k is given below

$$Y_{k+1,i} = \sum_{j \in \mathcal{V}} a_{ij} u_{k,ij} S_{k,j} + D_{k,i},$$

which means that agent i does not receive the state of agent j with probability τ. The result is demonstrated in Fig. 5(c). The case of $\tau = 0$ represents the original system. The estimation error becomes greater for larger τ, but the algorithm performs better than the two former cases. To sum up, our algorithm is insensitive to small unmodeled factors.

VI. Conclusion and Future Work

In this paper we studied a recursive network estimation problem based on binary data. Stability of the model and identifiability of the parameters were studied. In particular, we leveraged a maximum likelihood approach to address the network estimation problem for the model with real-valued weighted adjacency matrix under standard Gaussian noise, and proposed a recursive algorithm based on stochastic approximation techniques. The strong consistency of the algorithm was verified, and its convergence rate analyzed.

Future work includes investigation of asymptotic efficiency of the algorithm, as well as generalization of the model and noise conditions. Besides, the case with discrete noise is quite different from the current case with continuous noise, so it is worth studying in depth. In addition, network estimation problems based on partial observations or prior information about edge cardinality are also interesting.

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Appendix A
Proof of Lemma 1

Let $\tilde{P}$ be the transition probability matrix of $\{\tilde{S}_k\}$. From the definition of stationary distribution, we have that for $\tilde{s} \in \mathcal{S}^{2n}$

$$P\{\tilde{S} = \tilde{\boldsymbol{s}}\} = \sum_{\tilde{\boldsymbol{u}} \in \mathcal{S}^{2n}} P\{\tilde{S} = \tilde{\boldsymbol{u}}\} \tilde{P}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{s}}).$$

Define $\mathscr{S}_F := \{\tilde{\boldsymbol{u}} \in \mathcal{S}^{2n} : \tilde{\boldsymbol{u}}_{1:n} = \tilde{\boldsymbol{s}}_{n+1:2n}\}$, and it follows from the definition of $\{\tilde{S}_k\}$ that $\tilde{P}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{s}}) = 0$ for $\tilde{\boldsymbol{u}} \notin \mathscr{S}_F$. Hence,

$$P\{\tilde{S} = \tilde{\boldsymbol{s}}\} = \sum_{\tilde{\boldsymbol{u}} \in \mathscr{S}_F} P\{\tilde{S} = \tilde{\boldsymbol{u}}\} \tilde{P}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{s}}). \tag{14}$$

Similarly, we have that

$$\begin{aligned} &P\{\tilde{S}_{n+1:2n} = \tilde{\boldsymbol{s}}_{n+1:2n}\} \\ &= \sum_{\tilde{\boldsymbol{v}} \in \mathscr{S}_L} P\{\tilde{S} = \tilde{\boldsymbol{v}}\} = \sum_{\tilde{\boldsymbol{u}} \in \mathscr{S}_F} \sum_{\tilde{\boldsymbol{v}} \in \mathscr{S}_L} P\{\tilde{S} = \tilde{\boldsymbol{u}}\} \tilde{P}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}), \end{aligned} \tag{15}$$

where $\mathscr{S}_L := \{\tilde{\boldsymbol{v}} \in \mathcal{S}^{2n} : \tilde{\boldsymbol{v}}_{n+1:2n} = \tilde{\boldsymbol{s}}_{n+1:2n}\}$. Combining (4) with (14) and (15), respectively, it holds that

$$P\{\tilde{S} = \tilde{\boldsymbol{s}}\} = \sum_{\tilde{\boldsymbol{u}} \in \mathscr{S}_F} P\{\tilde{S} = \tilde{\boldsymbol{u}}\} P(\tilde{\boldsymbol{s}}_{n+1:2n}, \tilde{\boldsymbol{s}}_{1:n}),$$

$$
\begin{aligned}
&P\{\tilde{S}_{n+1:2n}=\tilde{\boldsymbol{s}}_{n+1:2n}\}\\
&=\sum_{\tilde{\boldsymbol{u}}\in\mathscr{S}_F}\sum_{\tilde{\boldsymbol{v}}\in\mathscr{S}_L}P\{\tilde{S}=\tilde{\boldsymbol{u}}\}P(\tilde{\boldsymbol{s}}_{n+1:2n},\tilde{\boldsymbol{v}}_{1:n})\\
&=\sum_{\tilde{\boldsymbol{u}}\in\mathscr{S}_F}\sum_{\boldsymbol{w}\in\mathcal{S}^n}P\{\tilde{S}=\tilde{\boldsymbol{u}}\}P(\tilde{\boldsymbol{s}}_{n+1:2n},\boldsymbol{w})\\
&=\sum_{\tilde{\boldsymbol{u}}\in\mathscr{S}_F}P\{\tilde{S}=\tilde{\boldsymbol{u}}\}.
\end{aligned}
$$

Hence, the conclusion follows from

$$
\begin{aligned}
&P\{\tilde{S}_{1:n}=\tilde{\boldsymbol{s}}_{1:n}|\tilde{S}_{n+1:2n}=\tilde{\boldsymbol{s}}_{n+1:2n}\}\\
&=\frac{P\{\tilde{S}=\tilde{\boldsymbol{s}}\}}{P\{\tilde{S}_{n+1:2n}=\tilde{\boldsymbol{s}}_{n+1:2n}\}}\\
&=\frac{\sum_{\tilde{\boldsymbol{u}}\in\mathscr{S}_F}P\{\tilde{S}=\tilde{\boldsymbol{u}}\}P(\tilde{\boldsymbol{s}}_{n+1:2n},\tilde{\boldsymbol{s}}_{1:n})}{\sum_{\tilde{\boldsymbol{u}}\in\mathscr{S}_F}P\{\tilde{S}=\tilde{\boldsymbol{u}}\}}\\
&=P(\tilde{\boldsymbol{s}}_{n+1:2n},\tilde{\boldsymbol{s}}_{1:n}).
\end{aligned}
$$

□

APPENDIX B
PROOF OF THEOREM 5

We first introduce the following property of the standard Gaussian distribution.

