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A simple self-triggered sampler for perturbed nonlinear systems[☆]

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ABSTRACT

Self-triggered control is a recent design paradigm for resource-constrained networked control systems. By allocating aperiodic sampling instances for a digital control loop, a self-triggered controller is able to utilize network resources more efficiently than conventional sampled-data systems. In this paper we propose a self-triggered sampler for perturbed nonlinear systems ensuring uniformly ultimately boundedness of trajectories. Robustness and time delays are considered. To reduce conservativeness, a disturbance observer for the self-triggered sampler is proposed. The effectiveness of the proposed method is shown by simulation.

1. Introduction

The emergence of new control and communication technologies is enabling the development of advanced applications in health care, intelligent transportation systems, process control, smart grids, etc. A common characteristic is in the coupling of heterogeneous systems which share common computing and network resources. Traditional control engineering seldom addresses the problem of distribution of shared resources. However, with the advent of these new applications, resource-sharing policies are of primary interest and must be included during the design phase. Control applications are in most cases developed for periodic implementation platforms, which may not be the best choice for shared-resources systems. For example, if a task running on a microprocessor is fed with the same input and it produces the same output, its periodic instantiation would utilize computation resources without providing any benefit to the system. Such an issue becomes more evident in modern applications, where the number of tasks running on the same microprocessor can be large. In the case of networked control systems (NCSs), a fixed periodicity limits the exchange rate of information. For example, a NCS in a steady-state communicates the same amount of data as if it is in a transition phase. Such a behavior clearly wastes network resources, as the system does not adapt to the need for the application.

A promising technique to deal with resource-constrained systems is event-triggered control [2–10]. With this technique, an action is taken when some information is available. A subsystem injects a packet in the network only when it has relevant information to send. It is not difficult to argue that the event-triggered paradigm allows the design of more efficient resource-sharing policies compared to periodic sampling. Event-triggered control often consists of constantly monitoring the output of the system, and to compute the new control only when a function of the output crosses a certain threshold. The selection of the output and the threshold should produce desired behavior from the closed-loop system.

While event-triggered control *reacts* to the detection of an event, self-triggered control *predicts* its occurrence based on a system model and the current measurement, see [11–18]. Most of the existing work on self-triggered control addresses

☆ A preliminary version of this work appeared in [1].

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linear systems, while nonlinear systems are still not greatly investigated. A preliminary attempt to design a self-triggered sampler for nonlinear systems can be found in [19], further extended in [20]. However, such a method addresses a *safety* problem, in which starting from an initial condition inside a ultimately bounded region, the self-triggered policy ensures the invariance of that region. This means that there is a severe limitation on the set of initial conditions that must belong to the ultimately bounded region. An alternative design framework is developed in [11] and extended in [21]. Such a method applies for any smooth system sampled according to a smooth sampling rule. It requires a so-called *homogenization* of both the closed-loop system and the sampling rule, and the computation of *isochronous manifolds*. Since in general it is not possible to compute an isochronous manifold in closed-form, the authors provide a method to approximate it. The conservativeness of the approach depends on the accuracy of the isochronous manifold approximation. However, although the homogenization procedure can be applied to any smooth system and to any smooth sampling rule, the proposed method to compute an isochronous manifold approximation does not always apply. Moreover, effects of external disturbances and time delays have not been addressed in [11].

In this paper we present a simple self-triggered sampler for perturbed nonlinear systems to ensure *uniformly ultimately boundedness* (UUB) of the closed-loop system. Our method does not require particular restriction on the initial condition set, and the computation of the next sampling instant is performed through a simple formula that can be easily implemented and evaluated. We show how the effects of external disturbances and time delay can be incorporated in the design. A method based on disturbance observers to reduce the conservativeness of the approach is further discussed. The design of our self-triggered sampler is based on a novel robust event-triggered sampling rule capable of ensuring UUB for any bounded disturbance. Robust event-based control has been also addressed in [22], but such results only apply to exponentially stable systems which are ISS with respect to measurement errors, which represent a quite restricted class of systems [23,24]. Our method extends the applicable cases. In the absence of disturbances and time delays, our sampling rule confines every trajectory into arbitrary small regions. Finally, as an additional contribution, we provide a result on the robustness of the Lebesgue sampling rule in [6] applied to a perturbed nonlinear system. We prove that under mild conditions, the Lebesgue sampling rule ensures UUB of the trajectories for any globally stabilizable nonlinear system, independently of the chosen sampling threshold.

The remainder of the paper is as follows: in the Section 2 some notation and preliminaries are introduced. In Section 3 the problem formulation is stated. In Section 4 an event-triggering rule to achieve UUB of the closed-loop system is proposed, while in Section 5 the self-triggered implementation of such an event-triggering rule is proposed. The theoretical results are validated by simulations in Section 7. Conclusions are provided in Section 8. Appendix which contains the proofs of the technical results is finally reported.

2. Preliminaries

We indicate with $\|v\|$ the Euclidean norm of vector $v \in \mathbb{R}^n$ and with $\mathcal{B}_r$ the closed ball center at the origin and radius r, i.e. $\mathcal{B}_r = \{v : \|v\| \le r\}$. Given a signal $s : \mathbb{R}^+ \to \mathbb{R}^n$, we denote s_k its realization at time $t = t_k$, i.e. $s_k := s(t_k)$, and with $\|s\|_{\mathcal{L}_\infty,k} := \sup_{t\ge t_k} \|s(t)\|$. A function $h : \mathcal{D}_h \to \mathbb{R}^n$, $\mathcal{D}_h \subseteq \mathbb{R}^m$, is said to be of class $\mathcal{C}^0(\mathcal{D}_h)$ if it is continuous over $\mathcal{D}_h$, and it is said to be $\mathcal{C}^r(\mathcal{D}_h)$, $r > 0$ if its derivatives are of class $\mathcal{C}^{r-1}(\mathcal{D}_h)$. A function $h : \mathcal{D}_p \times \mathcal{D}_q \to \mathbb{R}^n$ is said to be *Lipschitz continuous over* $\mathcal{D}_p \times \mathcal{D}_q$ if $\|h(p_1, q) - h(p_2, q)\| \le L_{h,p}\|p_1 - p_2\|$ for some $L_{h,p} > 0$ and for all $p_1, p_2 \in \mathcal{D}_p$, $q \in \mathcal{D}_p$ and $\|h(p, q_1) - h(p, q_2)\| \le L_{h,q}\|q_1 - q_2\|$ for some $L_{h,q} > 0$ and for all $q_1, q_2 \in \mathcal{D}_q$, $p \in \mathcal{D}_p$. The constants $L_{h,p}$ and $L_{h,q}$ are called *Lipschitz constants of h with respect to p* and *Lipschitz constants of h with respect to q*, respectively. A continuous function $\alpha : [0, a) \to +\infty$, $a > 0$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $a = \infty$ and $\alpha(r) \to +\infty$ for $r \to +\infty$, then α is said to be of class $\mathcal{K}_\infty$. Given a system $\dot{x} = f(t, x), x \in \mathbb{R}^n, x(t_0) = x_0, f : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^n$, where f is Lipschitz continuous with respect to x and piecewise continuous with respect to t, and where $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin, we say that the solutions are UUB if there exists three constants $a, b, T > 0$ independent of t_0 such that for all $\|x_0\| \le a$ it holds $\|x(t)\| \le b$ for all $t \ge t_0 + T$, and globally UUB (GUUB) if $\|x(t)\| \le b$ for all $t \ge t_0 + T$ and for arbitrarily large a. The value of b is referred as the *ultimate bound*.

3. Problem formulation

Consider a perturbed system of the form

$$\dot{x} = f(x, u, d), \tag{1}$$

where $x \in \mathcal{D}_x \subseteq \mathbb{R}^n$, $u \in \mathcal{D}_u \subseteq \mathbb{R}^p$, and d is a bounded external disturbance in a compact set $\mathcal{D}_d \subseteq \mathbb{R}^d$ with bound $\|d\| \le \bar{d}$. Assume that the domains $\mathcal{D}_x$, $\mathcal{D}_u$ and $\mathcal{D}_d$ contain the origin and consider the following assumption.

