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Event-Triggered Model Predictive Control of Discrete-Time Linear Systems Subject to Disturbances

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Abstract— **This paper presents an approach to event-triggered model predictive control for discrete-time linear systems subject to input and state constraints as well as exogenous disturbances. Stability properties are derived by evaluating the difference between the event-triggered implementation and the conventional time-triggered scheme. It is shown that the event-triggered implementation, in stationarity, is able to keep the state in an explicitly computable set given by the disturbance bound and the event threshold. Simulation results underline the effectiveness of the proposed scheme in terms of reducing the communication and computational effort while guaranteeing a desired performance.**

I. Introduction

Model predictive control (MPC) is a control scheme which at every sampling instant solves a finite horizon open-loop optimal control problem and applies the first part of the optimal input trajectory. By considering input and/or state constraints in the on-line optimization this procedure provides a well suited method for overcoming the potential performance degradation or stability problems imposed by these constraints [6], [19], [21], [24].

Considering networked control systems (NCS) [3] the use of predictive controllers has recently gained popularity [7], [16], [27], [33]. However, conventional time-triggered MPC is known to be computationally demanding and requires a periodic sampling. This is conflicting with the communication limitations and use of low-energy components in wireless networks [32].

A sampling scheme that solves these problems is given by event-triggered control. Event-triggered control aims at reducing the communication in NCS by only sending information if certain event conditions are satisfied [1], [2], [12], [18], [31]. As an unsatisfied event condition indicates that the actual plant behavior sufficiently coincides with a desired behavior, it also indicates that a re-calculation of the control input is not needed. Hence, the computational effort can be reduced significantly. Especially in the context of MPC this effect becomes obvious. In fact, as long as the behavior of the model used in the MPC scheme sufficiently coincides with the actual behavior of the plant a re-optimization at each discrete-time step is not required.

Adaptive sampling MPC has previously been proposed in [4], [9], [13], [23], [26], [29], [30] for the case of event-triggered sampling and in [17] for the case of asynchronous measurements. Here, [9], [23], [29] and [30] consider continuous-time nonlinear systems that are affected by exogenous disturbances [9], [29] or network delays [23], [30]. In these papers the feasibility and convergence of the MPC has been proven by means of a Lyapunov based analysis. The focus of [26] lies in combining an event-triggered state estimator and MPC. It has been shown that the resulting MPC closed-loop system is input-to-state stable with respect to the estimation error. In the context of wireless sensor networks a robust event-triggered MPC scheme for discrete-time systems has been proposed in [4], which is based on a min-max optimization. In [13] a self-triggered implementation has been developed. Here, at each sampling instant, the next sampling time is determined in advance and, hence, no monitoring of an event condition is required.

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The approach proposed in this paper considers an event-triggered implementation of MPC for systems subject to input and state constraints as well as exogenous disturbances. The novelty of the approach lies in the comparison of the stability behavior of the event-triggered scheme with the behavior of the conventional time-triggered implementation. An upper bound on the maximum stationary set-point deviation is derived for both of the sampling schemes. It turns out that this bound directly depends on the event threshold as well as indirectly on the disturbance characteristics. The effectiveness of the proposed scheme is illustrated by simulations.

The outline of the paper is as follows. Section II introduces the conventional time-triggered MPC which serves as an underlying system used to evaluate the performance of the event-triggered MPC scheme. The event-triggered implementation and its stability properties are discussed in Section III. Simulation results are presented in Section IV.

Notation: Throughout this paper a scalar is denoted by italic letters ($x \in \mathbb{R}$), a vector by bold italic letters ($\boldsymbol{x} \in \mathbb{R}^n$), a matrix by upper-case bold italic letters ($\boldsymbol{A} \in \mathbb{R}^{n\times n}$) and a signal at discrete time $k \in \mathbb{N}$ by $\boldsymbol{x}(k)$, where $\boldsymbol{x}_0$ is defined as the initial signal value at time $k = 0$. The absolute value of a scalar x is denoted by $|x|$ and $\|\boldsymbol{x}\|$ and $\|\boldsymbol{A}\|$ are used to denote an arbitrary vector norm or induced matrix norm. $\hat{\boldsymbol{x}}(k|k')$ denotes an estimate of $\boldsymbol{x}(k)$, given all available measurements up until time k' with $k \geq k'$. $\boldsymbol{A} > 0$ and $\boldsymbol{A} \geq 0$ mean that the matrix $\boldsymbol{A}$ is positive definite or positive semi-definite, respectively.