Lemma 2.
It holds that $G(x):=\dfrac{x\phi(x)\Phi(x)+\phi^2(x)}{\Phi^2(x)}\in(0,C)$, *for* $x\in\mathbb{R}$, *where* C *is a positive constant, and* $\phi(x)$ *(*$\Phi(x)$*) represents the probability density (cumulative distribution) function of the standard Gaussian random variable.*

Proof. For $x\geq 0$, $G(x)>0$ by definition. For $x<0$, from the inequality (Lemma 2.3.3 in [48]) $\Phi(x)<-\phi(x)/x$, $x<0$, it holds that $x\phi(x)\Phi(x)+\phi^2(x)>-\phi^2(x)+\phi^2(x)=0$, and hence $G(x)>0$. To prove that $G(x)$ has an upper bound, it suffices to note that $\lim_{x\to+\infty}G(x)=0$ and $\lim_{x\to-\infty}G(x)=1$. The latter follows from the L'Hôpital's rule and $\phi'(x)=-x\phi(x)$. □

Proof of Theorem 5*:* We divide the proof into three steps.
Step 1. We prove the Hessian of $E\{\log g_i(\tilde{S}|\theta^{(i)})\}$ is negative definite over $\mathbb{R}^{n+1}$, where $\theta^{(i)}=(A_i\ c_i)^T$.

Note that $\tilde{S}$ takes only finite values, so for $1\leq i\leq n$,

$$\nabla_{\theta^{(i)}}E\{\log g_i(\tilde{S}|\theta^{(i)})\}=E\{\nabla_{\theta^{(i)}}\log g_i(\tilde{S}|\theta^{(i)})\},\quad(16)$$

$$\nabla^2_{\theta^{(i)}}E\{\log g_i(\tilde{S}|\theta^{(i)})\}=E\{\nabla^2_{\theta^{(i)}}\log g_i(\tilde{S}|\theta^{(i)})\}.\quad(17)$$

The gradient of $\log g_i(\tilde{S}|\theta^{(i)})$ is given by

$$
\begin{aligned}
&\frac{\partial}{\partial a_{ij}}\log g_i(\tilde{S}|\theta^{(i)})\\
&=\frac{\partial}{\partial a_{ij}}\log\left[(1-\Phi(c_i-A_iS))^{\tilde{S}_i}\Phi(c_i-A_iS)^{1-\tilde{S}_i}\right]\\
&=\tilde{S}_{n+j}\left(\frac{\tilde{S}_i\phi(c_i-A_iS)}{1-\Phi(c_i-A_iS)}-\frac{(1-\tilde{S}_i)\phi(c_i-A_iS)}{\Phi(c_i-A_iS)}\right),\quad(18)\\
&\frac{\partial}{\partial c_i}\log g_i(\tilde{S}|\theta^{(i)})\\
&=\frac{\partial}{\partial c_i}\log\left[(1-\Phi(c_i-A_iS))^{\tilde{S}_i}\Phi(c_i-A_iS)^{1-\tilde{S}_i}\right]\\
&=-\frac{\tilde{S}_i\phi(c_i-A_iS)}{1-\Phi(c_i-A_iS)}+\frac{(1-\tilde{S}_i)\phi(c_i-A_iS)}{\Phi(c_i-A_iS)},\quad(19)
\end{aligned}
$$

where we denote $S:=\tilde{S}_{n+1:2n}$ for simplicity. The Hessian of $\log g_i(\tilde{S}|\theta^{(i)})$ is given by

$$
\begin{aligned}
&\frac{\partial^2}{\partial a_{ij}\partial a_{ik}}\log g_i(\tilde{S}|\theta^{(i)})\\
&=-\tilde{S}_{n+j}\tilde{S}_{n+k}\Big\{\frac{\tilde{S}_i\phi(c_i-A_iS)}{(1-\Phi(c_i-A_iS))^2}\\
&\quad\cdot\big[\phi(c_i-A_iS)-(c_i-A_iS)(1-\Phi(c_i-A_iS))\big]\\
&\quad+\frac{(1-\tilde{S}_i)\phi(c_i-A_iS)}{\Phi^2(c_i-A_iS)}\\
&\quad\cdot\big[(c_i-A_iS)\Phi(c_i-A_iS)+\phi(c_i-A_iS)\big]\Big\}\\
&=-\tilde{S}_{n+j}\tilde{S}_{n+k}[\tilde{S}_iG(A_iS-c_i)+(1-\tilde{S}_i)G(c_i-A_iS)]\\
&=-\tilde{S}_{n+j}\tilde{S}_{n+k}\bar{G}_i(\theta^{(i)}),\quad(20)\\
&\frac{\partial^2}{\partial a_{ij}\partial c_i}\log g_i(\tilde{S}|\theta^{(i)})\\
&=\tilde{S}_{n+j}[\tilde{S}_iG(A_iS-c_i)+(1-\tilde{S}_i)G(c_i-A_iS)]\\
&=\tilde{S}_{n+j}\bar{G}_i(\theta^{(i)}),\quad(21)\\
&\frac{\partial^2}{\partial^2 c_i}\log g_i(\tilde{S}|\theta^{(i)})=-\bar{G}_i(\theta^{(i)}),\quad(22)
\end{aligned}
$$

where $1\leq i,j,k\leq n$, $G(x)=(x\phi(x)\Phi(x)+\phi^2(x))/\Phi^2(x)$, and $\bar{G}_i(\theta^{(i)}):=[\tilde{S}_iG(A_iS-c_i)+(1-\tilde{S}_i)G(c_i-A_iS)]$. Lemma 2 and the boundedness of $\tilde{S}$ indicate that $\bar{G}_i(\theta^{(i)})\geq M(\theta^{(i)})>0$ for all $\theta^{(i)}\in\mathbb{R}^{n+1}$, where $M(\theta^{(i)})$ is a positive constant depending only on $\theta^{(i)}$.

Hence, from (17), (20), (21), and (22), we have that

$$
\begin{aligned}
&\nabla^2_{\theta^{(i)}}E\{\log g_i(\tilde{S}|\theta^{(i)})\}\\
&=E\left\{-\bar{G}_i(\theta^{(i)})\begin{pmatrix}\tilde{S}_{n+1:2n}\\-1\end{pmatrix}\begin{pmatrix}\tilde{S}^T_{n+1:2n}&-1\end{pmatrix}\right\}\\
&\leq E\left\{-M(\theta^{(i)})\begin{pmatrix}\tilde{S}_{n+1:2n}\\-1\end{pmatrix}\begin{pmatrix}\tilde{S}^T_{n+1:2n}&-1\end{pmatrix}\right\}\\
&=-M(\theta^{(i)})E\left\{\begin{pmatrix}\tilde{S}_{n+1:2n}\\-1\end{pmatrix}\begin{pmatrix}\tilde{S}^T_{n+1:2n}&-1\end{pmatrix}\right\}.
\end{aligned}
$$

We know from Theorem 2 that $P\{\tilde{S}_{n+1:2n}=\mathbf{0}_n\}>0$. Thus, $E\{\|(\tilde{S}^T_{n+1:2n}\ \ -1)\boldsymbol{x}\|^2\}\geq x^2_{n+1}P\{\tilde{S}_{n+1:2n}=\mathbf{0}_n\}>0$, for any $\boldsymbol{x}\in\mathbb{R}^{n+1}$ with $x_{n+1}\neq 0$. If $\boldsymbol{x}\in\mathbb{R}^{n+1}$ satisfies $x_{n+1}=0$ but $x_i\neq 0$ for some $1\leq i\leq n$, then similarly $E\{\|(\tilde{S}^T_{n+1:2n}\ \ -1)\boldsymbol{x}\|^2\}\geq x_i^2P\{\tilde{S}_{n+1:2n}=\boldsymbol{e_i}\}>0$.

