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Control of Multi-Agent Systems via Event-based Communication ⋆

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Abstract: In this paper, the average consensus problem for multi-agent systems is addressed. A novel event-based control strategy is proposed which renders both control signals and state measurements, which are broadcast over the network, piecewise constant. This enables implementation on digital platforms such as microprocessors. Different triggering conditions guaranteeing convergence to an adjustable region around the average consensus point or asymptotic convergence to this point, respectively, are discussed. Numerical simulations show the effectiveness of this approach, outperforming traditional time-scheduled control in terms of load on the communication medium. Both single- and double-integrator agents are covered.

Keywords: Networked Control Systems, Event-based Control, Multi-Agent Systems, Consensus

1. INTRODUCTION

Consensus problems for multi-agent systems have been in the focus of many researchers over the past years. This large interest is due to the variety of applications in engineering and science such as flocking, formation control and many more, cf. survey paper Olfati-Saber et al. (2007).

Another active research area within networked control is event-based control. In practice, controllers are usually implemented on digital computers. Hence, the control law is only updated at discrete time instances. These can either be prespecified by a constant period, which is referred to as time-scheduled control, or be determined by certain events that are triggered depending on the plant's behavior. The latter approach is called event-based control and was shown to be favorable in some cases by Åström and Bernhardsson (2002), and it was further developed by Tabuada (2007), Mazo Jr and Tabuada (2010).

Such event-based control strategies have recently been applied to the consensus problem by Dimarogonas and Johansson (2009), Dimarogonas and Frazzoli (2009). The authors use the event-based strategy introduced by Tabuada (2007) in order to schedule the agents' control updates. Each agent tracks its own and its neighbors' states in order to decide when to update the control law such that the overall system converges to average consensus asymptotically. Although this strategy renders each agent's control signal piecewise constant, the communication between neighboring agents is required to be continuous in time.

In this paper, we propose a novel event-based control strategy for multi-agent systems, which does not require continuous communication. On the contrary, each agent broadcasts its actual state over the network to its neighbors only at specific time instances, which are determined in an event-based fashion. A similar method was proposed by Wang and Lemmon (2008) for decentralized stabilization of physically coupled systems. The control law is updated whenever the agent sends or receives a new measurement value. Both the control signal and the broadcast states are rendered piecewise constant. We propose different triggering conditions which guarantee asymptotic convergence to average consensus or a specified region around the average consensus point, respectively. Numerical simulations show the effectiveness of the novel approach, outperforming traditional time-scheduled implementations.

The rest of this paper is organized as follows. Section 2 contains mathematical preliminaries and the problem statement for this work. Section 3 presents the novel event-based control strategy for multi-agent systems. In Section 4 numerical simulations are presented, including a comparison to the time-scheduled approach. The proposed strategy is extended to networks of double-integrator agents in Section 5, and Section 6 concludes the paper.

2. BACKGROUND AND PROBLEM STATEMENT

In this section some facts from algebraic graph theory are reviewed, cf. Godsil and Royle (2001). The system model is introduced and the problem statement is given.

2.1 Algebraic Graph Theory

Our notation is fairly standard, for the details we refer to Dimarogonas and Johansson (2009); Godsil and Royle (2001). For undirected graphs G with *vertex set* (or nodes) $\mathcal{V} = \{1, \ldots, N\}$ and *edge set* $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} : i,j \text{ adjacent }\}$, the *Laplacian matrix* L is defined as $L =$

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$D - A$, where D is the *degree matrix* and A the *adjacency matrix* of G. For connected graphs, L has exactly one zero eigenvalue $\lambda_1(G) = 0$ and the smallest non-zero eigenvalue $\lambda_2(G)$ is called *algebraic connectivity*.

2.2 System Model and Consensus Protocol

Consider the multi-agent system consisting of N agents with single-integrator dynamics $\dot{x}_i(t) = u_i(t)$, $i \in \mathcal{V}$, where u_i denote the control inputs. According to the associated communication graph G, each agent is assigned a *neighbor set* $N_i \subset \mathcal{V}$. With stack vectors $x = [x_1, ..., x_N]^T$ and $u = [u_1, ..., u_N]^T$ the overall system dynamics and initial conditions are given by

$$\dot{x}(t) = u(t), \qquad x(0) = x_0. \tag{1}$$

In this work the graphs G are assumed to be undirected and connected. However, the results extend easily to the case of directed graphs which are strongly connected and balanced, cf. Olfati-Saber and Murray (2004).