II. Time-Triggered MPC

Throughout the paper the plant is described by the discrete-time linear state-space model

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{u}(k)+\boldsymbol{E}\boldsymbol{d}(k),\quad \boldsymbol{x}(0)=\boldsymbol{x}_0 \quad (1)$$

with state $\boldsymbol{x}\in\mathbb{R}^n$, input $\boldsymbol{u}\in\mathbb{R}^m$ and disturbance $\boldsymbol{d}\in\mathbb{R}^r$. Further it is assumed that the state, input and disturbance belong to the compact sets

$$\boldsymbol{x}\in\mathcal{X},\quad \boldsymbol{u}\in\mathcal{U},\quad \boldsymbol{d}\in\mathcal{D}.$$

At each sampling time k, the time-triggered MPC solves

$$\underset{\boldsymbol{U}(k)}{\text{minimize}}\ J(\boldsymbol{x}(k),\boldsymbol{U}(k))=\underset{\boldsymbol{U}(k)}{\text{minimize}}\ \|\hat{\boldsymbol{x}}(k+N|k)\|^2_{\boldsymbol{Q}_\mathrm{N}} + \sum_{l=0}^{N-1}\left(\|\hat{\boldsymbol{x}}(k+l|k)\|^2_{\boldsymbol{Q}}+\|\hat{\boldsymbol{u}}(k+l|k)\|^2_{\boldsymbol{R}}\right) \quad (2)$$

subject to

$$\begin{aligned}\hat{\boldsymbol{x}}(k+l+1|k)&=\boldsymbol{A}\hat{\boldsymbol{x}}(k+l|k)+\boldsymbol{B}\hat{\boldsymbol{u}}(k+l|k),\\ \hat{\boldsymbol{x}}(k|k)&=\boldsymbol{x}(k),\quad \boldsymbol{x}\in\mathcal{X},\quad \boldsymbol{u}\in\mathcal{U} \qquad (3)\\ \boldsymbol{U}(k)&=\{\hat{\boldsymbol{u}}(k|k),\hat{\boldsymbol{u}}(k+1|k),\ldots,\hat{\boldsymbol{u}}(k+N-1|k)\}\end{aligned}$$

and $l\in\mathbb{N}$.

The weighting matrices $0\le\boldsymbol{Q}_\mathrm{N}$, $0\le\boldsymbol{Q}$ and $0<\boldsymbol{R}$ as well as the horizon length N are design variables. At each discrete-time instant k the optimization is repeated and the first control input $\boldsymbol{u}(k)=\hat{\boldsymbol{u}}(k|k)$ of the computed optimal control signal sequence $\boldsymbol{U}(k)$ is applied to the plant. For further reference the control signal $\boldsymbol{u}(k)$ is denoted by the nonlinear map

$$\boldsymbol{u}(k)=\hat{\boldsymbol{u}}(k|k)=\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))=\boldsymbol{G}(\boldsymbol{x}(k)). \quad (4)$$

The following result can be derived.

Lemma 1: Suppose that:

1) The optimization problem (2)–(3) is feasible for all $\boldsymbol{d}(k)\in\mathcal{D}$, $k\ge 0$.
2) $\exists\boldsymbol{K}$ s.t. $\lim_{k\to\infty}\|\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))-\boldsymbol{K}\hat{\boldsymbol{x}}(k|k)\|=0$ and $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is Schur.

Then, the closed-loop system is stable. Further it holds that

$$\lim_{k\to\infty}\boldsymbol{x}(k)\in\mathcal{R}=\{\boldsymbol{x}:\ \|\boldsymbol{x}\|<r\}$$

where

$$r=\sup_{\boldsymbol{d}\in\mathcal{D}}\left\|\sum_{j=0}^{\infty}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^j\boldsymbol{E}\boldsymbol{d}\right\|. \quad (5)$$

Proof: According to assumption 1 the problem is feasible, so the MPC is able to keep (1) within its specified constraints. With $\boldsymbol{u}(k)=\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))$ and $\hat{\boldsymbol{x}}(k|k)=\boldsymbol{x}(k)$ at each sampling instant k, the closed-loop system is given by

$$\begin{aligned}\boldsymbol{x}(k+1)&=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))+\boldsymbol{E}\boldsymbol{d}(k)\\ &=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{K}\hat{\boldsymbol{x}}(k|k)+\boldsymbol{E}\boldsymbol{d}(k)\\ &\quad+\boldsymbol{B}(\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))-\boldsymbol{K}\hat{\boldsymbol{x}}(k|k))\\ &=(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(k)+\boldsymbol{E}\boldsymbol{d}(k) \qquad (6)\\ &\quad+\boldsymbol{B}(\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))-\boldsymbol{K}\hat{\boldsymbol{x}}(k|k)).\end{aligned}$$