This fact means that $E\{(\tilde{S}^T_{n+1:2n}\ \ -1)^T(\tilde{S}^T_{n+1:2n}\ \ -1)\}$ is positive definite. Therefore, $\nabla^2_{\theta^{(i)}}E\{\log g_i(\tilde{S}|\theta^{(i)})\}$ is negative definite over $\mathbb{R}^{n+1}$.

Step 2. Note that

$$
\begin{aligned}
&\nabla_\theta E\{\textstyle\sum_{i=1}^n\log g_i(\tilde{S}|\theta^{(i)})\}\\
&=(\nabla_{\theta^{(1)}}E\{\log g_1(\tilde{S}|\theta^{(1)})\},\dots,\nabla_{\theta^{(n)}}E\{\log g_n(\tilde{S}|\theta^{(n)})\})^T,
\end{aligned}
$$

and $\nabla^2_\theta E\{\sum_{i=1}^n\log g_i(\tilde{S}|\theta^{(i)})\}$ is a block diagonal matrix with matrices

$$\nabla^2_{\theta^{(1)}}E\{\log g_1(\tilde{S}|\theta^{(1)})\},\dots,\nabla^2_{\theta^{(n)}}E\{\log g_n(\tilde{S}|\theta^{(n)})\}$$

at the diagonal line. So $\nabla^2_\theta E\{\sum_{i=1}^n\log g_i(\tilde{S}|\theta^{(i)})\}$ is negative definite over $\mathbb{R}^{n(n+1)}$, and $E\{\sum_{i=1}^n\log g_i(\tilde{S}|\theta^{(i)})\}$ is strictly concave from Proposition 1.2.6 in [49].

Step 3. Finally, we show that θ^* is a root of equation

$$\nabla_\theta E\{\sum_{i=1}^n \log g_i(\tilde{S}|\theta^{(i)})\} = \mathbf{0}_{n(n+1)}.$$

From Step 2, it suffices to show that for $1 \le i \le n$, $(\theta^*)^{(i)} = (A_i^* \ c_i^*)^T$ is a root of equation

$$\nabla_{\theta^{(i)}} E\{\log g_i(\tilde{S}|\theta^{(i)})\} = \mathbf{0}_{n+1}.$$

Again denote $S := \tilde{S}_{n+1:2n}$ and compute

$$\begin{aligned}
&E\left\{\frac{\partial}{\partial a_{ij}} \log g_i(\tilde{S}|\theta^{(i)})\right\} \\
&= \sum_{\tilde{\boldsymbol{s}}\in\mathcal{S}^{2n}} P\{\tilde{S}=\tilde{\boldsymbol{s}}\}\frac{\partial}{\partial a_{ij}} \log g_i(\tilde{\boldsymbol{s}}|\theta^{(i)}) \\
&= \sum_{\boldsymbol{s}\in\mathcal{S}^n, \tilde{\boldsymbol{s}}_i\in\mathcal{S}} P\{S=\boldsymbol{s}, \tilde{S}_i=\tilde{\boldsymbol{s}}_i\} \\
&\qquad \cdot \frac{\partial}{\partial a_{ij}} \log\left[(1-\Phi(c_i - A_i\boldsymbol{s}))^{\tilde{\boldsymbol{s}}_i}\Phi(c_i-A_i\boldsymbol{s})^{1-\tilde{\boldsymbol{s}}_i}\right] \\
&= \sum_{\boldsymbol{s}\in\mathcal{S}^n}\sum_{\tilde{\boldsymbol{s}}_i\in\mathcal{S}} P\{S=\boldsymbol{s}\}P\{\tilde{S}_i=\tilde{\boldsymbol{s}}_i|S=\boldsymbol{s}\} \\
&\qquad \cdot \frac{\partial}{\partial a_{ij}} \log\left[(1-\Phi(c_i - A_i\boldsymbol{s}))^{\tilde{\boldsymbol{s}}_i}\Phi(c_i-A_i\boldsymbol{s})^{1-\tilde{\boldsymbol{s}}_i}\right] \\
&= \sum_{\boldsymbol{s}\in\mathcal{S}^n} P\{S=\boldsymbol{s}\}\Bigg[P\{D_{1,i}\ge c_i^* - A_i^*\boldsymbol{s}\}\cdot\frac{\boldsymbol{s}_j\phi(c_i-A_i\boldsymbol{s})}{1-\Phi(c_i-A_i\boldsymbol{s})} \\
&\qquad - P\{D_{1,i} < c_i^* - A_i^*\boldsymbol{s}\}\cdot\frac{\boldsymbol{s}_j\phi(c_i-A_i\boldsymbol{s})}{\Phi(c_i-A_i\boldsymbol{s})}\Bigg] \\
&= \sum_{\boldsymbol{s}\in\mathcal{S}^n} P\{S=\boldsymbol{s}\}\Bigg[(1-\Phi(c_i^* - A_i^*\boldsymbol{s}))\cdot\frac{\boldsymbol{s}_j\phi(c_i-A_i\boldsymbol{s})}{1-\Phi(c_i-A_i\boldsymbol{s})} \\
&\qquad - \Phi(c_i^* - A_i^*\boldsymbol{s})\cdot\frac{\boldsymbol{s}_j\phi(c_i-A_i\boldsymbol{s})}{\Phi(c_i-A_i\boldsymbol{s})}\Bigg],
\end{aligned}$$

for $1 \le j \le n$, where the penultimate equation follows from Lemma 1 and $\mathcal{S} = \{0,1\}$. The above equation is zero when $\theta^{(i)} = (\theta^*)^{(i)}$. The argument is similar for $E\{\frac{\partial}{\partial c_i}\log g_i(\tilde{S}|\theta^{(i)})\}$. From step 2 and Proposition 2.1.2 in [49], θ^* is the unique global maximum. □

APPENDIX C
PROOF OF THEOREM 6

We need the following conditions to ensure convergence.

A1. $a_k > 0$, $\sum_{k=1}^\infty a_k = \infty$, $\sum_{k=1}^\infty a_k^2 < \infty$.