Assumption 3.1. There exists a differentiable state feedback law $\kappa : \mathcal{D}_x \to \mathcal{D}_u$ such that the origin of the unperturbed system

$$\dot{x} = f(x, \kappa(x), 0), \tag{2}$$

is the unique locally asymptotically stable equilibrium point in $\mathcal{D}_x$. □

Assumption 3.2. There exists a differentiable state feedback law $\kappa: \mathcal{D}_x \to \mathcal{D}_u$ such that the function $f(x, \kappa(x), d)$ is $\mathcal{C}^1(\mathcal{D}_x \times \mathcal{D}_u \times \mathcal{D}_d)$ with Lipschitz continuous derivatives over the set $\mathcal{D}_x \times \mathcal{D}_u \times \mathcal{D}_d$. □

We recall that from Assumption 3.1, converse theorems [25,26] ensure the existence of a Lyapunov function $V(x)$ for the system (1) such that

$$\begin{aligned} &\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \\ &\frac{\partial V(x)}{\partial x} f(x, \kappa(x), 0) \le -\alpha_3(\|x\|), \\ &\left\| \frac{\partial V(x)}{\partial x} \right\| \le \alpha_4(\|x\|), \end{aligned} \tag{3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are $\mathcal{K}$-class functions.

Suppose now to sample and hold the measurements $x(t)$ at time $t = t_k$ so that the control signal is piecewise constant of the form $u(t) = \kappa(x_k)$ for $t \in [t_k, t_{k+1})$. For such a time interval, the dynamics of the sampled-data system satisfies

$$\dot{x} = f(x, \kappa(x_k), d). \tag{4}$$

The problem we address in this paper is to design a self-triggered sampler such that the perturbed sampled-data system (4) is UUB, and such that the ultimate bound can be arbitrarily small when $\bar{d} = 0$ and there are no time delays. This means that without external disturbances and without time delays, all the trajectories must converge to an arbitrary small region around the equilibrium point. The proposed self-triggered sampler is derived from a novel event-triggered sampling rule which ensures UUB of the sampled-data system (4) and which is the argument of the next section.

4. Event-triggered sampler

In this section we present a novel event-triggering rule capable to ensure UUB of the sampled-data system (4). The design of such a sampling rule is based on the fact that the sampled-data system (4) can be rewritten as a nominal continuous-time system subject to a perturbation as

$$\dot{x} = f(x, \kappa(x), d) + g(t), \tag{5}$$

where

$$g(t) := f(x, \kappa(x_k), d) - f(x, \kappa(x), d). \tag{6}$$

It is then possible to resort to well-known results of perturbed nonlinear system theory to derive an event-based sampling rule to ensure UUB of the sampled-data system (4) as shown in the next result.

Proposition 4.1. *Suppose that Assumption 3.1 is satisfied and let δ be a positive constant such that*

$$\delta + L_{f,d}\bar{d} < \frac{\vartheta \alpha_3\left(\alpha_2^{-1}(\alpha_1(r))\right)}{\alpha_4(r)}, \tag{7}$$

where $\vartheta \in (0, 1)$ and $r > 0$. Then, by updating the control signal every time the triggering condition

$$\|g(t)\| \le \delta, \tag{8}$$

is violated, the sampled-data system (4) is UUB. □

An illustration of the sampling rule (8) is depicted in Fig. 1. Every time the function $\|g(t)\|$ hits the threshold δ, a new measurement is picked, the control law is updated and $\|g(t)\|$ is reset to zero. That way, the perturbation due to the sampling is bounded, and the Lyapunov function derivative along the trajectories of the system is enforced to be strictly negative in the region $\|x(t)\| > \mu(\delta)$ for all $t \ge t_0$, where $\mu(\cdot)$ is defined as

$$\mu(\delta) := \alpha_3^{-1}\left(\frac{\alpha_4(\|x\|_{\mathcal{L}_\infty,0})(\delta + L_{f,d}\bar{d})}{\vartheta}\right), \tag{9}$$

where $\vartheta \in (0, 1)$. This means that the trajectories must necessarily converge to a bounded invariant set. Although the determination of a tight ultimate bound is in general a difficult task, by following standard arguments we can get the bound

$$b(\delta) := \alpha_1^{-1}(\alpha_2(\mu(\delta))). \tag{10}$$

Inspection of (10) shows that the trajectories can be confined into smaller regions by decreasing the value of δ, and that such a region can be arbitrarily small when $\bar{d} = 0$. On the other hand, since $g(t)$ is continuous in the intervals $[t_k, t_{k+1})$, it is not difficult to see that the inter-sampling times decrease as δ decreases. Thus, the value of δ establishes a tradeoff between the number of the controller updates and the size of the ultimate bound guarantee. The limit case is obtained for $\bar{d} = 0$ and

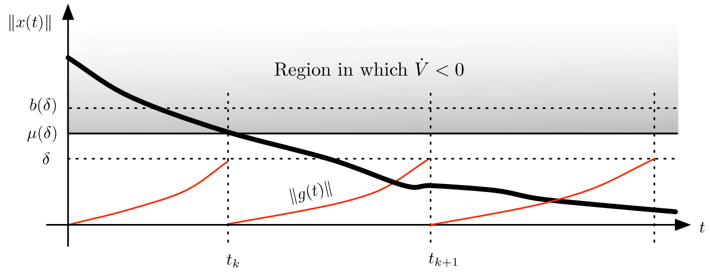

Fig. 1. The proposed event-triggered sampler. Every time the error function $\|g(t)\|$ hits the threshold δ, then the system is sampled again, and the error is reset. By doing that the Lyapunov derivative is strictly negative in the region $\|x\| > \mu(\delta)$, and then the trajectories must converge into an invariant set.

$\delta = 0$, for which we have $b(\delta) = 0$ and $t_{k+1} = t_k$ for all k, which corresponds to the case of continuous feedback control and asymptotic stability.

If Assumption 3.1 holds globally and if the right hand side of (7) is unbounded, then we can pick r so that (7) is fulfilled for any arbitrarily large δ. This means that we do not have a limitation in the choice of δ, and UUB is always ensured, as stated in the next result.

Theorem 4.1. *Let Assumption 3.1 hold globally, and assume that the right-hand side of (7) is unbounded. Then, the sampling rule (8) ensures GUUB of the closed-loop system for any $\delta > 0$.* □

The sampling-rule (8) requires knowledge of the system to define the function f. However, if this information is uncertain, it is possible to upper bound the proposed sampling rule with the Lebesgue sampling rule by observing that $\|f(x, \kappa(x_k), d) - f(x, \kappa(x), d)\| \leq L_{f,u}\|\kappa(x_k) - \kappa(x(t))\| \leq L_{f,u}L_{\kappa,x}\|x_k - x(t)\|$. However, because f and κ are substituted with their Lipschitz constant, the utilization of the Lebesgue sampling rule would provide more conservative inter-sampling times for a given δ, but on the other hand, it does not require an explicit knowledge of the model. Therefore, there is a certain analogy between the Lebesgue sampling rule and the proposed sampling rule, which relies on the conservativeness of the inter-sampling times and on the knowledge of the system model which is required to implement (8). Such an analogy, together with Theorem 4.1, highlights an important property of the Lebesgue sampling as shown in the next result.

Corollary 4.1. *Suppose the assumptions of Theorem 4.1 hold. Then, the sampling rule implicitly defined by*

$$h(t) = \|x_k - x(t)\| \leq \delta, \tag{11}$$

ensures GUUB of the sampled-data system (4) for any $\delta > 0$. □

The corollary states an important property of the Lebesgue sampling: for periodic sampling, there is an upper-limit in the choice of the sampling period after which we have instability, while in the Lebesgue sampling GUUB is achieved no matter how big the threshold δ is. On the other hand, enlarging δ corresponds to enlarging the ultimate bound. Nevertheless, GUUB for larger or smaller ultimate bounds is always achieved.

In the next section we show how to design a self-triggered implementation of the less conservative sampling rule (8), although a self-triggered sampler for the Lebesgue sampling can be designed by proceeding along the same line. However, before proceeding further, we wish to remark that the implementation of the event-triggered sampling rule (8) requires knowledge of external perturbation $d(t)$. Nevertheless, if this information is not available, it is possible to slightly modify such a sampling rule and obtain an equivalent result.

Proposition 4.2. *Assume that there exists a state feedback law $\kappa : \mathcal{D}_x \to \mathcal{D}_u$ such that the origin of the unperturbed system $\dot{x} = f(x, \kappa(x), 0)$ is a locally asymptotically stable equilibrium point. Let δ be an arbitrary positive constant. Then, by updating the control signal every time the triggering condition*

$$\|f(x(t), \kappa(x_k), 0) - f(x(t), \kappa(x(t)), 0)\| \leq \delta, \tag{12}$$

is violated, the sampled-data system (4) is UUB. □

The differentiability of the control function $\kappa(\cdot)$ and the fulfillment of Assumption 3.2 are not required. However, to design a self-triggered sampler, both Assumptions 3.1 and 3.2 must be satisfied, as we discuss next.