In stationarity, the state $\boldsymbol{x}(k)$ is given by

$$\lim_{k\to\infty}\|\boldsymbol{x}(k)\|=\lim_{k\to\infty}\left\|\bar{\boldsymbol{A}}^k\boldsymbol{x}_0+\sum_{j=0}^{k-1}\bar{\boldsymbol{A}}^{k-1-j}\boldsymbol{E}\boldsymbol{d}(j)+\sum_{j=0}^{k-1}\bar{\boldsymbol{A}}^{k-1-j}\boldsymbol{B}\boldsymbol{\Delta}(j,j)\right\|$$

with $\boldsymbol{\Delta}(j,j)=\boldsymbol{G}(\hat{\boldsymbol{x}}(j|j))-\boldsymbol{K}\hat{\boldsymbol{x}}(j|j)$ and $\bar{\boldsymbol{A}}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$. Using assumption 2, $\lim_{k\to\infty}\|\bar{\boldsymbol{A}}^k\|=0$ and $\lim_{k\to\infty}\|\boldsymbol{\Delta}(k,k)\|=0$, this equation can be rewritten as

$$\lim_{k\to\infty}\|\boldsymbol{x}(k)\|=\lim_{k\to\infty}\left\|\sum_{j=0}^{k-1}\bar{\boldsymbol{A}}^{k-1-j}\boldsymbol{E}\boldsymbol{d}(j)\right\|.$$

The right-hand side can be upper bounded by (5), leading to the conclusion of the lemma. ■

In the case that the constraints are not active, the following well known result applies.

Lemma 2: As long as no constraints are active, it holds that

$$\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k))=\boldsymbol{K}^*\hat{\boldsymbol{x}}(k|k),\quad\forall k$$

where $\boldsymbol{K}^*=-(\boldsymbol{B}^T\boldsymbol{P}_0\boldsymbol{B}+\boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_0\boldsymbol{A}$ and

$$\begin{aligned}\boldsymbol{P}_\mathrm{N}&=\boldsymbol{Q}_\mathrm{N}\\ \boldsymbol{P}_k&=\boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{A}+\boldsymbol{Q}\\ &\quad-\boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B}+\boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}.\end{aligned}$$

Proof: See, e.g., [5]. ■

Remark 1: Lemma 2 implies that if the MPC is able to drive the system into a region where the constraints are no longer active, the first part of assumption 2 in Lemma 1 is guaranteed to hold. However, it does not guarantee that $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is Schur. This property may be inferred by restricting $\boldsymbol{Q}_\mathrm{N}$ to fulfill certain properties, c.f., Lemma 4.

The results provide a basis for comparing the performance of the event-triggered implementation in relation to the performance of the time-triggered one.

Remark 2: For robust MPC the reader is referred to [8], [14], [15], [20], [22], [25], [28], where the convergence analysis is primarily carried out by showing that a Lyapunov function decreases over time despite the influence of unknown disturbances.

III. Event-Triggered MPC

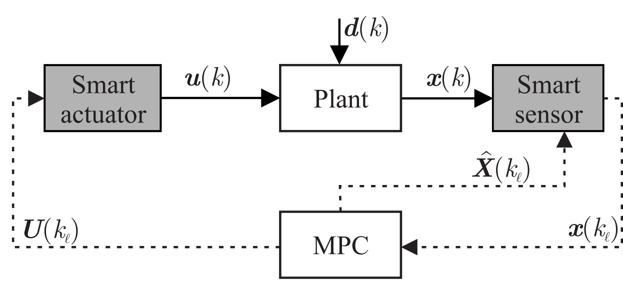

Fig. 1. Event-triggered model predictive control loop.

The control architecture for the event-triggered MPC is depicted in Fig. 1. It consists of the plant, a smart sensor incorporating the event condition, a MPC and a smart actuator. Information is sent over the feedback link only if the event condition, which is discussed next, is satisfied. The discrete-time instants at which this happens are denoted by k_ℓ, where $\ell \in \mathbb{N}$ is the event counter. In the following it is assumed that the first event $\ell = 0$ occurs at time $k_0 = 0$.

An event is generated at time $k_{\ell+1}$ whenever either the difference between the plant state $\boldsymbol{x}(k)$ and the state $\hat{\boldsymbol{x}}(k|k_\ell)$ predicted by the MPC exceeds a certain threshold, or the prediction horizon N has expired, i.e., when

$$\begin{aligned} &\|\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k_\ell)\| \geq \bar{e} \\ &\text{or } k \geq k_\ell + N, \end{aligned} \tag{7}$$

where $\bar{e} \geq 0$ is the threshold parameter. At event times k_ℓ the MPC state predictions are recalculated such that

$$\hat{\boldsymbol{x}}(k_\ell|k_\ell) = \boldsymbol{x}(k_\ell). \tag{8}$$

Introducing a change of variable $k = k_\ell + q$ with $q \in \{0, 1, \ldots, N\}$ the prediction error between two event times is given by

$$\boldsymbol{e}(q, k_\ell) = \boldsymbol{x}(k_\ell + q) - \hat{\boldsymbol{x}}(k_\ell + q|k_\ell) \tag{9}$$

which in turn is bounded according to the following theorem.