A2. There is a continuously differentiable function (not necessarily being nonnegative) $v(\cdot): \mathbb{R}^{n(n+1)} \to \mathbb{R}$ such that for $K(\theta) := E\{K(\theta,\tilde{S})\}$, where $\tilde{S}$ is subject to the stationary distribution of $\{\tilde{S}_k\}$, it holds that for any $d_2 > d_1 > 0$,

$$\sup_{d_1 \le d(\theta, J) \le d_2} K^T(\theta) v_\theta(\theta) < 0,$$

where $J := \{\theta \in \mathbb{R}^{n(n+1)} : K(\theta) = \mathbf{0}_{n(n+1)}\}$, $d(\theta, J) = \inf_\eta\{\|\theta-\eta\| : \eta \in J\}$, and v_θ denotes the gradient of v. Also, $v(J) := \{v(\theta) : \theta \in J\}$ is nowhere dense. Moreover, $v(0) < \inf_{\|\theta\|=d_0} v(\theta)$ for some $d_0 > 0$.

A3. $K(\cdot,\cdot)$ is locally Lipschitz-continuous in the first argument, i.e., for any fixed $L > 0$,

$$\|\big(K(\theta,\eta) - K(\kappa,\eta)\big)\mathbb{I}_{[\|\theta\|\le L, \|\kappa\|\le L]}\| \le c_L\|\theta-\kappa\|g(\eta), \quad (23)$$

where c_L is a constant depending on L, and $g(\eta)$ is a measurable function $\mathbb{R}^{2n} \to \mathbb{R}$.

A4. (i) $\{\tilde{S}_k\}$ is a ϕ-mixing process, i.e., for

$$\phi_k := \sup_{n\ge 1} \sup_{A\in\mathcal{F}_1^n, P(A)>0, B\in\mathcal{F}_{n+k}^\infty} \frac{|P(AB)-P(A)P(B)|}{P(A)},$$

$\phi_k \to 0$ as $k \to +\infty$, where $\mathcal{F}_i^j := \sigma(\tilde{S}_k, i \le k \le j)$.

(ii) $\mu^2 := \sup_k E\{(g^2(\tilde{S}_{k+1}) + \|K(0,\tilde{S}_{k+1})\|^2)|\mathcal{F}_1^k\} < \infty$,

$$\lambda^2 := E\{(g^2(\tilde{S}) + \|K(0,\tilde{S})\|^2)\} < \infty,$$

and $E\mu^2 < \infty$, where $g(\cdot)$ is defined in (23).

(iii) $\psi_k := \sup_{A\in\mathcal{B}^m} \big|P(\tilde{S}_k \in A) - P(\tilde{S}\in A)\big| \to 0$, $k \to \infty$, where $\mathcal{B}^m$ is the Borel σ-algebra of $\mathbb{R}^m$.

Under conditions ***A1***-***A4***, Theorem 2.5.1 in [44] ensures $d(\theta_k, J^*) \to 0$ a.s., as $k \to \infty$, where J^* is a connected subset of the closure of J. So to verify strong consistency of Algorithm (11), we validate the conditions ***A1***-***A4***.

Proof of Theorem 6: First note that condition ***A1*** is the same as Assumption 2.

Let $v(\theta) = -E\{\sum_{1\le i\le n}\log g_i(\tilde{S}|\theta^{(i)})\}$, and it is nonnegative by the definition of g_i in (7). From Step 1 in the proof of Theorem 5, $v_\theta(\theta) = -E\{K(\theta,\tilde{S})\} = -E\{\nabla_\theta \sum_{1\le i\le n}\log g_i(\tilde{S}|\theta^{(i)})\}$. Since $\tilde{S}$ takes finite number of values, Assumption 1′ implies that $v_\theta(\theta)$ is continuous.

Also, note that $J = \{K(\theta) = 0\} = \{\theta^*\}$ by Theorem 5, and

$$\sup_{d_1\le d(\theta,J)\le d_2} K^T(\theta)v_\theta(\theta) = -\|E\{K(\theta,\tilde{S})\}\|^2 < 0,$$

for any $d_2 > d_1 > 0$, because θ^* is the only root of $K(\theta)$ from Theorem 5.

From Theorem 2, $\pi_* := \min_{\tilde{s}\in\mathcal{S}^{2n}}\{P\{\tilde{S}=\tilde{s}\}\} > 0$. For the cumulative distribution function of standard Gaussian, $\Phi(\cdot)$, there exist constants $M_1 < 0$ and $M_2 > 0$ such that $\Phi(x) < \exp\{-v(0)/\pi_*\}$ for $x < M_1$ and $1-\Phi(x) < \exp\{-v(0)/\pi_*\}$ for $x > M_2$. Let $M = |M_1| \vee M_2$ and $d_0 = \sqrt{4n^2+n}(M+1)$. Then for $\|\theta_0\| = d_0$, if there exists c_j such that $|c_j| \ge M+1$, then supposing first $c_j \ge M+1 > M$, we have that

$$\begin{aligned}
v(\theta_0) &= -E\{\textstyle\sum_{i=1}^n \log g_i(\tilde{S}|\theta_0^{(i)})\} \\
&\ge -\textstyle\sum_{i=1}^n \log g_i(\tilde{\boldsymbol{u}}|\theta_0^{(i)})P\{\tilde{S}=\tilde{\boldsymbol{u}}\} \\
&\ge -\log g_j(\tilde{\boldsymbol{u}}|\theta_0^{(j)})P\{\tilde{S}=\tilde{\boldsymbol{u}}\} \\
&= -\log(1-\Phi(c_j))P\{\tilde{S}=\tilde{\boldsymbol{u}}\} > v(0),
\end{aligned}$$

where $\tilde{\boldsymbol{u}} \in \mathcal{S}^{2n}$ is a vector with $\tilde{\boldsymbol{u}}_j = 1$ and $\tilde{\boldsymbol{u}}_{n+1:2n} = \mathbf{0}_n$, the first and the second inequalities follow from $-\log g_i(\tilde{S}|\theta_0^{(i)}) \ge 0$, $1 \le i \le n$, and the second equation follows from the definition of g_i and $\tilde{\boldsymbol{u}}$. If $c_j < -(M+1)$, then choose $\tilde{\boldsymbol{u}}$ such that $\tilde{\boldsymbol{u}}_j = 0$ and $\tilde{\boldsymbol{u}}_{n+1:2n} = \mathbf{0}_n$. Hence, $v(\theta_0) \ge -\pi_*\log\Phi(c_j) > v(0)$.