The distributed control law (or consensus protocol)

$$u_i(t) = -\sum_{j \in N_i} (x_i(t) - x_j(t)) \tag{2}$$

globally asymptotically solves the average consensus problem, i.e., the average of all agents' states remains constant over time and for all $i \in \mathcal{V}$, $x_i(t) \overset{t\to\infty}{\longrightarrow} \frac{1}{N}\sum_{i\in\mathcal{V}} x_i(0)$. With (1) and (2), the closed-loop system is given by

$$\dot{x}(t) = -Lx(t), \qquad x(0) = x_0. \tag{3}$$

This solution to the average consensus problem is presented in Olfati-Saber and Murray (2004).

In practice, agents like e.g. mobile robots are equipped with digital microprocessors which coordinate measurement acquisition, communication with other agents and control actuation. Thus it is necessary to implement the continuous-time law (2) on a digital platform. The classical method is the time-scheduled control strategy, i.e., measurements are acquired at discrete instances via zero-order hold techniques and control laws are updated periodically according to a constant sampling period τ_s, i.e.,

$$u(t) = -Lx(t_k), \qquad t \in [t_k, t_{k+1}[\tag{4}$$

where $t_{k+1} = t_k + \tau_s$, $t_0 = 0$. Xie et al. (2009) prove that control law (4), updated with constant sampling period τ_s, globally asymptotically solves the average consensus problem if and only if the sampling period satisfies

$$0 < \tau_s < 2/\lambda_N(G). \tag{5}$$

This result serves as benchmark for the event-based control strategy proposed in the present work.

2.3 Problem Statement

Each agent consists of a digital microprocessor and dynamics as shown in Fig. 1. The microprocessor monitors $x_i(t)$ continuously. Based on local information, it decides when to broadcast the actual measurement over the network. The latest broadcast value of agent i is a piecewise constant function $\hat{x}_i(t) = x_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i[$, where $t_0^i, t_1^i, ...$ is the sequence of events of agent i. Whenever one agent sends or receives a new measurement value, it updates its control law immediately, thus rendering the control signal

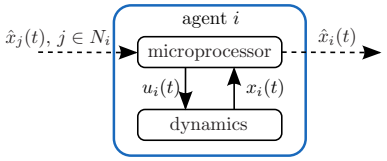

Fig. 1. Single agent in event-based control setup.

piecewise constant. Analogously to (2), the control law is defined as

$$u(t) = -L\hat{x}(t). \tag{6}$$

The problem is now to find a ruling which determines, based on local information, when agent i has to trigger and broadcast a new measurement value to its neighbors.

3. EVENT-BASED CONTROL STRATEGY

Define a *trigger function* $f_i(\cdot)$ which depends on local information of agent i only and map to $\mathbb{R}$. An event for agent i is triggered as soon as the *trigger condition*

$$f_i\left(t, x_i(t), \hat{x}_i(t), \bigcup_{j\in N_i} \hat{x}_j(t)\right) > 0 \tag{7}$$

is fulfilled. The sequence of events for agent i is thus defined iteratively by $t_{k+1}^i = \inf\left\{t : t > t_k^i, f_i(t) > 0\right\}$ where t_0^i is the first instance when (7) is fulfilled. Therefore, for each agent there is a monotonically increasing sequence of events $0 \le t_0^i \le t_1^i \le t_2^i \le \cdots$. It remains to derive suitable $f_i(\cdot)$, such that the closed-loop system reaches average consensus. Since the system under consideration is hybrid, well-posedness has to be guaranteed. In particular, it has to be shown that there are not infinitely many events in finite time, which is referred to as Zeno behavior. Zeno behavior can be excluded by proving that there is a positive lower bound on the inter-event times.

Before suitable trigger functions are presented, some useful variables are introduced. Define for each $i \in \mathcal{V}$ and $t \ge 0$ the *measurement error*

$$e_i(t) = \hat{x}_i(t) - x_i(t) \tag{8}$$

and denote the stack vector $e(t) = [e_1(t), ..., e_N(t)]^T$. The closed-loop system is then given by

$$\dot{x}(t) = -L\hat{x}(t) = -L(x(t) + e(t)). \tag{9}$$

Define the *average value* $a(t) = \frac{1}{N}\sum_{i\in\mathcal{V}} x_i(t)$. The initial average is $a(0)$. The time derivative of $a(t)$ is given by $\dot{a}(t) = (1/N)\mathbf{1}^T L\hat{x}(t) \equiv 0$ since $\mathbf{1}^T L = \mathbf{0}^T$. Therefore it holds that $a(t) = a(0) = a$ for all $t \ge 0$ and the state $x(t)$ can be decomposed according to $x(t) = a\mathbf{1} + \delta(t)$ with *disagreement vector* $\delta(t)$, following the notation of Olfati-Saber and Murray (2004). By definition, the disagreement vector has zero average, i.e., $\mathbf{1}^T\delta(t) \equiv 0$.