5. Self-triggered sampler

In this section we present the self-triggered implementation of the sampling rule (8). Without loss of generality, we present a self-triggered sampler assuming that Assumption 3.1 holds globally and that the right-hand side of (7) is unbounded. However, if these assumptions are not met, it is not difficult to obtain a local result by proceeding along the same line as we described in this section. We first analyze the case without time delays, and then we show how to include them in the design. In this case, we assume a sufficiently small time delay bounded by a maximum time delay $\tau_{\max}$. The idea of the self-triggered sampler we propose, is to predict when condition (8) is violated. To do that, we exploit an upper-bound $\hat{g}(x^*, d^*, x_k, t - t_k)$ of the function $\|g(t)\|$.

5.1. Without time delays

Lemma 5.1. *Consider the system (4) and suppose that Assumptions 3.1 and 3.2 hold. Then, for all $t \in [t_k, t_{k+1})$ the function $g(t)$ is upper-bounded with*

$$\|g(t)\| \leq \frac{1}{2}\|f(x^*, \kappa(x_k), d^*)\|\left(e^{2L(t-t_k)} - 1\right) := \hat{g}(x^*, d^*, x_k, t - t_k), \tag{13}$$

where $L = L_{f,u}L_{\kappa,x}$ and $(x^*, d^*) := \arg\max_{(y_1,y_2)\in\mathbb{R}^n\times\mathcal{D}_d} \|f(y_1, \kappa(x_k), y_2)\|$. □

A self-triggered sampler to ensure GUUB of the sampled-data system (4) is given by the next result.

Theorem 5.1. *Consider the sampled-data system (4). Suppose that Assumptions 3.1 and 3.2 hold and let $\delta > 0$. Then, the self-triggered sampler*

$$t_{k+1} = t_k + \frac{1}{2L}\ln\left(1 + \frac{2\delta}{\|f(x^*, \kappa(x_k), d^*)\|}\right), \tag{14}$$

ensures GUUB of the closed-loop system. □

According to the self-triggered sampling paradigm, at time $t = t_k$ it is possible to predict the next time t_{k+1} by which the system must be sampled to ensure GUUB.

Remark 5.1. The proposed self-triggered sampler can be used provided the values of L, d^* and x^* are computed, and their computation is performed by considering the set $\mathbb{R}^n \times \mathcal{D}_d$. However, since the trajectories of the system are upper-bounded with $\|\dot{x}\| \leq \|f(x, \kappa(x), 0)\| + \delta + L_{f,d}\bar{d}$, for all $t \geq t_0$, at every sampling time it is possible to re-compute L and x^* over the region $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$. In addition, since the sampled-data system (4) with the self-triggered sampler (14) is UUB, it follows that $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}} \subset \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$ outside the ultimately bounded region. Then, the sequence of sets $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}}$ is decreasing outside the ultimately bounded region, and L and x^* computed over such a sequence of sets is non-increasing, see Fig. 2. By doing this adaptation, we dynamically subtract conservativeness to the computation of L and x^*, and the proposed self-triggered sampler would better approximate the sampling rule (8). In addition to the worst-case values of x^* and L used in (14) another source of conservativeness comes from the utilization of d^*. However, to reduce such a conservativeness, it is possible to replace the component d^* with an estimate of the disturbance d_k, as we will explain in Section 6. □

The implementation of (14) requires the computation of (x^*, d^*) which is achieved by solving an online optimization problem. Nevertheless, this can be avoided by slightly modifying the self-triggered sampler (14), as shown by the next result.

Corollary 5.1. *Suppose that the assumptions of Theorem 5.1 hold. Let (p, q) be any point in $\in \mathbb{R}^n \times \mathcal{D}_d$ such that $\min_{y\in\mathbb{R}^n} \|f(p, y, q)\| > m$ for some $m > 0$. Then, the self-triggered sampler*

$$t_{k+1} = t_k + \frac{1}{2L}\ln\left(1 + \frac{2\delta}{\|f(p, \kappa(x_k), q)\|}\right), \tag{15}$$

ensures GUUB of the closed-loop system. □

Remark 5.2. From a comparison between the self-triggered samplers (14)–(15) it is easy to see that there are some points (p, q) such that (15) provides larger inter-sampling times compared to (14). This is because the utilization of (15) would correspond to consider the time it takes for $\hat{g}(x^*, d^*, x_k, t - t_k)$ to go from 0 to a certain $\delta' > \delta$, where δ' depends on the choice of the point (p, q). However, when the utilization of (15) provides larger inter-sampling times compared to (14), then the ultimate bound guarantee is, in general, larger. □

Remark 5.3. By assumption, the origin is the unique equilibrium point of the system (2). Then, it follows that the inter-sampling times provided the self-triggered samplers (15)–(14) are upper-bounded, being $\|f(x^*, y, d^*)\| > m > 0$ for all y. Moreover, since the trajectories are ultimately bounded, then there exists a k' such that $\kappa(x_k)$ is also bounded for all $k \geq k'$. This implies the existence of two constants $0 < \Delta_m \leq \Delta_M$ such that $\Delta_m \leq t_{k+1} - t_k \leq \Delta_M$ for all $k \geq k'$. □

Notice how the self-triggered sampler (15) does not require any numerical method and how it can be easily implemented on digital platforms and can be used in all the situations in which existing self-triggered samplers are difficult to apply. It also provides a certain degree of robustness with respect to external perturbations.

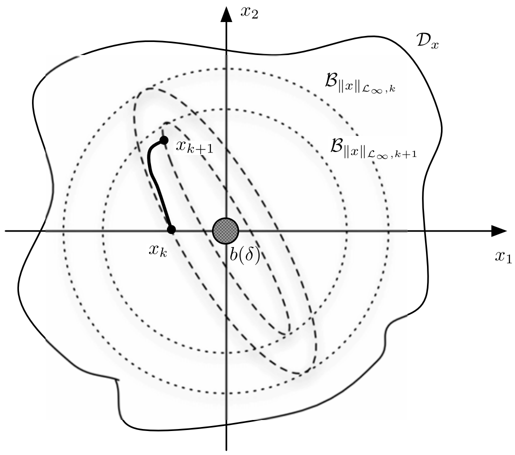

Fig. 2. Adaptation of the operating region. The dashed-line represents the Lyapunov surface levels, the dotted-line the adapted operating region and $b(\delta)$ denotes the ultimate bound. For $t \in [t_k, t_{k+1})$, the trajectories cannot escape from $\mathcal{B}_{\mathcal{L}_\infty,k}$, while for $t \in [t_{k+1}, t_{k+2})$ the trajectories cannot escape from $\mathcal{B}_{\mathcal{L}_\infty,k+1}$. Since the Lyapunov function is strictly negative outside the region delimited by b, we have $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}} \subset \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$.

In both the proposed event and self-triggered samplers there is still the open issue of finding a lower-bound of the inter-sampling times. Indeed, it is possible to prove the existence of such a lower-bound by observing that the trajectories are bounded with $\|x(t)\| \leq \|x\|_{\mathcal{L}_\infty,0}$ for all $t \geq t_0$. Hence, the minimum inter-sampling interval provided by the proposed self-triggered samplers can be obtained by solving the optimization problem max $\|f(y_1, \kappa(y_2), y_3)\|$ with $y_1, y_2 \in \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,0}}$ and $y_3 \in \mathcal{D}_d$. Now since f is continuous and $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,0}} \times \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,0}} \times \mathcal{D}_d$ is a compact set, then the optimization problem admits a finite maximum, and therefore a lower-bound of the inter-sampling times exists. We have then proved the following result.

Corollary 5.2. *Suppose that the assumptions of Corollary 5.1 hold. Then, the inter-sampling times provided self-triggered samplers (15) and (14) are lower-bounded with bound*

$$t_{k+1} - t_k \geq \frac{1}{2L} \ln\left(1 + \frac{2\delta}{c}\right) \quad \forall k, \tag{16}$$

where

$$c := \min_{\substack{y_1, y_2 \in \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,0}} \\ y_3 \in \mathcal{D}_d}} \|f(y_1, \kappa(y_2), y_3)\|. \quad \square \tag{17}$$

The self-triggered sampler (14) represents a conservative approximation of the event-triggered rule (8). This means the inter-sampling times provided by the self-triggered implementation (14) are shorter than or equal to the inter-sampling times provided by the event-triggered rule (8) for all k. Hence, since the inter-sampling times of the self-triggered sampler (14) are lower-bounded, then the inter-sampling times provided by the event-triggered rule defined in (8) are also lower-bounded by some positive constant. This means that eventual Zeno behaviors of the inter-sampling times are avoided even when the proposed event-triggered strategy is employed.