If $|c_i| < M+1$ for all $1 \le i \le n$, then there must exist a_{ij} such that $|a_{ij}| \ge 2(M+1)$. Otherwise, $\|\theta_0\|^2 < n^2 4(M+1)^2 + n(M+1)^2 = d_0^2$. Suppose that $|a_{11}| > 2(M+1)$ for convenience, and as above suppose further that $a_{11} \ge 2(M+1)$. Then $c_1 - a_{11} \le -(M+1) < -M$ since

$|c_i| < M+1$. Thus, selecting a vector $\tilde{\boldsymbol{w}} \in \mathcal{S}^{2n}$ such that $\tilde{\boldsymbol{w}}_j = 0$ and $\tilde{\boldsymbol{w}}_{n+1:2n} = \mathbf{1}_n$, analogously we have that $v(\theta) \geq -\pi_* \log \Phi(c_1 - a_{11}) > v(0)$. Therefore, we have showed that there exists $d_0 > 0$ such that $v(0) < \inf_{\|\theta_0\|=d_0} v(\theta_0)$ and hence validate ***A2***.

Apropos of ***A3***, for $\bar{\theta}$ and $\hat{\theta}$ such that $\|\bar{\theta}\|, \|\hat{\theta}\| \leq L$ with $L > 0$ fixed and $\tilde{\boldsymbol{z}} \in \mathbb{R}^{2n}$,

$$\begin{aligned}
&\|K(\bar{\theta}, \tilde{\boldsymbol{z}}) - K(\hat{\theta}, \tilde{\boldsymbol{z}})\| \leq \sum_{i=1}^{n} \|K_i(\bar{\theta}^{(i)}, \tilde{\boldsymbol{z}}) - K_i(\hat{\theta}^{(i)}, \tilde{\boldsymbol{z}})\| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n+1} \Big\| \frac{\partial}{\partial \theta_j^{(i)}} \log g_i(\tilde{\boldsymbol{z}}|\theta^{(i)})\big|_{\theta^{(i)}=\bar{\theta}^{(i)}} \\
&\qquad\qquad - \frac{\partial}{\partial \theta_j^{(i)}} \log g_i(\tilde{\boldsymbol{z}}|\theta^{(i)})\big|_{\theta^{(i)}=\hat{\theta}^{(i)}} \Big\| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n+1} \Big\| \nabla_{\theta^{(i)}} \big(\frac{\partial}{\partial \theta_j^{(i)}} \log g_i(\tilde{\boldsymbol{z}}|\theta^{(i)}) \big)\big|_{\theta^{(i)}=\tilde{\theta}^{(i)}} \Big\| \\
&\qquad\qquad \cdot \|\bar{\theta}^{(i)} - \hat{\theta}^{(i)}\| \\
&\leq \varepsilon_L h(\tilde{\boldsymbol{z}}) \sum_{i=1}^{n} \|\bar{\theta}^{(i)} - \hat{\theta}^{(i)}\| \\
&\leq \varepsilon_L \cdot \sqrt{n} h(\tilde{\boldsymbol{z}}) \cdot \|\bar{\theta} - \hat{\theta}\| := \varepsilon_L \cdot g(\tilde{\boldsymbol{z}}) \cdot \|\bar{\theta} - \hat{\theta}\|, \qquad (24)
\end{aligned}$$

where the third inequality follows from the mean value theorem, $\tilde{\theta}^{(i)} = (1-\lambda)\bar{\theta}^{(i)} + \lambda\hat{\theta}^{(i)}$ for $1 \leq i \leq n$ and some $\lambda \in (0,1)$, and the fourth inequality can be obtained from the boundedness of (20), (21), and (22) for $\|\theta\| \leq L$, with some bounded function $\varepsilon_L h(\tilde{\boldsymbol{z}})$.

Since $\{\tilde{S}_k\}$ is an aperiodic irreducible finite-state Markov chain from Theorem 2, it is ϕ-mixing [50]. We also have that $g^2(\tilde{S}_{k+1}) + \|K(0, \tilde{S}_{k+1})\|^2$ and $E\{g^2(\tilde{S}) + \|K(0,\tilde{S})\|^2\}$ are bounded because $\tilde{S}_k$ takes value only in $\mathcal{S}^{2n}$. In addition, Theorem 4.9 in [51] and Theorem 2 imply that $\psi_k \to 0$ as $k \to \infty$. Therefore, ***A4*** holds, and the conclusion follows from Theorem 2.5.1 in [44] by noticing that $J = \{\theta^*\}$. □

APPENDIX D
PROOF OF THEOREM 7

Recall $K(\theta) = E\{K(\theta, \tilde{S})\}$, where $K(\theta, \tilde{S})$ is defined in (10) and $\tilde{S}$ is subject to the stationary distribution of $\{\tilde{S}_k\}$. We know from Theorem 5 that $K(\theta)$ has a single root θ^*. In addition, it is differentiable at θ^*, so its Taylor expansion at θ^* is $K(\theta) = F(\theta - \theta^*) + \delta(\theta)$, where $\delta(\theta^*) = 0$ and $\delta(\theta) = o(\|\theta - \theta^*\|)$ as $\theta \to \theta^*$.

Consider the following conditions.

A1'. $a_k > 0$, $a_k \to 0$ as $k \to \infty$, $\sum_{k=1}^{\infty} a_k = \infty$, and

$$\frac{a_k - a_{k+1}}{a_k a_{k+1}} \to \alpha \geq 0, \ k \to \infty. \qquad (25)$$

A3'. $K(\theta)$ is measurable and locally bounded, and is differentiable at θ^* such that as $\theta \to \theta^*$

$$K(\theta) = F(\theta - \theta^*) + \delta(\theta), \ \delta(\theta^*) = 0, \ \delta(\theta) = o(\|\theta - \theta^*\|). \qquad (26)$$

The matrix F is stable (all its eigenvalues are with negative real parts). In addition, $F + \alpha\delta I$ is also stable, where α and δ are given by (25) and (27), respectively.

A4'. For the sample path ω under consideration the observation noise $\varepsilon_k := K(\theta_{k-1}, \tilde{S}_k) - K(\theta_{k-1})$, $k \geq 1$, can be decomposed into two parts $\varepsilon_k = \varepsilon_k' + \varepsilon_k''$ such that

$$\sum_{k=1}^{\infty} a_k^{1-\delta} \varepsilon_{k+1}' < \infty, \ \varepsilon_{k+1}'' = O(a_k^{\delta}), \qquad (27)$$

for some $\delta \in (0, 1]$.

From Appendix C we know that under Assumption 1′, condition ***A2*** holds with $J = \{\theta^*\}$. The following result characterizes convergence rate of Algorithm (11) under the preceding conditions.