3.1 Static Trigger Function

In this subsection, we propose a static trigger function, which guarantees asymptotic convergence of all agents to a specified region around the consensus point. Before stating this result, we prove the following lemma.

Lemma 1. Suppose L is the Laplacian of an undirected, connected graph G. Then, for all $t \ge 0$ and $v \in \mathbb{R}^N$ with $\mathbf{1}^T v = 0$, it holds that $\|\exp(-Lt)v\| \le \exp(-\lambda_2(G)t)\|v\|$.

Proof. Since the graph G is undirected, its Laplacian L is symmetric, i.e., $L = L^T$. It is diagonalizable with an orthogonal matrix $T = [v_1, v_2, \dots, v_N]$, consisting of eigenvectors v_i corresponding to eigenvalues $\lambda_i = \lambda_i(G)$, $i \in \mathcal{V}$. Consequently it holds that $\exp(-Lt) = T\text{diag}(1, \exp(-\lambda_2 t), \dots, \exp(-\lambda_N t))T^T$. With $v_1 = (1/\sqrt{N})\mathbf{1}$, it follows that

$$e^{-Lt}v = \frac{1}{N}\mathbf{1}\mathbf{1}^T v + T \operatorname{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right) T^T v.$$

By assumption, $\mathbf{1}^T v = 0$, and consequently

$$\begin{aligned}\|e^{-Lt}v\| &= \|T \operatorname{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right) T^T v\| \\ &\leq \|T\|\|\text{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right)\|\|T^T\|\|v\| \\ &= e^{-\lambda_2 t}\|v\|.\end{aligned}$$

Theorem 2. Consider system (1) with control law (6) and undirected, connected graph G. Define the static trigger function

$$f_i(e_i(t)) = |e_i(t)| - c_0 \tag{10}$$

with $c_0 > 0$. Then, for all $x_0 \in \mathbb{R}^N$ and $t \geq 0$, it holds that

$$\|\delta(t)\| \leq \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0 + e^{-\lambda_2(G)t}\left(\|\delta(0)\| - \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0\right) \tag{11}$$

and the closed-loop system does not exhibit Zeno behavior.

The choice of trigger function (10) is intuitive, since an event is triggered as soon as the measurement error $e_i(t)$ crosses the specified threshold c_0. The errors are thus bounded by c_0. The system state converges to a region around the consensus point which scales with c_0.

Proof. The disagreement dynamics are given by

$$\dot{\delta}(t) = \dot{x}(t) = -Lx(t) - Le(t) = -L\delta(t) - Le(t) \tag{12}$$

with initial condition $\delta(0) = x(0) - a\mathbf{1}$. The analytical solution is $\delta(t) = \exp(-Lt)\delta(0) - \int_0^t \exp(-L(t-s))Le(s)ds$. Thus, the disagreement is bounded by

$$\|\delta(t)\| \leq \|e^{-Lt}\delta(0)\| + \int_0^t \|e^{-L(t-s)}Le(s)\|ds.$$

The vector $Le(t)$ has zero average since $L\mathbf{1} = \mathbf{0}$ and Lemma 1 can be applied. This yields

$$\|\delta(t)\| \leq e^{-\lambda_2(G)t}\|\delta(0)\| + \int_0^t e^{-\lambda_2(G)(t-s)}\|Le(s)\|ds. \tag{13}$$

Since $\|Le(t)\| \leq \|L\|\|e(t)\|$ and the trigger condition enforces $e_i(t) \leq c_0$, it holds that $\|Le(t)\| \leq \|L\|\sqrt{N}c_0$ and

$$\begin{aligned}\|\delta(t)\| &\leq e^{-\lambda_2(G)t}\|\delta(0)\| + \|L\|\sqrt{N}c_0 \int_0^t e^{-\lambda_2(G)(t-s)}ds \\ &= \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0 + e^{-\lambda_2(G)t}\left(\|\delta(0)\| - \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0\right).\end{aligned}$$

In order to exclude Zeno behavior, it remains to show that the inter-event times are lower-bounded by a positive constant τ. Assume that agent i triggers at time $t^* \geq 0$. Then it holds that $e_i(t^*) = 0$. Note that $f_i(0) = -c_0 < 0$ and therefore agent i cannot trigger again at the same instance of time. From (8), it follows that

$$\dot{e}_i(t) = -\dot{x}_i(t) = -u_i(t) \tag{14}$$

between the trigger events. We conclude that the next inter-event time is strictly positive through the following argument. Observe that

$$\begin{aligned}|u_i(t)| \leq \|u(t)\| &= \|L(x(t) + e(t))\| = \|L(\delta(t) + e(t))\| \\ &\leq \|L\|\left(\|\delta(t)\| + \|e(t)\|\right) \leq \|L\|\left(\|\delta(t)\| + \sqrt{N}c_0\right)\end{aligned} \tag{15}$$