In the next section we show how to obtain an expression similar to (15) in the case of time delay.

5.2. Time delays

In this section we show how to design a self-triggered sampler when time delays are present. We assume that time delays are smaller than the inter-sampling times. For generic time delays, we can set up the system as described in [27] and proceed with the analysis as we describe next. If time delays are smaller than the inter-sampling times, the input applied to the system is piecewise constant, and satisfies

$$\begin{cases} u = \kappa(x_{k-1}) & \text{for } t \in [t_k, t_k + \tau_k), \\ u = \kappa(x_k) & \text{for } t \in [t_k + \tau_k, t_{k+1}), \end{cases} \tag{18}$$

where τ_k is the elapsed time between the instant when the measurement x_k is picked and the instant in which the actuator is updated. The dynamics of the sampled-data system (4) are split into

$$\dot{x} = f(x, \kappa(x), d) + f(x, \kappa(x_{k-1}), d) - f(x, \kappa(x), d), \tag{19}$$

for $t \in [t_k, t_k + \tau_k)$ and

$$\dot{x} = f(x, \kappa(x), d) + f(x, \kappa(x_k), d) - f(x, \kappa(x), d), \tag{20}$$

for $t \in [t_k + \tau_k, t_{k+1})$. The perturbation due to the sampling for $t \in [t_k, t_{k+1})$ is also split into two terms: the first term depends on the measurement x_{k-1} and it acts for $t \in [t_k, t_k + \tau_k)$; the second term depends on the measurement x_k and it acts for $t \in [t_k + \tau_k, t_{k+1})$. In both cases there is also a perturbative term due to the external disturbance. To design a self-triggered sampler when time delays are present, it is convenient to consider time intervals of the form $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$. As done in the previous section, the self-triggered sampler is designed by exploiting an upper-bound of the perturbation term due to the sampling. By proceeding along the same line as in the proof of Lemma 5.1, we have

$$\|g(t)\| \le \hat{g}(p, q, x_{k-1}, \tau_k)e^{2L(t-t_k-\tau_k)} + \frac{1}{2}\|f(p, \kappa(x_k), q)\|\left(e^{2L(t-t_k-\tau_k)} - 1\right), \tag{21}$$

for $t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1})$. As we discussed, the underlying idea of the presented self-triggered samplers is to predict the time needed to the function $\|g(t)\|$ to go from 0 to δ. In the case without time delays, the function is reset to zero, and it holds $\|g(t_k^+)\| = 0$. Since $\delta > 0$, by exploiting the continuity of $\|g(t)\|$ we have shown that the inter-sampling times are lower-bounded. In the case with time delays, the situation is slightly different. Indeed, we should predict the time it takes for $\|g(t)\|$ to go from $\|\hat{g}(t_k + \tau_k)\|$ to δ. However, for $\tau_k > 0$, it holds $\|\hat{g}(t_k + \tau_k)\| \neq 0$ which leads to $t_{k+1} - t_k \le 0$ if $\|\hat{g}(t_k + \tau_k)\| \ge \delta$. Thus, in the case with time delays, we need an additional condition to ensure a finite lower-bound of the inter-sampling times. Such a condition is given by the next result.

Corollary 5.3. *Suppose that the assumptions of Corollary 5.1 hold and consider the self-triggered sampler*

$$t_{k+1} = t_k + \tau_k - \tau_{\max} + \frac{1}{2L}\ln\left(\frac{2\delta + \|f(p, \kappa(x_k), q)\|}{2\hat{g}(p, q, x_{k-1}, \tau_k) + \|f(p, \kappa(x_k), q)\|}\right). \tag{22}$$

If δ is chosen such that

$$\delta > \hat{g}(p, q, \bar{x}, \tau_{\max}) + \varepsilon, \tag{23}$$

where $\varepsilon >$ is an arbitrary positive constant and $\bar{x} = \arg\max_{y \in \mathcal{B}_{\|x\|_{\mathcal{L}_\infty},0}} \hat{g}(p, q, y, \tau_{\max})$, then the trajectories of the closed loop system (19)–(20) are GUUB and there exists a constant $c(\varepsilon) > 0$ such that $t_{k+1} - t_k > c(\varepsilon)$ for all k. $\square$

Notice that the implementation of the self-triggered sampler (22) requires the knowledge of the time delay τ_k. However, if this information is not available it is possible to use a slightly more conservative implementation of (22) by replacing τ_k with $\tau_{\max}$.

Remark 5.4. Corollary 5.3 gives a tradeoff among the inter-sampling time, the size of the ultimate bound region and the maximum allowable time delay. For example, given a maximum time delay $\tau_{\max}$, condition (23) can be fulfilled just by increasing the value of δ. On the other hand, an increase of δ leads to an increase of the ultimate bound region size, but it also enlarges the inter-sampling times, since the function $\hat{g}(p, q, x_k, t - t_k)$ would take more time to reach the triggering threshold δ. $\square$

In the next section we show how to reduce the conservativeness of the proposed self-triggered sampler by means of disturbance observers.

6. Disturbance observers

In this section we present a method based on disturbance observers to reduce the conservativeness of the approximation of the event-based strategy (8) through the proposed self-triggered samplers. All the presented observers provide a zero-order estimate $\hat{d}_k$ of a disturbance acting for $t \in [t_k, t_{k+1})$.

The utilization of disturbance observers is motivated as follows: in the self-triggered sampler (14), information about the maximum disturbance d^* explicitly appears in the formulas to compute the next sampling times. For all the times in which there are no external disturbances acting on the systems, the computed next sampling time may be rather conservative since the self-triggered sampler has been devised by implicitly assuming a worst-case disturbance d^* acting for all the times. On the other hand, by neglecting possible disturbances, i.e. by setting $d^* = (0, \ldots, 0)^T$, what happens in reality is that the system is sampled when the condition $\|f(x, k(x_k), d) - f(x, k(x), d)\| \le \delta'$ is violated, where $\delta' > \delta$, although the self-triggered sampler "believes" to sample on δ. However, since the external disturbance d is assumed to be bounded, then δ' is also bounded. Therefore, the perturbation due to the sampling is still bounded and we can conclude that the UUB property of the closed loop system does not change either if we consider a disturbance acting for all the time, or if we neglect it at all. What it does change is the behavior of the system between the inter-sampling times. By neglecting external disturbances, we achieve larger inter-sampling times but the system response may exhibit large peaks if a disturbance suddenly enters the system between two distant sampling instants. On the other hand, such peaks can be reduced by assuming a maximum disturbance affecting the system for all the time, but in this case the system is unnecessary over-sampled when there are no external disturbances acting on the system.

A tradeoff between the system response performance and the conservativeness of the inter-sampling times can be achieved by means of disturbance observers. That way, the model used by the self-triggered sampler is kept as close as possible to the real model of the system, and we would expect the best performance in terms of system response and approximation of the event-triggered sampling rule. However, being the system sampled aperiodically, the design of a disturbance observer may be a tricky task. Although we believe that the design of disturbance observers for an aperiodically sampled nonlinear system is an interesting topic for future research, for the sake of completeness we provide here three methods to design them. Without loss of generality, we consider the case without time delays, and, in all the presented observers, the goal is to estimate a constant disturbance $\hat{d}_k$ acting on the system for $t \in [t_k, t_{k+1})$, which is used in the expressions of the self-triggered samplers instead of d^*. The estimate of $\hat{d}_k$ is performed at time $t = t_{k+1}$ based on the past and the current measurement, on an estimate of the current measurement. The disturbance estimate $\hat{d}_k$ is used to determine the next time t_{k+2}.