Lemma 3. *(Theorem 3.1.1 in [44])*
*Assume **A1'**, **A3'**, and **A4'** hold. In addition, **A2** holds with $J = \{\theta^*\}$. Then for those sample paths for which* (27) *holds, θ_k given by Algorithm* (11) *converges to θ^* with the following convergence rate:*

$$\|\theta_k - \theta^*\| = o(a_k^{\delta}), \qquad (28)$$

where δ is the one given in (27)*.*

We now prove the following auxiliary lemma.

Lemma 4. *For fixed $\theta \in \mathbb{R}^{n(n+1)}$ and $\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}$, the series*

$$\hat{K}(\theta, \tilde{\boldsymbol{z}}) := \sum_{k=0}^{\infty} \Big(\sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} K(\theta, \tilde{\boldsymbol{z}}') \tilde{P}^k(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') - K(\theta) \Big) \qquad (29)$$

converges, and it is a solution of the following Poisson equation

$$K(\theta, \tilde{\boldsymbol{z}}) - K(\theta) = \hat{K}(\theta, \tilde{\boldsymbol{z}}) - \sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \hat{K}(\theta, \tilde{\boldsymbol{z}}') \tilde{P}(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}'), \qquad (30)$$

where $\tilde{P}(\cdot,\cdot)$ and $\tilde{P}^k(\cdot,\cdot)$ are the transition probability matrix and k-step transition probability matrix of $\{\tilde{S}_k\}$, respectively. $\tilde{P}^0(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') = 1$ if $\tilde{\boldsymbol{z}}' = \tilde{\boldsymbol{z}}$, and $\tilde{P}^0(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') = 0$ otherwise.

Proof. Note that

$$\begin{aligned}
&\Big\| \sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} K(\theta, \tilde{\boldsymbol{z}}') \tilde{P}^k(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') - K(\theta) \Big\| \\
&= \Big\| \sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} K(\theta, \tilde{\boldsymbol{z}}') \tilde{P}^k(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') - K(\theta, \tilde{\boldsymbol{z}}') \tilde{\pi}(\tilde{\boldsymbol{z}}') \Big\| \\
&\leq \sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \|K(\theta, \tilde{\boldsymbol{z}}')\| \cdot |\tilde{P}^k(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') - \tilde{\pi}(\tilde{\boldsymbol{z}}')| \\
&\leq \max_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \|K(\theta, \tilde{\boldsymbol{z}}')\| \cdot \sum_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} |\tilde{P}^k(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}') - \tilde{\pi}(\tilde{\boldsymbol{z}}')| \\
&\leq \max_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \|K(\theta, \tilde{\boldsymbol{z}}')\| \cdot C_1 \rho^k,
\end{aligned}$$

where $\tilde{\pi}$ is the stationary distribution of $\{\tilde{S}_k\}$, and the last inequality follows from the convergence theorem of finite-state Markov chains (Theorem 4.9 in [51]) for some $C_1 > 0$, $\rho \in (0,1)$, and any $\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}$. Hence,

$$\begin{aligned}
\|\hat{K}(\theta, \tilde{\boldsymbol{z}})\| &\leq \max_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \|K(\theta, \tilde{\boldsymbol{z}}')\| \cdot \sum_{k=0}^{\infty} C_1 \rho^k \\
&:= C_2 \cdot \max_{\tilde{\boldsymbol{z}}' \in \mathcal{S}^{2n}} \|K(\theta, \tilde{\boldsymbol{z}}')\|, \qquad (31)
\end{aligned}$$

where C_2 is a positive constant. Thus (30) is obtained from

$$\begin{aligned}
&\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}} \hat{K}(\theta,\tilde{\boldsymbol{z}}')\tilde{P}(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')\\
&=\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}''\in\mathcal{S}^{2n}}(K(\theta,\tilde{\boldsymbol{z}}'')\tilde{P}^k(\tilde{\boldsymbol{z}}',\tilde{\boldsymbol{z}}'')-K(\theta))\tilde{P}(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')\\
&=\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}''\in\mathcal{S}^{2n}}\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}(K(\theta,\tilde{\boldsymbol{z}}'')\tilde{P}^k(\tilde{\boldsymbol{z}}',\tilde{\boldsymbol{z}}'')-K(\theta))\tilde{P}(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')\\
&=\sum_{k=0}^{\infty}\Big(\sum_{\tilde{\boldsymbol{z}}''\in\mathcal{S}^{2n}}K(\theta,\tilde{\boldsymbol{z}}'')\tilde{P}^{k+1}(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}'')-K(\theta)\Big)\\
&=\hat{K}(\theta,\tilde{\boldsymbol{z}})-(K(\theta,\tilde{\boldsymbol{z}})-K(\theta)).
\end{aligned}$$

□

Proof of Theorem 7: Note that Assumption 2′ satisfies the conditions in Assumption 2, so under the assumptions of Theorem 7, Algorithm (11) converges a.s. Thus, the algorithm is bounded a.s. That is, $\|\theta_k\|\le L$, $\forall k\ge 0$, where L is a positive random variable.

First note that $\frac{a_k-a_{k+1}}{a_ka_{k+1}}\to 0$ when $\beta\in(0,1/2)$. Thus ***A1'*** holds with $\alpha=0$. In addition, ***A2*** has been verified in the proof of Theorem 6. From the definition of $K(\theta)$, we know that F in (26) is in fact the Hessian of $E\{\sum_{i=1}^n\log g_i(\tilde{S}|\theta^{(i)})\}$ at θ^*. It follows that F is negative definite and consequently stable from the proof of Theorem 5. Thus ***A3'*** holds as $\alpha=0$ for $\beta\in(0,1/2)$.

Now we show that ***A4'*** holds a.s. for $\delta\in(0,1-\frac{1}{2(1-\beta)})$ by using (30) and decomposing the noise into three parts:

$$\begin{aligned}
\varepsilon_k&=K(\theta_{k-1},\tilde{S}_k)-K(\theta_{k-1})\\
&=\hat{K}(\theta_{k-1},\tilde{S}_k)-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})\\
&=I_k^{(1)}+I_k^{(2)}+I_k^{(3)},
\end{aligned}$$

where for $k\ge 2$,

$$\begin{aligned}
I_k^{(1)}&:=\hat{K}(\theta_{k-1},\tilde{S}_k)-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_{k-1},\tilde{\boldsymbol{z}}),\\
I_k^{(2)}&:=\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_{k-1},\tilde{\boldsymbol{z}})\\
&\qquad-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-2},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_{k-1},\tilde{\boldsymbol{z}}),\\
I_k^{(3)}&:=\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-2},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_{k-1},\tilde{\boldsymbol{z}})\\
&\qquad-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}}).
\end{aligned}$$