for all $i \in \mathcal{V}$ and $t \geq 0$. From inequality (11) follows that $\|\delta(t)\| \leq \|\delta(0)\| + \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0$ for all $t \geq 0$, and with (15),

$$|u_i(t)| \leq \|L\|\left(\sqrt{N}c_0 + \|\delta(0)\| + \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0\right)$$

where the right-hand side is independent of t. With (14),

$$\begin{aligned}|e_i(t)| &\leq \int_{t^*}^t |u_i(s)|\,ds \\ &\leq \|L\|\left(\sqrt{N}c_0 + \|\delta(0)\| + \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0\right)(t - t^*)\end{aligned} \tag{16}$$

for all $t \geq t^*$. The next event is triggered as soon as (10) crosses zero, i.e., $|e_i(t)| > c_0$. From (16) it can be concluded that this is not fulfilled before

$$\|L\|\left(\sqrt{N}c_0 + \|\delta(0)\| + \frac{\|L\|}{\lambda_2(G)}\sqrt{N}c_0\right)(t - t^*) = c_0.$$

Thus, a lower bound on the inter-event times is given by $\tau = t - t^*$ which solves the latter equation. This bound holds for all times t^* and all agents i, therefore Zeno behavior of the closed-loop system is excluded.

3.2 Time-dependent Trigger Function

The next theorem presents a time-dependent trigger function, which drives the overall system to consensus asymptotically while guaranteeing that the inter-event times for all agents are lower-bounded by a positive quantity.

Theorem 3. Consider system (1) with control law (6) and undirected, connected graph G. Define the time-dependent trigger function

$$f_i(t, e_i(t)) = |e_i(t)| - c_1 e^{-\alpha t}, \tag{17}$$

where $c_1 > 0$ and $0 < \alpha < \lambda_2(G)$. Then, for all $x_0 \in \mathbb{R}^N$, the overall system converges to average consensus asymptotically and the closed-loop system does not exhibit Zeno behavior.

Proof. From (13) and with $\|Le(t)\| \leq \|L\|\sqrt{N}c_1\exp(-\alpha t)$, it follows that

$$\|\delta(t)\| \leq e^{-\lambda_2(G)t}\|\delta(0)\| + \frac{\|L\|\sqrt{N}c_1}{\lambda_2(G) - \alpha}\left(e^{-\alpha t} - e^{-\lambda_2(G)t}\right).$$

Since $\alpha > 0$, average consensus is reached asymptotically. Analogously to the reasoning in the proof of Theorem 2, it can be shown that the inter-event times for all agents are lower bounded by a positive constant τ. Assume again that agent i triggers at time $t^* \geq 0$. Analogously to (15) it holds that $|u_i(t)| \leq \|L\|(\|\delta(t)\| + \sqrt{N}c_1\exp(-\alpha t))$. With the bound on $\|\delta(t)\|$ and positive constants $k_1 = \|L\|\|\delta(0)\|$, $k_2 = \|L\|\sqrt{N}c_1\left(1 + \|L\|/(\lambda_2(G) - \alpha)\right)$ it follows that

$$|u_i(t)| \leq e^{-\lambda_2(G)t}k_1 + e^{-\alpha t}k_2 \leq e^{-\lambda_2(G)t^*}k_1 + e^{-\alpha t^*}k_2$$

for all $t \geq t^*$ and therefore

$$|e_i(t)| \leq \left(e^{-\lambda_2(G)t^*}k_1 + e^{-\alpha t^*}k_2\right)(t - t^*).$$

The next event will not be triggered before $|e_i(t)| = c_1\exp(-\alpha t)$. Thus, a lower bound on the inter-event intervals is given by $\tau = t - t^*$ which solves $[\exp((\alpha - \lambda_2)t^*)k_1 +$

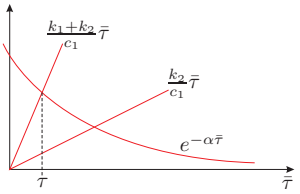

Fig. 2. Solution of the implicit equation for τ.

$k_2]\tau = c_1 \exp(-\alpha\tau)$. For $\alpha < \lambda_2$ the term in brackets is upper bounded by $k_1 + k_2$ and lower bounded by k_2. For all $t^* \geq 0$ the solutions $\tau(t^*)$ of this equation are greater or equal to τ given by $(k_1 + k_2)\tau = c_1 \exp(-\alpha\tau)$, which is strictly positive, as illustrated in Fig. 2.