The first disturbance observer we present is the nonlinear version of the disturbance observer for linear system provided in [5]. By discretizing the system $\dot{x} = f(x, \kappa(x_k), \hat{d}_k)$ over the time interval $[t_k, t_{k+1})$, we obtain the following discrete-time system

$$\tilde{x}_{k+1} = \tilde{f}(\tilde{x}_k, \kappa(\tilde{x}_k), \hat{d}_k), \tag{24}$$

where $\tilde{x}_k = x_k$. At time $t = t_{k+1}$ i.e. when the measure x_{k+1} is available, we can pick $\hat{d}_k$ such that $x_{k+1} = \tilde{x}_{k+1}$. Notice that to obtain an expression of the observer in closed form, we need to discretize the nonlinear system $\dot{x} = f(x, \kappa(x_k), \hat{d}_k)$ and to compute the inverse of $\tilde{f}$, and, with exception of few lucky cases, such computations require numerical methods.

Another method is to jointly use the sensitivity function [26] and the triggering rule. A first-order disturbance observer is given by

$$\hat{d}_k = \hat{d}_{k-1} + S^{-1}(t_{k+1})(x_{k+1} - \hat{x}_{k+1}), \tag{25}$$

where $\hat{x}_{k+1}$ is an estimate of the state at time $t = t_{k+1}$ and $S(t_{k+1})$ is the realization of the sensitivity function at time $t = t_{k+1}$, whose dynamics satisfies

$$\dot{S}(t) = A(x(t))S(t) + F(x(t)), \qquad S(t_k) = 0, \tag{26}$$

where

$$A(x(t)) := \left.\frac{\partial f(x, \kappa(x_k), d)}{\partial x}\right|_{\substack{x=x(t-t_k,x_k)\\ d=\hat{d}_{k-1}}}, \tag{27}$$

and

$$F(x(t)) := \left.\frac{\partial f(x, \kappa(x_k), d)}{\partial d}\right|_{\substack{x=x(t-t_k,x_k)\\ d=\hat{d}_{k-1}}}, \tag{28}$$

where $x(t - t_k, x_k)$ is the solution of (4) with initial condition x_k. To design the observer (25), we need the estimate $\hat{x}_{k+1}$. Nevertheless, such an estimate can be obtained by picking $\hat{x}_{k+1}$ such that $\|f(x_k, \kappa(x_k), 0) - f(\hat{x}_{k+1}, \kappa(x_k), 0)\| = \delta$. Unfortunately, we have infinity values of $\hat{x}_{k+1}$ which satisfy the previous condition. To guess which is the closest value $\hat{x}_{k+1}$ to the real state x_{k+1}, we linearize the system $\dot{x} = f(x, \kappa(x_k), \hat{d}_k)$ around x_k and $\hat{d}_{k-1}$, and we compute the value of the state at time $t = t_{k+1}$ of the linearized system $\dot{x}^\ell = f(x_k, \kappa(x_k), d_{k-1}) + A(x_k)(x^\ell - x_k)$.

Therefore, to design the observer (25), it is enough solve the following optimization problem

$$\begin{aligned}
&\min_{\hat{x}_{k+1}} \|\hat{x}_{k+1} - x^\ell_{k+1}\| \\
&\text{s.t. } \|f(x_k, \kappa(x_k), 0) - f(\hat{x}_{k+1}, \kappa(x_k), 0)\| = \delta, \\
&x^\ell_{k+1} = A_d(x_k)x^\ell_k + B_d(x_k),
\end{aligned} \tag{29}$$

where

$$A_d(x_k) = \exp(A(x_k)(t_{k+1} - t_k)), \tag{30}$$

and

$$B_d(x_k) = \int_{t_k}^{t_{k+1}} \exp(A(x_k)(t_{k+1} - \sigma))\big(f(x_k, \kappa(x_k), d_{k-1}) - A(x_k)x_k\big)\, d\sigma. \tag{31}$$

Unfortunately, this method requires the solution $x(t - t_k, x_k)$ of the system (4), which is in general unknown.

Finally, a last method to design a disturbance observer resorts to Lyapunov analysis. By taking a look at the proof of Proposition 4.1, it is not difficult to get the inequality $\dot{V} \le \Phi(V, \bar{d})$, where the function Φ depends on the comparison functions $\alpha_{1,\ldots,4}$. Let us now consider the differential equation, for $t \in [t_k, t_{k+1})$

$$\dot{\tilde{V}} = \Phi(\tilde{V}, \|\hat{d}_k\|), \tag{32}$$

and let $\phi(\tilde{V}(t_k), \|\hat{d}_k\|, t - t_k)$ be a solution of the previous differential equation with initial condition $\tilde{V}(t_k) = V(t_k)$. At time $t = t_{k+1}$ we take $\|\hat{d}_k\|$ such that $|V(t_{k+1}) - \phi(\tilde{V}(t_k), \|\hat{d}_k\|, t_{k+1} - t_k)| = 0$. The advantage of this method is that the computation the solution of (32) may be easier than the computation of the solution of $\dot{x} = f(x, \kappa(x_k), \hat{d}_k)$. The drawback of this method is that it provides only an estimate of the magnitude $\|d_k\|$ of the disturbance d. However, since $\|f(x, u, d)\| \leq \|f(x, u, 0)\| + L_{f,d}\bar{d}$, we can replace the terms $\|f(x^*, \kappa(x_k), d^*)\|$ and $\|f(p, \kappa(x_k), q)\|$ with $\|f(x^*, \kappa(x_k), 0)\| + L_{f,d}\|\hat{d}_k\|$ and $\|f(p, \kappa(x_k), 0)\| + L_{f,d}\|\hat{d}_k\|$ in the self-triggered samplers (14) and (15), respectively, and exploit information about the estimate $\|\hat{d}_k\|$ obtained with this last presented approach. However, depending on the Lipschitz constant $L_{f,d}$ and the quality of estimations $\|\hat{d}_k\|$, we may achieve a more or less accurate approximation of the sampling rule (8) when using this observer. The quality of the estimations $\|\hat{d}_k\|$ with this last method depends on how tight the inequalities (3) are.

Notice that using the values $d_k = 0$, $d_k = d^*$ or using any other estimate of the disturbance in the self-triggered sampler expressions, does not change the stability properties of the sampled-data system, but it changes the accuracy of the model used by the self-triggered sampler.

7. Simulation results

7.1. Perturbed system

In this example we borrow the system used in [23] for which a controller is used that renders the closed-loop system globally asymptotically stable, but does not render the closed-loop ISS with respect to measurement errors. Moreover, we add an external disturbance and we compare the results when a disturbance observer is used or not, and we further compare our self-triggered sampler with a continuous-time implementation of the controller. The dynamics of the perturbed system are given by

$$\dot{x} = \left(I + 2\Theta\left(\frac{\pi}{2}\right)xx^T\right)\Theta(x^Tx)\cdot\left(\begin{bmatrix}-1 & 0\\ 0 & x^Tx\end{bmatrix}\Theta(-x^Tx)x + \begin{bmatrix}0\\1\end{bmatrix}(u+d)\right), \tag{33}$$

where

$$\Theta(\theta) := \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

By considering the coordinate transformation $z = \Theta(-x^Tx)x$, system (33) in the z-coordinates becomes

$$\begin{aligned}\dot{z}_1 &= -z_1,\\ \dot{z}_2 &= (z_1^2 + z_2^2)z_2 + u + d.\end{aligned} \tag{34}$$

By using such a coordinate transform, a stabilizing control for (33) is given by

$$u = H(\Theta(-x^Tx)x), \tag{35}$$

where the function $H : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$H(z) := -(1 + z_1^2 + z_2^2)z_2. \tag{36}$$

Since the coordinate transformation is a diffeomorphism, and since $\|x\| = \|z\|$, if we achieve UUB for the system in the new coordinate, then the original system is UUB. Hence, we design the self-triggered sampler to ensure UUB of (34) to determine when to update the control (35) for the system (33).

A Lyapunov function for the system (34) with the control (35), is given by $V(z) = 0.5z^Tz$. Then, we get $\alpha_1(\|z\|) = \alpha_s(\|z\|) = 0.5\|z\|^2$ and $\alpha_4(\|z\|) = \|z\|$ which implies $\|z(t)\|_{\mathcal{L}_{\infty,0}} = \|z_0\|$ and then $\|z(t)\|_{\mathcal{L}_{\infty,k}} \leq \max\{\|z_k\|, b\}$. Then, at each sampling instant we adapt z^* with $z^* = (\|z(t)\|_{\mathcal{L}_{\infty,k}}, \|z(t)\|_{\mathcal{L}_{\infty,k}})^T$. By considering an operating region $\mathcal{B}_r$, where $r = 5$, we get $L_{f,u} = L_{f,d} = 1$ and , $L_{\kappa,x} \leq \sqrt{16\|z(t)\|^4_{\mathcal{L}_{\infty,k}} + 12\|z(t)\|^2_{\mathcal{L}_{\infty,k}} + 1}$ for all $t \geq t_k$, and then we adapt the Lipschitz constant $L_{\kappa,x}$ at each sampling instant accordingly. The simulation is performed by considering $x_0 = (4, -3)^T$ as initial condition, and we considered an external disturbance of $d = \bar{d} = 0.4$ acting for $t \in [5.3, 8]$ s and $d = 0$ otherwise. By choosing $\delta = 0.25$ and $\vartheta = 0.999$ we get the ultimate bound $b = 0.5/\vartheta(\delta + \bar{d}) = 0.325$.