It follows that $\hat{K}(\theta_k,\tilde{S}_{k+1})I_{[\|\theta_k\|\le N]}$ is bounded a.s. for fixed $N>0$ from Lemma 4. Denote $\mathcal{F}_i^j:=\sigma\{\tilde{S}_k,i\le k\le j\}$, it holds that for $N>0$

$$\begin{aligned}
&E\{\hat{K}(\theta_k,\tilde{S}_{k+1})I_{[\|\theta_k\|\le N]}|\mathcal{F}_1^k\}(\omega)\\
&=\int\hat{K}(\theta_k(\omega),\tilde{\boldsymbol{z}})I_{[\|\theta_k\|\le N]}dF_{k+1}^{\omega}(\tilde{\boldsymbol{z}};\mathcal{F}_1^k)\\
&=\int\hat{K}(\theta_k(\omega),\tilde{\boldsymbol{z}})I_{[\|\theta_k\|\le N]}dF_{k+1}^{\omega}(\tilde{\boldsymbol{z}};\mathcal{F}_k^k)\\
&=\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_k(\omega),\tilde{\boldsymbol{z}})I_{[\|\theta_k\|\le N]}\tilde{P}(\tilde{S}_k(\omega),\tilde{\boldsymbol{z}}),
\end{aligned}$$

where ω is a sample path, $F_{k+1}^{\omega}(\cdot;\mathcal{F}_1^k)$ is the conditional distribution of $\tilde{S}_{k+1}$ given $\mathcal{F}_1^k$, and the second equality follows from the Markov property of $\{\tilde{S}_k\}$. Thus

$$\begin{aligned}
I_k^{(1,N)}&:=\hat{K}(\theta_{k-1},\tilde{S}_k)I_{[\|\theta_{k-1}\|\le N]}\\
&\quad-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})I_{[\|\theta_{k-1}\|\le N]}\tilde{P}(\tilde{S}_{k-1},\tilde{\boldsymbol{z}})
\end{aligned}$$

is a martingale difference sequence for any $N>0$. For $\delta\in(0,1-\frac{1}{2(1-\beta)})$ and $\beta\in(0,1/2)$, it holds that $2(1-\beta)(1-\delta)>1$, so $\sum_{k=1}^{\infty}a_k^{2(1-\delta)}<\infty$ for $a_k=a/(k^{1-\beta}+\gamma)$, and $\sum_{k=1}^{\infty}a_k^{1-\delta}I_{k+1}^{(1,N)}<\infty$ for $N>0$ by Theorem B.6.1 in [44].

As discussed above, from Theorem 6, for a fixed sample path $\omega\in\Omega_0$ with $P(\Omega_0)=1$, $\theta_k(\omega)\to\theta^*$ as $k\to\infty$. So there exists an integer $L=L(\omega)>0$ such that for all $k\ge 0$, $\|\theta_k(\omega)\|\le L$. Hence set $N=L$, we have that for $\delta\in(0,1-\frac{1}{2(1-\beta)})$

$$\sum_{k=1}^{\infty}a_k^{1-\delta}I_{k+1}^{(1)}(\omega)=\sum_{k=1}^{\infty}a_k^{1-\delta}I_{k+1}^{(1,L)}(\omega)<\infty.$$

Therefore, $\sum_{k=1}^{\infty}a_k^{1-\delta}I_{k+1}^{(1)}<\infty$ a.s.

To analyze $I_k^{(2)}$, first we have for $\bar{\theta}$, $\hat{\theta}$ with $\|\bar{\theta}\|,\|\hat{\theta}\|\le L$ and $\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}$,

$$\begin{aligned}
&\|\hat{K}(\bar{\theta},\tilde{\boldsymbol{z}})-\hat{K}(\hat{\theta},\tilde{\boldsymbol{z}})\|\\
&=\Big\|\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}(K(\bar{\theta},\tilde{\boldsymbol{z}}')-K(\hat{\theta},\tilde{\boldsymbol{z}}'))(\tilde{P}^k(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')-\tilde{\pi}(\tilde{\boldsymbol{z}}'))\Big\|\\
&\le\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}\|K(\bar{\theta},\tilde{\boldsymbol{z}}')-K(\hat{\theta},\tilde{\boldsymbol{z}}')\|\cdot|\tilde{P}^k(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')-\tilde{\pi}(\tilde{\boldsymbol{z}}')|\\
&\le\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}\varepsilon_L\cdot g(\tilde{\boldsymbol{z}}')\cdot\|\bar{\theta}-\hat{\theta}\|\cdot|\tilde{P}^k(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')-\tilde{\pi}(\tilde{\boldsymbol{z}}')|\\
&=\varepsilon_L\cdot\max_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}g(\tilde{\boldsymbol{z}})\cdot\|\bar{\theta}-\hat{\theta}\|\cdot\sum_{k=0}^{\infty}\sum_{\tilde{\boldsymbol{z}}'\in\mathcal{S}^{2n}}|\tilde{P}^k(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{z}}')-\tilde{\pi}(\tilde{\boldsymbol{z}}')|\\
&\le\varepsilon_LC_2\max_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}g(\tilde{\boldsymbol{z}})\cdot\|\bar{\theta}-\hat{\theta}\|,
\end{aligned}\tag{32}$$

where the second inequality follows from (24) in the proof of Theorem 6, the last inequality is obtained as (31) in the proof of Lemma 4 with the constant C_2, and $\tilde{\pi}$ is the stationary distribution of $\{\tilde{S}_k\}$.

Hence, for the fixed sample path ω such that $\|\theta_k(\omega)\|\le L$, $\forall k\ge 1$, (we omit ω in the following for simplicity)

$$\begin{aligned}
&\Big\|\sum_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_k,\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})-\sum\nolimits_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})\Big\|\\
&\le\sum_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\|\hat{K}(\theta_k,\tilde{\boldsymbol{z}})I_{[\|\theta_k\|\le L]}-\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})I_{[\|\theta_{k-1}\|\le L]}\|\\
&\qquad\cdot\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})\\
&\le\varepsilon_LC_2\cdot\max_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}g(\tilde{\boldsymbol{z}})\cdot\|\theta_k-\theta_{k-1}\|\cdot\sum_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})\\
&=\varepsilon_LC_2\cdot\max_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}g(\tilde{\boldsymbol{z}})\cdot a_k\|K(\theta_{k-1},\tilde{S}_k)\|\\
&=\varepsilon_LC_2\cdot\max_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}g(\tilde{\boldsymbol{z}})\cdot a_k\|K(\theta_{k-1},\tilde{S}_k)I_{[\|\theta_{k-1}\|\le L]}\|\le C_3a_k,
\end{aligned}$$

where the second inequality follows from (32), and C_3 is a constant depending on ω. Hence, $\|I_{k+1}^{(2)}\|\le C_3a_k$, and $I_{k+1}^{(2)}=O(a_k^{\delta})$ for all $\delta\in(0,1]$.