Theorem 3 shows that it is possible to drive the overall system to average consensus asymptotically in an event-based fashion, rendering both the control signals and the broadcast state measurements piecewise constant. The condition $\alpha < \lambda_2(G)$ is intuitive, because the states should converge faster than the threshold decreases. However, from a practical point of view there might arise problems for trigger function (17), e.g. in presence of measurement noise. For large times t, arbitrarily small noise amplitudes will cause events. Furthermore, numerical problems arise with increasing time since checking the trigger condition leads to comparing very small numbers. Therefore it might be preferable to use a combination of the constant and the exponentially decreasing threshold on the measurement error as stated in the following theorem.

Theorem 4. Consider system (1) with control law (6) and undirected, connected graph G. Define the time-dependent trigger function

$$f_i(t, e_i(t)) = |e_i(t)| - \left(c_0 + c_1 e^{-\alpha t}\right), \qquad (18)$$

where $c_0, c_1 > 0$ and $0 < \alpha$. Then, for all $x_0 \in \mathbb{R}^N$, $\|\delta(t)\|$ converges to the region

$$\|\delta\| \leq \frac{\|L\|}{\lambda_2(G)} \sqrt{N} c_0 \qquad (19)$$

and the closed-loop system does not exhibit Zeno behavior.

Proof. Due to space limitations this proof is omitted. However, analogously to the previous proofs, convergence to the given region can be concluded from the analytical solution of the disagreement dynamics and a positive lower bound τ on the inter-event times is obtained by the same arguments as in proof of Theorem 2 since $c_0 > 0$.

The parameter c_0 can be used to adjust the size of this region, as well as to avoid problems due to numerics or measurement noise. Parameter c_1 can be tuned such that the events are not too dense for small times t. Thus, Theorem 4 provides a very flexible event-based control strategy for multi-agent systems.

4. SIMULATION RESULTS

In order to demonstrate the event-based control strategy, the multi-agent system with communication graph G given in Fig. 3 is considered. The initial conditions $x(0)$ are chosen such that all modes of the system are excited, i.e., if $v_1, ..., v_N$ are the normalized eigenvectors corresponding to the eigenvalues $\lambda_1(G), ..., \lambda_N(G)$, the initial conditions are set to $x(0) = (v_2 + \cdots + v_N)/\|v_2 + \cdots + v_N\|$.

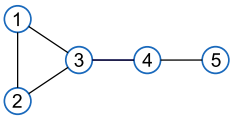

Fig. 3. Communication graph G.

For trigger function (10), the constant threshold is set to $c_0 = 0.03$. The simulation results, which are consistent with Theorem 2, are shown in Fig. 4.

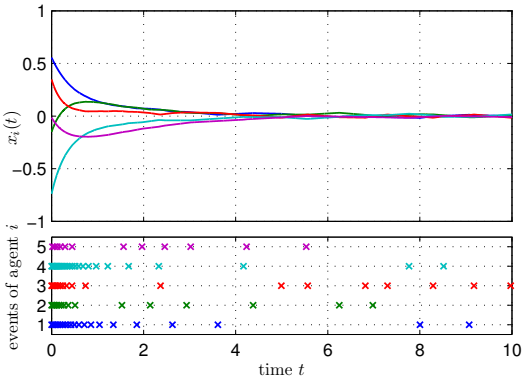

Fig. 4. Simulation result with static trigger function (10).

For trigger function (18), the constants are set to $c_0 = 0.001$, $c_1 = 0.25$ and $\alpha = 0.9\lambda_2(G)$. A small c_0 results in a small region (19), while α determines the speed of convergence and c_1 decreases the event density for small t. This is supported by the results shown in Fig. 5.

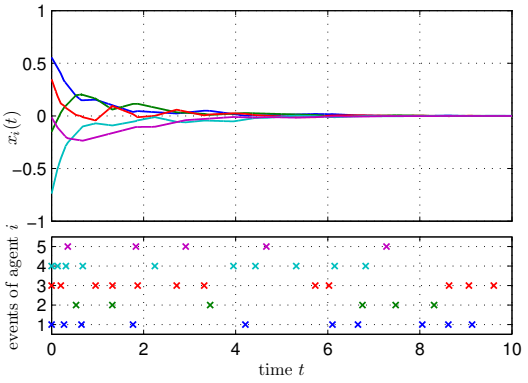

Fig. 5. Simulation result with trigger function (18).