Theorem 1: The prediction error $\boldsymbol{e}(q, k_\ell)$ in (9) is bounded as

$$\boldsymbol{e}(q, k_\ell) \in \mathcal{E} = \{\boldsymbol{e} : \|\boldsymbol{e}\| \leq e_{\max}\},\ \forall k_\ell,\ q \in \{0, 1, \ldots, N\}$$

with

$$e_{\max} = \|\boldsymbol{A}\|\bar{e} + \|\boldsymbol{E}\| \max_{\boldsymbol{d} \in \mathcal{D}} \|\boldsymbol{d}\|. \tag{10}$$

Proof: The evolution of the prediction error is given by

$$\begin{aligned} \boldsymbol{e}(q+1, k_\ell) &= \boldsymbol{x}(k_\ell + q + 1) - \hat{\boldsymbol{x}}(k_\ell + q + 1|k_\ell) \\ &= \boldsymbol{A}\boldsymbol{e}(q, k_\ell) + \boldsymbol{E}\boldsymbol{d}(k_\ell + q) \end{aligned}$$

with $\boldsymbol{e}(0, k_\ell) = 0$ according to (8) and (9). Consider a given time instant $k' = k_\ell + q'$ such that no event is generated and, hence, $\|\boldsymbol{e}(q', k_\ell)\| < \bar{e}$. There always exists such a k' as $q' = 0$ is allowed. Consequently, the maximum possible prediction error at the next sampling instant can be bounded by

$$\begin{aligned} \max \|\boldsymbol{e}(q'+1, k_\ell)\| &= \max \|\boldsymbol{A}\boldsymbol{e}(q', k_\ell) + \boldsymbol{E}\boldsymbol{d}(k_\ell + q')\| \\ &\leq e_{\max} \end{aligned}$$

with $e_{\max}$ given by (10). ∎

At each event time k_ℓ, the optimization problem (2)–(3) is executed and the control sequence

$$\boldsymbol{U}(k_\ell) = \{\hat{\boldsymbol{u}}(k_\ell|k_\ell), \hat{\boldsymbol{u}}(k_\ell + 1|k_\ell), \ldots, \hat{\boldsymbol{u}}(k_\ell + N - 1|k_\ell)\}$$

is sent to the actuator. The sequence of predicted states

$$\hat{\boldsymbol{X}}(k_\ell) = \{\hat{\boldsymbol{x}}(k_\ell|k_\ell), \hat{\boldsymbol{x}}(k_\ell + 1|k_\ell), \ldots, \hat{\boldsymbol{x}}(k_\ell + N|k_\ell)\}$$

is sent to the sensor. In the following let $\hat{\boldsymbol{u}}(k|k_\ell)$ be denoted by the non-linear operator

$$\hat{\boldsymbol{u}}(k|k_\ell) = \boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)), \quad k \geq k_\ell.$$

The principle of the event generator is illustrated in Fig. 2 for $N = 3$. Here, the first and third event are triggered by the difference between the predicted and measured state exceeding the event threshold $\bar{e}$, whereas the second event results from an expired prediction horizon N.

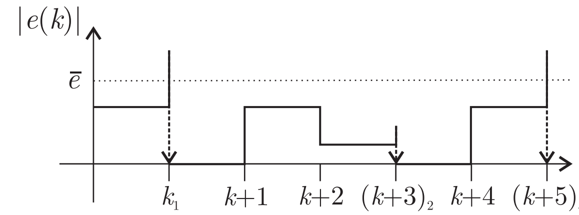

Fig. 2. Working principle of the event generator.

A. *Event-triggered* MPC *subject to large disturbances*

Considering large persistent disturbances d the following result can be obtained.

Lemma 3: Suppose that the disturbance $\boldsymbol{d}(k)$ satisfies

$$\|\boldsymbol{E}\boldsymbol{d}(k)\| \geq \bar{e},\ \forall k. \tag{11}$$

Then the event-triggered implementation of the MPC (2)–(3) using event condition (7) gives the same control sequence $\boldsymbol{U}(k)$ as the time-triggered implementation.

Proof: Assume that an event has been generated at time k_ℓ and hence $\hat{\boldsymbol{x}}(k_\ell|k_\ell) = \boldsymbol{x}(k_\ell)$. A new event is detected at the next sampling instant $k_\ell + 1$ if

$$\|\boldsymbol{x}(k_\ell + 1) - \hat{\boldsymbol{x}}(k_\ell + 1|k_\ell)\| = \|\boldsymbol{E}\boldsymbol{d}(k_\ell)\| \geq \bar{e}$$

holds. If $\boldsymbol{d}(k_\ell)$ satisfies this condition for all $k \in \mathbb{N}$, an event is generated at each sampling instant. Therefore, the event-triggered implementation uses the same state information for the optimization as the time-triggered MPC and, consequently, produces the same input sequence $\boldsymbol{U}(k)$. ∎

Remark 3: Note that instead of considering sufficiently large disturbances $\boldsymbol{d}(k)$ the same effect can be obtained by considering sufficiently small event thresholds $\bar{e}$. For $\bar{e} = 0$, the event-triggered model predictive controller always has the same behavior as its time-triggered counterpart.

B. *Event-triggered* MPC *subject to small disturbances*

In the case that the disturbance d is small there is a trade-off between the event threshold $\bar{e}$ and the control performance, as stated in the following result.