Due to the simplicity of the Lyapunov function $V(z)$, we designed a disturbance observer by exploiting such a $V(z)$. Using standard arguments we get $\dot{V} \leq -2V + \sqrt{2V}(\|d\| + \delta)$. Now let $\dot{\tilde{V}} = -2\tilde{V} + \sqrt{2\tilde{V}}(\|d\| + \delta)$ and let $W = \sqrt{\tilde{V}}$. It holds $\dot{W} = -1/2W + \sqrt{2}/2(\|d\| + \delta)$, and then we get the disturbance observer

$$\|\hat{d}_k\| = \min\left(\bar{d}, \frac{2}{\sqrt{2}}\left|\frac{W(t_{k+1}) - W(t_k)\exp\left(-\frac{1}{2}(t_{k+1} - t_k)\right)}{\left(1 - \exp\left(-\frac{1}{2}(t_{k+1} - t_k)\right)\right)} - \delta\right|\right). \tag{37}$$

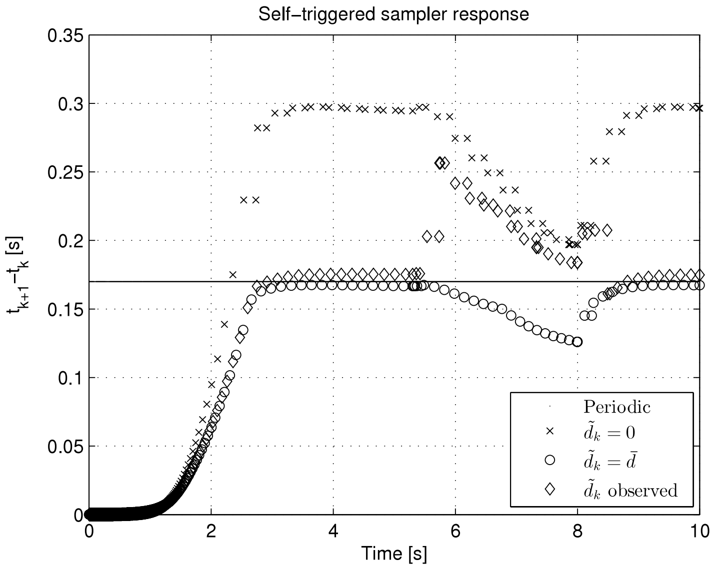

Fig. 3. State evolution with the self-triggered and with the continuous execution of the controller.

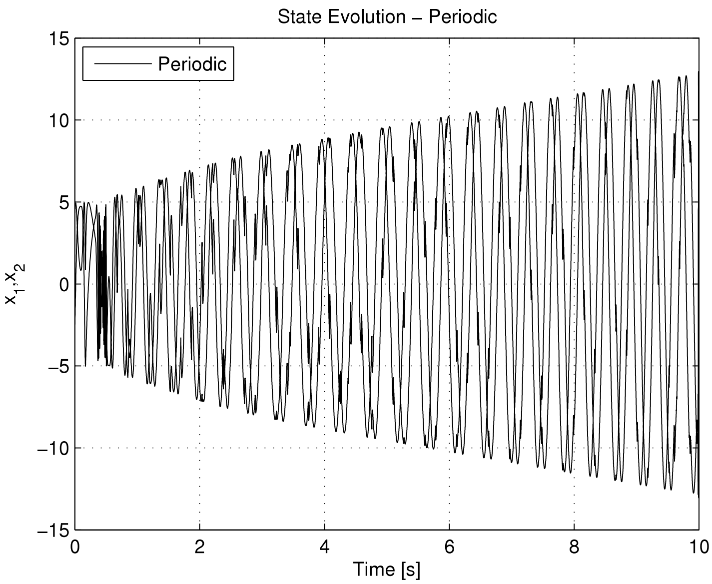

Fig. 4. State evolution with the periodic implementation of the controller with period 170 ms.

Finally we compare the number of the controller updates with a quadratic performance index J given by

$$J = \int_0^{t_{\text{sim}}} \|x(\sigma)\|^2 + |u(\sigma)|^2 \, d\sigma.$$

The simulation result is depicted in Figs. 3–5 and some numerical values are summarized in Table 1 for a simulation time $t_{\text{sim}} = 10$ s. For all the self-triggered implementations we have roughly the same performance of the closed-loop system, which is surprisingly lower compared to the continuous-time case. However, the number of controller updates changes between the self-triggered samplers. As expected, we have the largest number of updates when $\|d_k\| = \bar{d}$ and the smallest when $\|d_k\| = 0$. The same happens for the overshoot when the disturbance enters the system. A tradeoff is achieved when the disturbance observer is employed. The behavior of both the inter-sampling times and the system response when $\hat{d}_k = \bar{d}$ or when $\|\hat{d}_k\|$ is observed is similar because of the conservativeness of the disturbance observer (37). Finally, note that the inter-sampling times provided by both the self-triggered samplers with $\|d_k\| = \bar{d}$ and with $\|d_k\|$ observed converge to a steady-state value of $t_{k+1} - t_k$ of $\sim$170 ms. However, by using a periodic implementation of the controller with period 170 ms the closed-loop system is unstable, as shown in Fig. 4.

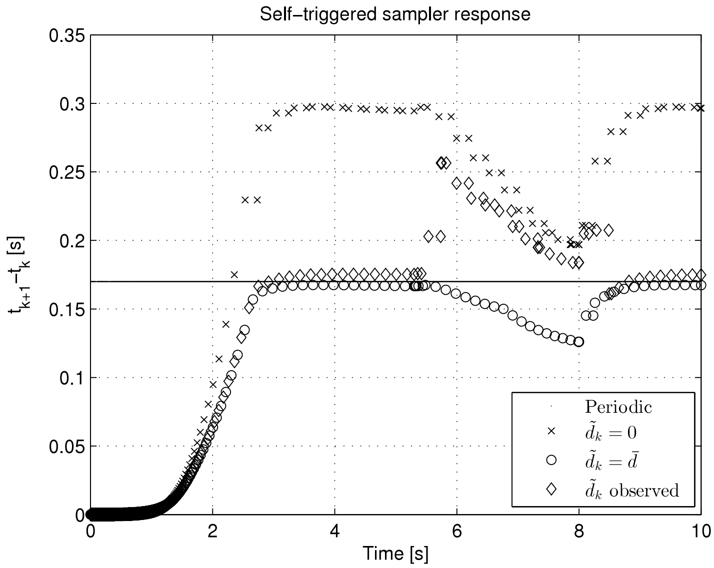

Fig. 5. Inter-sampling times obtained with the self-triggered and with the periodic implementation of the controller.

Table 1
The table shows the number of controller updates and the value of J for all the considered cases for a simulation time of 10 s.