In order to illustrate the effectiveness of the proposed control strategy, we compare it to the time-scheduled implementation of (2). Fig. 6 shows the latter simulation in comparison with time-scheduled control. The constant sampling period τ_s is chosen such that both strategies yield similar performance in terms of the convergence of $\|\delta(t)\|$. We set $\tau_s = 0.35$ while the average sampling period for event-based control over $t \in [0,\ 20]$ and over all agents is $\tau_{\text{avg}} = 1.44$. According to (5), the maximum stabilizing sampling period is $2/\lambda_N(G) = 0.4796$. This shows that time-scheduled control with sampling period such that the number of samples was the same as in the event-based case, would render the closed-loop system unstable. The average sampling period resulting from event-based control is more than two times higher than the maximum

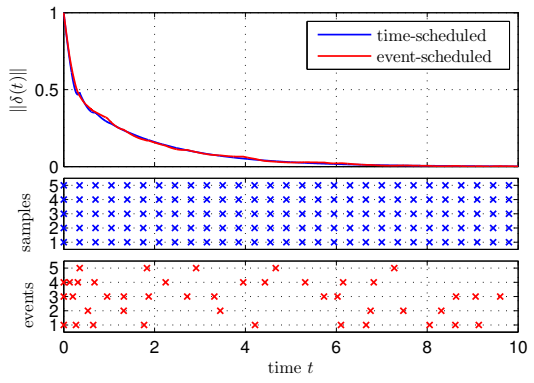

Fig. 6. Comparison to time-scheduled control.

stabilizing sampling period $2/\lambda_N(G)$. Similar results were obtained for various communication graphs. Consequently, the proposed event-scheduled control strategy is preferable to the classical time-scheduled approach.

5. EXTENSION TO DOUBLE-INTEGRATORS

A broad class of agents, e.g. holonomic mobile robots, require second-order dynamic models. Therefore the novel control strategy is extended to agents with double-integrator dynamics. Each agent $i \in \mathcal{V}$ is described by

$$\dot{\xi}_i(t) = \zeta_i(t), \quad \dot{\zeta}_i(t) = u_i(t). \tag{20}$$

The distributed continuous-time consensus protocol proposed by Ren and Atkins (2007) is given by

$$u_i(t) = -\sum_{j\in N_i}(\xi_i(t)-\xi_j(t)) - \gamma\sum_{j\in N_i}(\zeta_i(t)-\zeta_j(t)) \tag{21}$$

with $\gamma > 0$. The closed-loop dynamics can be written as

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = \Gamma\begin{bmatrix}\xi\\ \zeta\end{bmatrix}, \quad \text{where} \quad \Gamma = \begin{bmatrix}0 & I\\ -L & -\gamma L\end{bmatrix}.$$

It is shown by Ren and Atkins (2007) that consensus is reached asymptotically if and only if Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. In case of undirected connected G, all eigenvalues of L are real, and thus, by Ren and Atkins (2007), Lemma 4.2, it follows that consensus is achieved for all $\gamma > 0$. Define the initial average values $a = (1/N)\mathbf{1}^T\xi(0)$, $b = (1/N)\mathbf{1}^T\zeta(0)$. Then, for all $i \in \mathcal{V}$, it holds that $\xi_i(t) \to a + bt$ and $\zeta_i(t) \to b$ as $t \to \infty$.

Analogously to the single-integrator case, the broadcast states are defined by $\hat{\xi}_i(t) = \xi_i(t_k^i)$ and $\hat{\zeta}_i(t) = \zeta_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i[$, with corresponding stack vectors $\hat{\xi}$ and $\hat{\zeta}$. We propose the control law

$$u(t) = -L\left(\hat{\xi}(t) + \mathrm{diag}(t-t_k^1,...,t-t_k^N)\hat{\zeta}(t) + \gamma\hat{\zeta}(t)\right). \tag{22}$$

as event-based implementation of (21). It incorporates first-order hold instead of zero-order hold for the position $\xi_i(t)$, i.e., it is approximated by $\hat{\xi}_i(t) + (t-t_k^i)\hat{\zeta}_i(t)$ for $t \in [t_k^i, t_{k+1}^i[$, which respects the agent dynamics. We define the *measurement errors*

$$e_\xi(t) = \hat{\xi}(t) + \mathrm{diag}(t-t_k^1,...,t-t_k^N)\hat{\zeta}(t) - \xi(t)$$
$$e_\zeta(t) = \hat{\zeta}(t) - \zeta(t),$$

corresponding to the agents' positions ξ and velocities ζ. This yields $u(t) = -L\left(\xi(t) + \gamma\zeta(t) + e_\xi(t) + \gamma e_\zeta(t)\right)$. Consequently, the closed-loop dynamics are