Theorem 2: Suppose that:

1) The disturbance $\boldsymbol{d}(k)$ satisfies

$$\|\boldsymbol{E}\boldsymbol{d}(k)\| \leq \bar{e},\ \forall k. \tag{12}$$

2) The optimization problem (2)–(3) using event condition (7) is feasible for all $\boldsymbol{d}(k) \in \mathcal{D}$, $k \geq 0$ and keeps $\boldsymbol{e}(k) \in \mathcal{E}$, $k \geq 0$.

3) $\exists \boldsymbol{K}$ s.t. $\lim_{k\to\infty} \|\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)) - \boldsymbol{K}\hat{\boldsymbol{x}}(k|k_\ell)\| = 0$ and $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is Schur.

Then the closed-loop system is stable. Further it holds

$$\lim_{k\to\infty} \boldsymbol{x}(k) \in \mathcal{R}_{\mathrm{e}} = \{\boldsymbol{x}:\ \|\boldsymbol{x}\| < r_e\}$$

where

$$r_{\mathrm{e}} = r + \sum_{j=0}^{\infty} \left\|(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^j \boldsymbol{B}\boldsymbol{K}\right\| e_{\max} = r + r_\delta(\bar{e}) \quad (13)$$

and

$$r = \sup_{\boldsymbol{d}\in\mathcal{D}} \left\| \sum_{j=0}^{\infty} (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^j \boldsymbol{E}\boldsymbol{d} \right\|.$$

Proof: According to assumption 2 the problem is feasible and the controller is able to keep the system within its specified constraints. The closed-loop system is given by

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)) + \boldsymbol{E}\boldsymbol{d}(k)\\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}(\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)) - \boldsymbol{K}\hat{\boldsymbol{x}}(k|k_\ell))\\ &\quad + \boldsymbol{B}\boldsymbol{K}\hat{\boldsymbol{x}}(k|k_\ell) + \boldsymbol{E}\boldsymbol{d}(k). \qquad (14)\end{aligned}$$

Using the change of variables $k = k_\ell + q$ and the prediction error (9), this may be written as

$$\begin{aligned}\boldsymbol{x}(k+1) = {}& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(k)\\ &+ \boldsymbol{B}(\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)) - \boldsymbol{K}\hat{\boldsymbol{x}}(k|k_\ell))\\ &- \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(q,k_\ell) + \boldsymbol{E}\boldsymbol{d}(k).\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} \|\boldsymbol{x}(k)\| = \lim_{k\to\infty} \Bigg\| & \bar{\boldsymbol{A}}^k \boldsymbol{x}_0 + \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{B}\boldsymbol{\Delta}(j,k_\ell)\\ & - \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(j-k_\ell,k_\ell)\\ & + \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{E}\boldsymbol{d}(j) \Bigg\|\end{aligned}$$

which with $\boldsymbol{\Delta}(j,k_\ell) = \boldsymbol{G}(\hat{\boldsymbol{x}}(j|k_\ell)) - \boldsymbol{K}\hat{\boldsymbol{x}}(j|k_\ell)$ can be rewritten, using assumption 3, as

$$\begin{aligned}\lim_{k\to\infty} \|\boldsymbol{x}(k)\| = \lim_{k\to\infty} \Bigg\| & - \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(j-k_\ell,k_\ell)\\ & + \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{E}\boldsymbol{d}(j) \Bigg\|.\end{aligned}$$

By using (10) in Theorem 1, the right-hand side of this equation can be bounded by

$$\begin{aligned}\lim_{k\to\infty} \|\boldsymbol{x}(k)\| &\le r + \lim_{k\to\infty} \sum_{j=0}^{k-1} \left\|\bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(j-k_\ell,k_\ell)\right\|\\ &\le r + \lim_{k\to\infty} \sum_{j=0}^{k-1} \left\|\bar{\boldsymbol{A}}^{k-1-j} \boldsymbol{B}\boldsymbol{K}\right\| e_{\max}.\end{aligned}$$

Hence, as $k\to\infty$, $\boldsymbol{x}(k) \in \mathcal{R}_e$. ■

The theorem shows how the event-triggered MPC approximates the time-triggered implementation through the parameter $e_{\max}$ in (10) depending on the event threshold $\bar{e}$. By increasing $\bar{e}$ the bound $r_\delta(\bar{e})$ becomes larger leading to an increase of the approximation error r_δ. Note that a small $\bar{e}$ generally violates assumption 1. In that case Lemma 3 applies instead.

Remark 4: Methods to analyze the feasibility and convergence properties of event-triggered MPC subject to exogenous disturbances can be found in [4], [9].