	Continuous	$\|\hat{d}_k\| = 0$	$\|\hat{d}_k\| = \bar{d}$	$\|\hat{d}_k\|$ obsv.
Ctrl. updates	–	12 803	12 907	12 891
J	1525.2	1503.6	1503.6	1503.6

7.2. Unperturbed system — comparison with [21]

Now, we apply the method in [21] to design a self-triggered implementation of the proposed sampling rule and we compare with our self-triggered implementation in the disturbance-free case. In the disturbance-free case, the application of our self-triggered sampler gives an ultimate bound $b = 0.125$. In order to apply [21], we have to render both the continuous-time closed-loop system and the sampling rule homogeneous. The closed-loop system with continuous control is given by

$$\begin{aligned} \dot{z}_1 &= -z_1, \\ \dot{z}_2 &= -z_2. \end{aligned} \tag{38}$$

To render (38) homogeneous, it is enough to add a state variable $\dot{w} = 0$, with $w(t_0) = 1$. The homogeneous system associated to (38) satisfies

$$\begin{aligned} \dot{z}_1 &= -wz_1, \\ \dot{z}_2 &= -wz_2, \\ \dot{w} &= 0. \end{aligned} \tag{39}$$

The sampling function $\|g(t)\|$ for the system to (34) is given by $\|H(z_k) - H(z)\|$ which is not homogeneous. However, by defining $\Gamma(z_1, z_2) := \|H(z_k) - H(z)\| - \delta$ we have the sampling rule implicitly defined by $\tilde{\Gamma}(z_1, z_2, w) \le 0$, where $\tilde{\Gamma}(z_1, z_2, w) = w\Gamma(w^{-1}z_1, w^{-1}z_2)$ is homogeneous. By computing the Lie derivative of $\tilde{\Gamma}(z_1, z_2, w)$ along the trajectories of (39) it holds

$$\begin{pmatrix} \dfrac{\partial \tilde{\Gamma}}{\partial z} & \dfrac{\partial \tilde{\Gamma}}{\partial w} \end{pmatrix}^T \begin{pmatrix} -wz_1 \\ -wx_2 \\ 0 \end{pmatrix} = \frac{H(z_k) - H(z)}{|H(z_k) - H(z)|} \begin{pmatrix} 2w^{-2}z_1z_2 & 3w^{-2}z_2^2 \end{pmatrix}^T \begin{pmatrix} -wz_1 \\ -wx_2 \end{pmatrix} \tag{40}$$

$$= \operatorname{sign}(H(z_k) - H(z))(-2w^{-1}z_1z_2 - 3w^{-1}z_2^3). \tag{41}$$

Hence, there does not exist a finite value of χ_0 which satisfies

$$\operatorname{sign}(H(z_k) - H(z))(-2w^{-1}z_1z_2 - 3w^{-1}z_2^3) \le \chi_0 w|H(z_k) - H(w^{-1}z)| - w\delta, \tag{42}$$

for any z, and then we cannot use the method proposed in [21] to compute an isochronous manifold. However, we can consider a ball as an approximation of the isochronous manifold, which is the method proposed in [11]. For any point $z \in \mathcal{B}_1$,

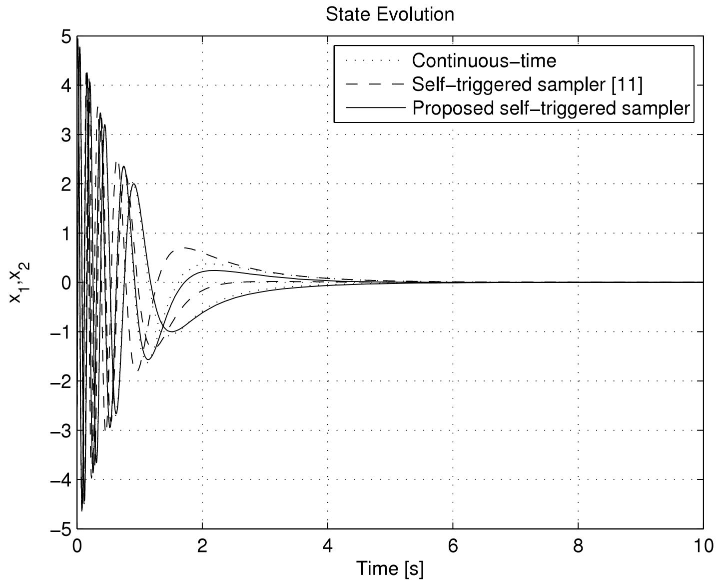

Fig. 6. State evolution with the proposed and the self-triggered in [11].

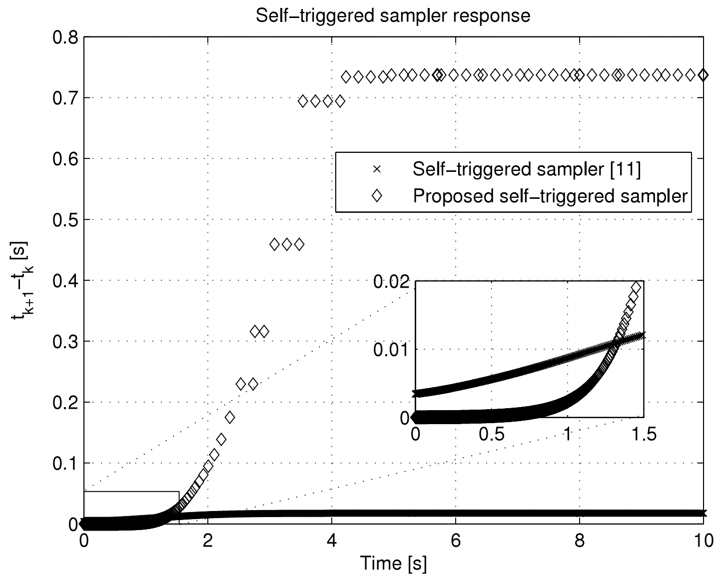

Fig. 7. Inter-sampling times obtained with the proposed self-triggered and with the self-triggered sampler [11].

a lower bound of the inter-sampling times provided by the sampling rule implicitly defined by $\tilde{\Gamma}(z_1, z_2, w) \leq 0$ is $t_* = 17.5$ ms, which has been found by using our self-triggered sampler. Then, the self-triggered sampler with the method proposed in [11] is given by

$$t_{k+1} = t_k + \frac{t^*}{\|(z_{1,k}, z_{2,k}, 1)\|}. \tag{43}$$

Finally, the ultimate bound provided when using our self-triggered sampler in the disturbance free case is $b = 0.125$.

The simulation results are depicted in Figs. 6 and 7. At the beginning the self-triggered sampler (43) provides large inter-sampling times, but after $t \simeq 1.4$ s we have the opposite situation. Moreover, the proposed self-triggered sampler converges to a sampling period which is larger compared to the self-triggered sampler (43). Regarding the number of updates and the system performance, we get 12 786 and 736 updates and $J = 1502.94$ and $J = 2545$ with the proposed self-triggered sampler and with the self-triggered sampler (43), respectively. This means that the utilization of proposed self-triggered sampler provides a better system response with respect to (43), but, on the other hand, it requires a larger number of samples. However, this is true only during the transient phase, since in steady-state the proposed self-triggered sampler converges to a larger value. Finally, note how the system response with our self-triggered sampler is closer to the system response with continuous control, for which we obtained $J \simeq 1524$.

Note that a source of conservatism of the self-triggered sampler (43) is due because of the homogenization procedure of the closed-loop system, which enforces $\min_k \|(z_{1,k}, z_{2,k}, 1)\| = 1$. Better results can be achieved by exploiting isochronous manifolds, but, to the best of our knowledge, the only method to approximate an isochronous manifold is given in [21], which unfortunately does not apply to our case.

8. Conclusions

In this paper we presented a simple self-triggered sampler for perturbed nonlinear systems subject to bounded external disturbances and small time delay. To reduce the conservativeness of the proposed approach, techniques based on disturbance observers are discussed, and different disturbance observers have been proposed. Finally, a discussion about the proposed sampling rule has been performed, and a result related to the robustness of the well-known Lebesgue sampling has been provided. Simulation results showed the effectiveness of the proposed method.

Acknowledgments

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Appendix

Proof of Proposition 4.1. Let $V(x)$ be a Lyapunov function which satisfies (3). The time derivative of V along the trajectories of the sampled-data system (4), for $t \in (t_k, t_{k+1})$, satisfy

$$\dot{V} = \frac{\partial V}{\partial x} f(x, \kappa(x), 0) + \frac{\partial V}{\partial x}\big(f(x, \kappa(x_k), d) - f(x, \kappa(x), d)\big) \tag{44}$$

$$+ \frac{\partial V}{\partial x}\big(f(x, \kappa(x), d) - f(x, \kappa(x), 0)\big) \tag{45}$$

$$\leq -\alpha_3(\|x\|) + \alpha_4\big(\|x\|)(\|g\| + L_{f,d}\bar{d}\big). \tag{46}$$