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = \Gamma\begin{bmatrix}\xi\\ \zeta\end{bmatrix} - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t). \tag{23}$$

with overall measurement error $e(t) = [e_\xi^T(t)\ \ \gamma e_\zeta^T(t)]^T$. It can easily be verified that the average velocity b of all agents remains constant over time and the average position is given by $a+bt$. Thus, the state vector can be decomposed according to

$$\xi(t) = a\mathbf{1} + bt\mathbf{1} + \delta_\xi(t)$$
$$\zeta(t) = b\mathbf{1} + \delta_\zeta(t) \tag{24}$$

such that the *disagreement vectors* $\delta_\xi(t)$ and $\delta_\zeta(t)$ have zero average, i.e., $\mathbf{1}^T\delta_\xi(t) \equiv \mathbf{1}^T\delta_\zeta(t) \equiv 0$. Derivation of (24) with respect to time t yields

$$\begin{bmatrix}\dot{\delta}_\xi\\ \dot{\delta}_\zeta\end{bmatrix} = \Gamma\begin{bmatrix}a\mathbf{1} + bt\mathbf{1} + \delta_\xi\\ b\mathbf{1} + \delta_\zeta\end{bmatrix} - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t) - \begin{bmatrix}b\mathbf{1}\\ 0\end{bmatrix}.$$

With stack vector $\delta(t) = [\delta_\xi^T(t)\ \delta_\zeta^T(t)]^T$ the disagreement dynamics are given by

$$\dot{\delta}(t) = \Gamma\delta(t) - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t). \tag{25}$$

Before we state the main result, we derive a counterpart to Lemma 1 for the double-integrator case.

Matrix Γ has exactly two zero eigenvalues and only one linearly independent eigenvector corresponding to the zero eigenvalues, cf. Ren and Atkins (2007). The eigenvalues are denoted by $0 = \lambda_1(\Gamma) = \lambda_2(\Gamma) > \mathrm{Re}(\lambda_3(\Gamma)) \geq \cdots \geq \mathrm{Re}(\lambda_{2N}(\Gamma))$. It can be verified that $v_1 = 1/\sqrt{N}[\mathbf{1}^T\mathbf{0}^T]^T$ and $v_2 = 1/\sqrt{N}[\mathbf{0}^T\mathbf{1}^T]^T$ are a left eigenvector and generalized left eigenvector of Γ, respectively, corresponding to eigenvalue zero. Using non-singular matrix V consisting of v_1^T, v_2^T, and normalized eigenvectors v_j^T corresponding to eigenvalues $\lambda_j(\Gamma)$, $j = 3,...,2N$, matrix Γ can be transformed to Jordan normal form J. Therefore it holds that $\exp(\Gamma t) = V^{-1}\exp(Jt)V$ and it can be verified that $\|\exp(\Gamma t)v\| \leq \exp(\mathrm{Re}(\lambda_3(\Gamma))t)\|V^{-1}\|\|V\|\|v\|$ for all vectors $v \in \mathbb{R}^{2N}$ with $[\mathbf{1}^T\mathbf{0}^T]v = [\mathbf{0}^T\mathbf{1}^T]v = 0$, since $v_1^Tv = v_2^Tv = 0$. This proves the following lemma.

Lemma 5. Suppose L is the Laplacian of an undirected, connected graph G and $\gamma > 0$. Then, for all $t \geq 0$ and $v \in \mathbb{R}^{2N}$ with $[\mathbf{1}^T\mathbf{0}^T]v = [\mathbf{0}^T\mathbf{1}^T]v = 0$, it holds that $\|\exp(\Gamma t)v\| \leq \exp(\mathrm{Re}(\lambda_3(\Gamma))t)c_V\|v\|$, where $c_V = \|V^{-1}\|\|V\|$.

The main result for double-integrator agents is given in the following theorem. It covers both static and time-dependent trigger functions.

Theorem 6. Consider system (20) with control law (22) and undirected, connected graph G. Define the time-dependent trigger function

$$f_i(t, e_{\xi,i}(t), e_{\zeta,i}(t)) = \left\|\begin{bmatrix}e_{\xi,i}(t)\\ \gamma e_{\zeta,i}(t)\end{bmatrix}\right\| - \left(c_0 + c_1e^{-\alpha t}\right) \tag{26}$$

where $c_0, c_1 \geq 0$, $c_0 + c_1 > 0$, and $0 < \alpha < |\mathrm{Re}(\lambda_3(\Gamma))|$. Then, for all $\xi_0, \zeta_0 \in \mathbb{R}^N$, $\|\delta(t)\|$ converges to the region

$$\|\delta\| \leq c_0c_V\sqrt{2N}\|L\|/|\mathrm{Re}(\lambda_3(\Gamma))| \tag{27}$$

and the closed-loop system does not exhibit Zeno behavior.