Lemma 4: Assumption 3 of Theorem 2 holds, as long as no constraints are active, the pair $(\boldsymbol{A},\boldsymbol{B})$ is controllable, $\boldsymbol{Q}_{\mathrm{N}}$ in (2)–(3) is chosen to satisfy the Algebraic Riccati Equation

$$\begin{aligned}\boldsymbol{Q}_{\mathrm{N}} = {}& \boldsymbol{A}^T\boldsymbol{Q}_{\mathrm{N}}\boldsymbol{A} + \boldsymbol{Q}\\ &- \boldsymbol{A}^T\boldsymbol{Q}_{\mathrm{N}}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{Q}_{\mathrm{N}}\boldsymbol{B}+\boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{Q}_{\mathrm{N}}\boldsymbol{A},\end{aligned}$$

and $\boldsymbol{K}$ is chosen as

$$\boldsymbol{K} = -(\boldsymbol{B}^T\boldsymbol{Q}_{\mathrm{N}}\boldsymbol{B}+\boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{Q}_{\mathrm{N}}.$$

In fact it then holds that

$$\boldsymbol{G}(\hat{\boldsymbol{x}}(k|k_\ell)) = \boldsymbol{K}\hat{\boldsymbol{x}}(k|k_\ell), \quad \forall k, k_\ell\ (k \ge k_\ell).$$

Proof: See, e.g., [5]. ■

C. Relations to model-based event-triggered control

Without constraints the optimal control problem (2)–(3), is equivalent to a state-feedback controller which can be explicitly determined, see Lemma 2.

By incorporating the MPC component on the actuator node this method shows some interesting relations to existing event-triggered control schemes denoted model-based event-triggered control, see [10], [18] for the continuous-time case and [11] for the discrete-time scenario. In fact by considering no constraints, the input provided by the MPC is the same as the input provided by a time-triggered state-feedback loop when the controller matrix $\boldsymbol{K}$ has been obtained from the optimal control problem (2)–(3).

IV. SIMULATION EVALUATION

Let the plant be described by the second-order linear discrete-time model

$$\begin{aligned}\boldsymbol{x}(k+1) = {}& \begin{pmatrix} 1 & -0.5\\ 0.5 & 0\end{pmatrix}\boldsymbol{x}(k)\\ &+ \begin{pmatrix} 0.5\\ 0\end{pmatrix} u(k) + \begin{pmatrix} 0.5\\ 0\end{pmatrix} d(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0\end{aligned}$$

subject to the following constraints:

$$\begin{aligned}&-10 < x_i < 10,\ i = 1,2\\ &-2 < u < 2\\ &-2 < d < 2.\end{aligned}$$

The prediction horizon is chosen to be $N = 10$ with the weights

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \quad R = 0.1.$$

The final state weight $\boldsymbol{Q}_{\mathrm{N}}$ is chosen, according to Lemma 4, as

$$\boldsymbol{Q}_{\mathrm{N}} = \begin{pmatrix} 1.554 & -0.151 \\ -0.151 & 1.080 \end{pmatrix}$$

with the corresponding state-feedback gain

$$\boldsymbol{K} = \begin{pmatrix} 1.513 & -0.795 \end{pmatrix}.$$

The event threshold for the event-triggered implementation is set to $\bar{e} = 0.5$. Considering these parameters the assumptions of Lemma 1 and Theorem 2 are satisfied.

A. Plant subject to large disturbances

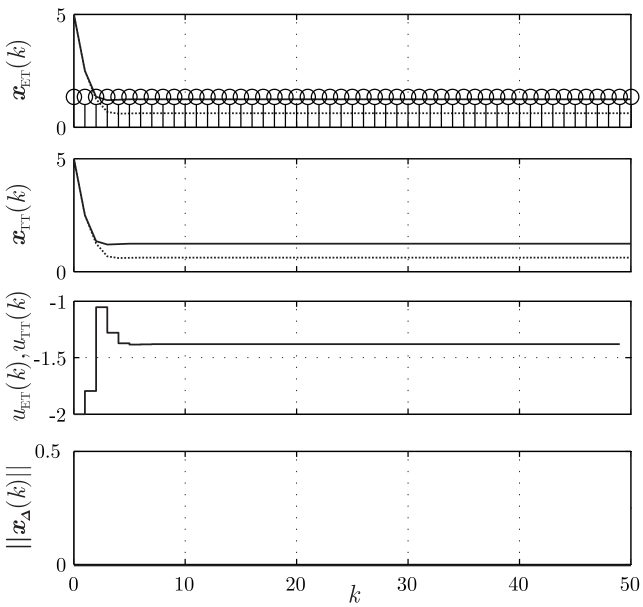

Fig. 3. Behavior of event-triggered and time-triggered MPC for a large disturbance; $x_{\mathrm{ET},1}, x_{\mathrm{TT},1}, u_{\mathrm{ET}}$: solid line; $x_{\mathrm{ET},2}, x_{\mathrm{TT},2}, u_{\mathrm{TT}}$: dashed line.