By using the triggering rule (8), and since α_4 is increasing it holds

$$\begin{aligned}\dot{V} &\leq -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\delta + L_{f,d}\bar{d}) \\ &= -(1-\vartheta)\alpha_3(\|x\|) + \vartheta\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\delta + L_{f,d}\bar{d}),\end{aligned}$$

for any $\vartheta \in (0, 1)$. Then, we have $\dot{V} \leq -(1-\vartheta)\alpha_3(\|x\|)$ if

$$\|x\| \geq \alpha_3^{-1}\left(\frac{\alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\delta + L_{f,d}\bar{d})}{\vartheta}\right), \tag{47}$$

and the system is UUB provided that the perturbative term $\delta + L_{f,d}\bar{d}$ is sufficiently small to satisfy (7). By observing that at each sampling time $t = t_k$ it holds $\dot{V} \leq -\alpha_3(\|x_k\|) + \alpha_4(\|x_k\|)L_{f,d}\bar{d}$ or, if (47) is satisfied, it holds $\dot{V} \leq -(1-\vartheta)\alpha_3(\|x_k\|)$, and that $\|g(t)\|$ is continuous over $[t_k, t_{k+1})$, then by triggering accordingly to (8) it follows that the system is UUB over the set $\mathcal{D}_x$. $\square$

Proof of Proposition 4.2. It is enough to rewrite $f(x, \kappa(x_k), d)$ as $f(x, \kappa(x_k), d) = f(x, \kappa(x), 0) + f(x, \kappa(x_k), 0) - f(x, \kappa(x), 0) + f(x, \kappa(x_k), d) - f(x, \kappa(x_k), 0)$ and proceed analogously as in the proof of Proposition 4.1. $\square$

Proof of Lemma 5.1. Let $g(t) = f(x(t), u_k, d(t)) - f(x(t), u(t), d(t))$, where $u(t) = \kappa(x(t))$. It holds

$$\begin{cases} \dfrac{d}{ds}g(s) = \dfrac{d}{ds}f(x(t), u(s), d(t)) = -\dfrac{\partial}{\partial u}f(x(t), u_k, d(t))\dfrac{\partial\kappa(x(s))}{\partial x}\dot{x}(s) \\ \quad = -\dfrac{\partial}{\partial u}f(x(t), u(s), d(t))\dfrac{\partial\kappa(x(s))}{\partial x}f(x(s), u_k, d(s)), \\ g(s_k) = 0. \end{cases} \tag{48}$$

The solution of the previous differential equation is given by

$$g(s) = \int_{s_k}^{s} -\frac{\partial f}{\partial u}\frac{\partial\kappa}{\partial x}f(x(\sigma), u_k, d(\sigma))d\sigma. \tag{49}$$

By adding and subtracting the term $\int_{s_k}^{s} \frac{\partial f}{\partial u}\frac{\partial \kappa}{\partial x} f(x(\sigma), u(\sigma), d(\sigma))d\sigma$ and by taking the norm of both sides it holds:

$$\begin{aligned}\|g(s)\| &\le \int_{s_k}^{s} L\|f(x(\sigma), u_k, d(\sigma)) - f(x(\sigma), u(\sigma), d(\sigma))\|d\sigma + \int_{s_k}^{s} L\|f(x(\sigma), u(\sigma), d(\sigma))\|d\sigma \\ &\le \int_{s_k}^{s} L\|f(x(\sigma), u_k, d(\sigma)) - f(x(\sigma), u(\sigma), d(\sigma))\|d\sigma \\ &\quad + \int_{s_k}^{s} L\|f(x(\sigma), u_k, d(\sigma)) - f(x(\sigma), u(\sigma), d(\sigma))\|d\sigma + \int_{s_k}^{s} L\|f(x(\sigma), u_k, d(\sigma))\|d\sigma \\ &= \int_{s_k}^{s} 2L\|g(\sigma)\|d\sigma + \int_{s_k}^{s} L\|f(x(\sigma), u_k, d(\sigma))\|d\sigma. \end{aligned} \tag{50}$$

By using the Leibniz Theorem, we get

$$\begin{aligned}\frac{d}{ds}\|g(s)\| &\le 2L\|g(s)\| + L\|f(x(s), u_k, d(s))\| \\ &\le 2L\|g(s)\| + L\|f(x^*, u_k, d^*)\|, \end{aligned} \tag{51}$$

and then

$$\|g(s)\| \le \frac{1}{2}\|f(x^*, u_k, d^*)\|\left(e^{2L(s-s_k)} - 1\right), \tag{52}$$

for all $s \in [s_k, s_{k+1})$. □

Proof of Theorem 5.1. By considering the Lyapunov function (3), we have, for $t \in (t_k, t_{k+1})$

$$\begin{aligned}\dot V &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)(\|g\| + L_{f,d}\bar d) \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\hat g(x^*, d^*, x_k, t - t_k) + L_{f,d}\bar d).\end{aligned}$$

Since $\hat g(x^*, d^*, x_k, t - t_k)$ is continuous and it is strictly increasing with t, then there exists a time t_{k+1} satisfying

$$\hat g(x^*, d^*, x_k, t_{k+1} - t_k) = \delta.$$

Hence, the Lyapunov derivative is further bounded with

$$\dot V \le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\delta + L_{f,d}\bar d).$$

By following the same line as in the proof of Proposition 4.1, it is easy to prove uniformly ultimate boundedness. Hence, the next triggering time is given by the inverse of $\hat g(x^*, d^*, x_k, t_{k+1} - t_k)$, which gives (14). □

Proof of Corollary 5.1. It is enough to observe that, for $t \in (t_k, t_{k+1})$, the Lyapunov derivative can be rewritten as

$$\begin{aligned}\dot V &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\|g\| + L_{f,d}\bar d) \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\big(\hat g(p, q, x_k, t - t_k) + L_{f,d}\bar d + \hat g(x^*, d^*, x_k, t - t_k) - \hat g(p, q, x_k, t - t_k)\big).\end{aligned}$$

By using the sampling rule (15), it holds

$$\hat g(x^*, d^*, x_k, t_{k+1} - t_k) - \hat g(p, q, x_k, t_{k+1} - t_k) \le \delta\left(\frac{\|f(x^*, \kappa(x_k), d^*)\|}{\|f(p, \kappa(x_k), q)\|} - 1\right).$$

Thus, the Lyapunov derivative can be bounded in the time interval (t_k, t_{k+1}) with

$$\dot V \le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\left(\delta\frac{\|f(x^*, \kappa(x_k), q)\|}{\|f(p, \kappa(x_k), q)\|} + L_{f,d}\bar d\right). \tag{53}$$

Now, since it holds

$$\begin{aligned}\frac{\|f(x^*, \kappa(x_k), d^*)\|}{\|f(p, \kappa(x_k), q)\|} &\le 1 + \frac{\|f(x^*, \kappa(x_k), d^*) - f(p, \kappa(x_k), q)\|}{\|f(p, \kappa(x_k), q)\|} \\ &\le 1 + \frac{L_{f,x}\|x^* - p\| + L_{f,d}\|d^* - q\|}{m} := M,\end{aligned}$$

the Lyapunov derivative can be further upper-bounded with

$$\dot V \le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})(\delta' + L_{f,d}\bar d),$$

where $\delta' = \delta M$, and then GUUB can be proved by following the same steps as in the proof of Theorem 5.1.

Finally, since it holds that $f(x, u, d) = 0$ if, and only if $x = u = d = 0$ and since f is continuous in the origin, it is enough to take $(p, q) \in \mathbb{R}^n \setminus \mathcal{S}_x \times \mathcal{D}_d \setminus \mathcal{S}_d$, where $\mathcal{S}_x \times \mathcal{S}_d$ is a non-empty compact set which contains the origin, to ensure the existence of $m > 0$ such that $\|f(p, y, q)\| > m > 0$ for all $y \in \mathbb{R}^p$. $\square$

Proof of Corollary 5.3. Since the GUUB property by using the self-triggered sampler (22) can be proved by using the same argument as in the proof of Corollary 5.1, here we prove only the existence of a lower-bound of the inter-sampling times. First of all, notice that to achieve $t_{k+1} - t_k > 0$, the argument of the logarithm in (22) must be greater than 1, and this happens if, and only if $\delta > \hat{g}(p, x_{k-1}, \tau_k)$, $\forall k$. Since it holds that $\hat{g}(p, \bar{x}, \tau_{\max}) \geq \hat{g}(p, \bar{x}, \tau_k) \geq \hat{g}(p, x_{k-1}, \tau_k)$ for all k, a sufficient condition to have $t_{k+1} - t_k > 0$ for all k is $\delta > \hat{g}(p, \bar{x}, \tau_{\max})$. Finally, since the sampling rule (22) is an increasing continuous function of δ, it follows that if $\delta > \hat{g}(p, \bar{x}, \tau_{\max}) + \varepsilon$, $\varepsilon > 0$, then there exists a $c(\varepsilon)$ such that $t_{k+1} - t_k > c(\varepsilon)$ for all k. $\square$

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