As for $I_k^{(3)}$, rewrite $\sum_{k=1}^{\infty}a_k^{1-\delta}I_{k+1}^{(3)}$ as

$$\sum_{k=1}^{\infty}\sum_{\tilde{\boldsymbol{z}}\in\mathcal{S}^{2n}}a_k^{1-\delta}\Big(\hat{K}(\theta_{k-1},\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_k,\tilde{\boldsymbol{z}})-\hat{K}(\theta_k,\tilde{\boldsymbol{z}})\tilde{P}(\tilde{S}_{k+1},\tilde{\boldsymbol{z}})\Big)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} a_k^{1-\delta} \hat{K}(\theta_{k-1}, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_k, \tilde{\boldsymbol{z}}) \\
&\quad - \sum_{k=1}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} a_k^{1-\delta} \hat{K}(\theta_k, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_{k+1}, \tilde{\boldsymbol{z}}) \\
&= \sum_{k=0}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} a_{k+1}^{1-\delta} \hat{K}(\theta_k, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_{k+1}, \tilde{\boldsymbol{z}}) \\
&\quad - \sum_{k=1}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} a_k^{1-\delta} \hat{K}(\theta_k, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_{k+1}, \tilde{\boldsymbol{z}}) \\
&= \sum_{k=1}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} (a_{k+1}^{1-\delta} - a_k^{1-\delta}) \hat{K}(\theta_k, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_{k+1}, \tilde{\boldsymbol{z}}) \\
&\quad + a_1^{1-\delta} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} \hat{K}(\theta_0, \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_1, \tilde{\boldsymbol{z}}). \qquad (33)
\end{aligned}
$$

For the fixed sample path ω such that $\|\theta_k(\omega)\| < L$, $\forall k > 0$, the first term in (33) is bounded by

$$
\begin{aligned}
&\left\| \sum_{k=1}^{\infty} \sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} (a_{k+1}^{1-\delta} - a_k^{1-\delta}) \hat{K}(\theta_k(\omega), \tilde{\boldsymbol{z}}) \tilde{P}(\tilde{S}_{k+1}(\omega), \tilde{\boldsymbol{z}}) \right\| \\
&\le \sum_{k=1}^{\infty} |a_k^{1-\delta} - a_{k+1}^{1-\delta}| \left(\sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} \|\hat{K}(\theta_k(\omega), \tilde{\boldsymbol{z}})\| \tilde{P}(\tilde{S}_{k+1}(\omega), \tilde{\boldsymbol{z}}) \right) \\
&= \sum_{k=1}^{\infty} |a_k^{1-\delta} - a_{k+1}^{1-\delta}| \\
&\quad \cdot \left(\sum_{\tilde{\boldsymbol{z}} \in \mathcal{S}^{2n}} \|\hat{K}(\theta_k(\omega), \tilde{\boldsymbol{z}}) \mathbb{I}_{\|\theta_k(\omega)\| \le L} \| \tilde{P}(\tilde{S}_{k+1}(\omega), \tilde{\boldsymbol{z}}) \right) \\
&\le C_4 \sum_{k=1}^{\infty} |a_k^{1-\delta} - a_{k+1}^{1-\delta}| = O\Big(\sum_{k=1}^{\infty} \frac{1}{k^{1+(1-\beta)(1-\delta)}} \Big),
\end{aligned}
$$

where the second inequality follows from (31) in the proof of Lemma 4 for a positive constant C_4, and the last equation is obtained from the following fact. For $\delta \in (0,1)$, $a_k = \frac{a}{k^{1-\beta}+\gamma}$ and $\beta \in (0, 1/2)$, it holds that

$$
\begin{aligned}
&a_k^{1-\delta} - a_{k+1}^{1-\delta} \\
&= a^{1-\delta} \frac{1}{((k+1)^{1-\beta}+\gamma)^{1-\delta}} \Big(\Big(\frac{(k+1)^{1-\beta}+\gamma}{k^{1-\beta}+\gamma} \Big)^{1-\delta} - 1 \Big) \\
&\sim a^{1-\delta} \frac{1}{k^{(1-\beta)(1-\delta)}} \Big(\Big(1 + \frac{1}{k} \Big)^{(1-\beta)(1-\delta)} - 1 \Big) \\
&= O\Big(\frac{1}{k^{1+(1-\beta)(1-\delta)}} \Big),
\end{aligned}
$$

where for two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ with $\beta_k \neq 0$, $k \ge 1$, $\alpha_k \sim \beta_k$ means that $\lim_{k\to\infty} \alpha_k/\beta_k = 1$. The second term of (33) is finite because the algorithm starts with a fixed initial state θ_0.

To sum up, we have shown that $\sum_{k=1}^{\infty} a_k^{1-\delta} \varepsilon'_{k+1} < \infty$ a.s., where $\varepsilon'_k := I_k^{(1)} + I_k^{(3)}$, for $\delta \in (0, 1 - \frac{1}{2(1-\beta)})$ and $\beta \in (0, \frac{1}{2})$. Also, $\varepsilon''_{k+1} := I_{k+1}^{(2)} = O(a_k^{\delta})$ for all $\delta \in (0,1]$. By Lemma 3, $\|\theta_k - \theta^*\| = o(a_k^{\delta}) = O(k^{-\eta})$, $\eta = (1-\beta)\delta$. The conclusion follows from $\eta = (1-\beta)\delta \in (0, \frac{1}{2} - \beta)$ for $\delta \in (0, 1 - \frac{1}{2(1-\beta)})$.

When $\beta = 0$, $\frac{a_k - a_{k+1}}{a_k a_{k+1}} \to \frac{1}{a} = \alpha$. Similar to the above argument, we know that ***A4'*** holds a.s. for $\delta \in (0, 1/2)$. According to ***A3'***, $F + \alpha\delta I$ has to be stable. But the maximum eigenvalue of F depends on the parameter vector θ^*. Nevertheless, from the negative definiteness of F, there exists $\delta' \in (0, 1/2]$ such that $F + \alpha\delta I < 0$ for all $\delta \in (0, \delta')$. In fact, $\delta' = (-a\lambda_{\max}(F)) \wedge (1/2)$, where $\lambda_{\max}(F)$ is the largest eigenvalue of F. So for $\delta \in (0, \delta')$ we have that $\|\theta_k - \theta^*\| = o(k^{-\delta})$. □

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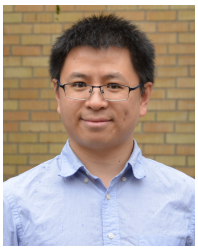

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