Fig. 3 shows the behavior of the event-triggered MPC (first plot) and the time-triggered MPC (second plot) for the disturbance

$$d(k) = 2,\ \forall k$$

which satisfies condition (11) at each sampling instant and

$$\boldsymbol{x}_0 = \begin{pmatrix} 5 & 5 \end{pmatrix}^T.$$

The subscript ET is used to indicate the event-triggered implementation whereas TT indicates the time-triggered scheme. As expected from Lemma 3, the behavior of both schemes coincide as shown in the fourth plot with

$$\boldsymbol{x}_{\Delta}(k) = \boldsymbol{x}_{\mathrm{ET}}(k) - \boldsymbol{x}_{\mathrm{TT}}(k)$$

because at each time instant k an event is generated. These generated events are indicated by the circles in the upper plot.

The third plot shows the control inputs u_{ET} and u_{TT} which likewise are identical for both schemes. It is shown that due to the large initial state the input is saturated which, however, does not affect the stability.

B. Plant subject to small disturbances

The behavior of both control schemes for the disturbance

$$d(k) = \begin{cases} 0, & \text{for } 0 \le k \le 20 \\ 0.5, & \text{for } k > 20 \end{cases}$$

and

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$$

is depicted in Fig. 4. For $k \le 20$ the disturbance is zero and, hence, $\boldsymbol{x}_{\mathrm{ET}}(k)$ and $\boldsymbol{x}_{\mathrm{TT}}(k)$ coincide as the model state and the measured state are the same. The two events generated in the event-triggered scenario during this period, at $k_1 = 10$ and $k_2 = 20$, are due to that the prediction horizon of the MPC expires. At $k = 21$, the magnitude of the disturbance changes and the following events are caused by the deviation of the model and the measured state, which occurs every six discrete-time steps. Consequently, the inputs u_{ET} and u_{TT} also deviate (third plot).

Nevertheless, the difference between the states $\boldsymbol{x}_{\mathrm{ET}}(k)$ and $\boldsymbol{x}_{\mathrm{TT}}(k)$ remains bounded and with $\max|d| = 0.5$ it holds

$$\|\boldsymbol{x}_{\mathrm{ET}}(k) - \boldsymbol{x}_{\mathrm{TT}}(k)\| \le r_{\delta}(0.5) = 0.44$$

which is illustrated in the lower plot.

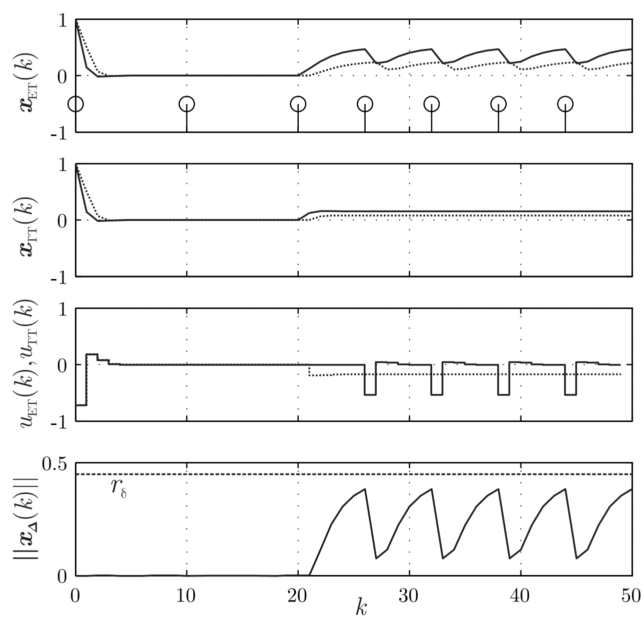

Fig. 4. Behavior of event-triggered and time-triggered MPC for a small disturbance; $x_{\mathrm{ET},1}, x_{\mathrm{TT},1}, u_{\mathrm{ET}}$: solid line; $x_{\mathrm{ET},2}, x_{\mathrm{TT},2}, u_{\mathrm{TT}}$: dashed line.

V. Conclusions

The paper investigated the stationary behavior of event-triggered MPC and evaluated the difference to a conventional time-triggered implementation. It was shown how the event-triggered approach is affected by the event condition as well as the disturbance magnitude. The analytical results were illustrated by simulations.

Future work will include the evaluation of the transient behavior and the incorporation of integral action in the MPC.

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REFERENCES

[1] A. Anta and P. Tabuada. To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9):2030–2042, 2010.

[2] K. J. Åström and B. Bernhardsson. Comparison of Riemann and Lebesgue sampling for first order stochastic systems. In *Proceedings of IEEE Conference on Decision and Control*, pages 2011–2016, Las Vegas, 2002.

[3] A. Bemporad, W. P. M. H. Heemels, and M. Johansson, editors. *Networked Control Systems*. Lecture Notes in Control and Information Sciences. Springer Verlag, 2010.

[4] D. Bernardini and A. Bemporad. Energy-aware robust model predictive control based on noisy wireless sensors. *Automatica*, 48(1):36–44, 2012.

[5] D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, 1995.

[6] E. F. Camacho and C. Bordons. *Model Predictive Control*. Advanced Textbooks in Control and Signal Processing. Springer-Verlag London, 2nd ed. edition, 2007.

[7] A. Casavola, E. Mosca, and M. Papini. Predictive teleoperation of constrained dynamic systems via internet-like channels. *IEEE Transactions on Control Systems Technology*, 14(4):681–694, 2006.

[8] L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37(7):1019–1028, 2001.

[9] A. Eqtami, D. V. Dimarogonas, and K. J. Kyriakopoulos. Novel event-triggered strategies for model predictive controllers. In *Proceedings of IEEE Conference on Decision and Control and European Control Conference*, pages 3392–3397, Orlando, 2011.

[10] E. Garcia and P. J. Antsaklis. Model-based event-triggered control with time-varying network delays. In *Proceedings of IEEE Conference on Decision and Control*, Orlando, 2011.

[11] L. Grüne, S. Jerg, O. Junge, D. Lehmann, J. Lunze, F. Müller, and M. Post. Two complementary approaches to event-based control. *Automatisierungstechnik*, 58(4):173–182, 2010.

[12] W. P. M. H. Heemels, J. Sandee, and P. van den Bosch. Analysis of event-driven controllers for linear systems. *Intern. J. of Control*, 81(4):571–590, 2008.

[13] E. Henriksson, D. E. Quevedo, H. Sandberg, and K. H. Johansson. Self-triggered model predictive control for network scheduling and control. In *Proceedings of International Symposium on Advanced Control of Chemical Processes*, Singapore, 2012.

[14] J.-S. Kim, T.-W. Yoon, A. Jadbabaie, and C. De Persis. Input-to-state stable finite horizon mpc for neutrally stable linear discrete-time systems with input constraints. *Systems & Control Letters*, 55:293–303, 2006.

[15] D. Limon, I. Alvarado, T. Alamo, and E. Camacho. Robust tube-based mpc for tracking of constrained linear systems with additive disturbances. *Journal of Process Control*, 20(3):248 – 260, 2010.

[16] G. P. Liu, D. Rees J. X. Mu, and S. C. Chai. Design and stability analysis of networked control systems with random communication time delay using the modified MPC. *International Journal of Control*, 79:288–297, 2006.

[17] J. Liu, D. Muñoz de la Peña, and P. D. Christofides. Distributed model predictive control of nonlinear systems subject to asynchronous and delayed measurements. *Automatica*, 46:52 – 61, 2010.

[18] J. Lunze and D. Lehmann. A state-feedback approach to event-based control. *Automatica*, 46(1):211–215, 2010.

[19] J. M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, 2002.

[20] D. L. Marruedo, T. Alamo, and E. F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties. In *Proceedings of IEEE Conference on Decision and Control*, pages 4619–4624, Las Vegas, 2002.

[21] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[22] D.Q. Mayne, M.M. Seron, and S.V. Raković. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219 – 224, 2005.

[23] D. Muñoz de la Peña and P. D. Christofides. Lyapunov-based model predictive control of nonlinear systems subject to data losses. *Automatic Control, IEEE Transactions on*, 53(9):2076–2089, 2008.

[24] J. B. Rawlings and D. Q. Mayne. *Model Predictive Control: Theory and Design*. Nob Hill Publishing, Madison, 2009.

[25] J. A. Rossiter, B. Kouvaritakis, and M. J. Rice. A numerically robust state-space approach to stable predictive control strategies. *Automatica*, 34(1):6573, 1998.

[26] J. Sijs, M. Lazar, and W. P. M. H. Heemels. On integration of event-based estimation and robust MPC in a feedback loop. In *Hybrid Systems: Computation and control*, pages 31–40, Stockholm, 2010.

[27] P. L. Tang and C. W. de Silva. Compensation for transmission delays in an ethernet-based control network using variable-horizon predictive control. *IEEE Transactions on Control Systems Technology*, 14(4):707 –718, 2006.

[28] P. Trodden and A. Richards. Robust distributed model predictive control using tubes. In *2006 American Control Conference*, pages 2034–2039. Ieee, 2006.

[29] P. Varutti, T. Faulwasser, B. Kern, M. Kögel, and R. Findeisen. Event-based reduced-attention predictive control for nonlinear uncertain systems. In *Computer-Aided Control System Design (CACSD), 2010 IEEE International Symposium on*, pages 1085 –1090, 2010.

[30] P. Varutti, B. Kern, T. Faulwasser, and R. Findeisen. Event-based model predictive control for networked control systems. In *Proceedings of IEEE Conference on Decision and Control*, pages 567–572, Shanghai, 2009.

[31] X. Wang and M. D. Lemmon. Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability. *IEEE Transactions on Automatic Control*, 54(3):452–467, 2009.

[32] A. Willig. Recent and emerging topics in wireless industrial communications: A selection. *IEEE Transactions on Industrial Informatics*, 4(2):102 –124, May 2008.

[33] Y.-B. Zhao, G. P. Liu, and D. Rees. Improved predictive control approach to networked control systems. *IET Control Theory and Applications*, 2(8):675 –681, 2008.