Proof. The analytical solution of (25) is given by

$$\delta(t) = e^{\Gamma t}\delta(0) - \int_0^t e^{\Gamma(t-s)} \begin{bmatrix} 0 & 0 \\ L & L \end{bmatrix} e(s)ds.$$

With Lemma 5 and $\|e(t)\| \leq \sqrt{N}(c_0 + c_1 \exp(-\alpha t))$, we have

$$\|\delta(t)\| \leq k_1 + k_2 e^{-\alpha t} + k_3 e^{\mathrm{Re}(\lambda_3(\Gamma))t} \tag{28}$$

with positive constants $k_1 = c_0 c_V \sqrt{2N}\|L\|/|\mathrm{Re}(\lambda_3(\Gamma))|$, $k_2 = c_1 c_V \sqrt{2N}\|L\|/(|\mathrm{Re}(\lambda_3(\Gamma))+\alpha|)$, $k_3 = c_V\|\delta(0)\|$. Note that $\mathrm{Re}(\lambda_3(\Gamma)) < -\alpha < 0$. Therefore $\|\delta(t)\|$ converges asymptotically to the region (27) as $t \to \infty$.

In order to exclude Zeno behavior we show that there exists a positive lower bound on the inter-event times. Assume that agent i triggers at time $t^* \geq 0$. Then the measurement errors are reset to zero and agent i cannot trigger again at the same instance of time since $f_i(t^*, 0, 0) < 0$ for all $t^* \geq 0$. Observe that for $t \geq t^*$,

$$\left\| \begin{bmatrix} e_{\xi,i}(t) \\ \gamma e_{\zeta,i}(t) \end{bmatrix} \right\| \leq \int_{t^*}^t \left\| \begin{bmatrix} \dot{e}_{\xi,i}(s) \\ \gamma \dot{e}_{\zeta,i}(s) \end{bmatrix} \right\| ds \leq \int_{t^*}^t \|\dot{e}(s)\| ds.$$

The time-derivative of $e(t)$ is given by

$$\dot{e}(t) = \begin{bmatrix} \dot{e}_\xi(t) \\ \gamma \dot{e}_\zeta(t) \end{bmatrix} = \begin{bmatrix} \dot{\hat{\zeta}}(t) - \dot{\xi}(t) \\ -\gamma \dot{\zeta}(t) \end{bmatrix} = \begin{bmatrix} \dot{\hat{\zeta}}(t) - \zeta(t) \\ -\gamma u(t) \end{bmatrix} = \begin{bmatrix} e_\zeta(t) \\ -\gamma u(t) \end{bmatrix}$$

and therefore $\|\dot{e}(t)\| \leq 1/\gamma \|e(t)\| + \gamma\|u(t)\|$. The control $u(t)$ is bounded by $\|u(t)\| = \|[-L - \gamma L]\delta(t) - [L\ L]e(t)\| \leq \sqrt{1+\gamma^2}\|L\|\|\delta(t)\| + \sqrt{2}\|L\|\|e(t)\|$, and with inequality (28) and $\|e(t)\| \leq \sqrt{N}(c_0 + c_1 \exp(-\alpha t))$, $\|u(t)\| \leq \sqrt{1+\gamma^2}\|L\|(k_1 + k_2 \exp(-\alpha t) + k_3 \exp(\mathrm{Re}(\lambda_3(\Gamma))t)) + \sqrt{2N}\|L\|(c_0 + c_1 \exp(-\alpha t))$. Therefore it holds that $\|\dot{e}(t)\| \leq (1/\gamma + \gamma\sqrt{2}\|L\|)\sqrt{N}(c_0 + c_1 \exp(-\alpha t)) + \gamma\sqrt{1+\gamma^2}\|L\|(k_1 + k_2 \exp(-\alpha t) + k_3 \exp(\mathrm{Re}(\lambda_3(\Gamma))t))$. Two different cases depending on c_0 are distinguished:

Case 1 Assume $c_0 \neq 0$. Then $\|\dot{e}(t)\| \leq (1/\gamma + \gamma\sqrt{2}\|L\|)\sqrt{N}(c_0 + c_1) + \gamma\sqrt{1+\gamma^2}\|L\|(k_1 + k_2 + k_3) = C = const.$ An upper bound on the measurement error for $t \geq t^*$ is given by $\|[e_{\xi,i}(t), \gamma e_{\zeta,i}(t)]^T\| \leq \int_{t^*}^t \|\dot{e}(s)\| ds \leq (t - t^*)C$. The next event will not be triggered before (26) crosses zero, and not before $(t - t^*)C = c_0$. Thus, a positive lower bound τ on the inter-event times is given by $\tau = c_0/C$.

Case 2 Assume $c_0 = 0$. Then $k_1 = 0$ and $\|\dot{e}(t)\| \leq (1/\gamma + \gamma\sqrt{2}\|L\|)\sqrt{N}c_1 \exp(-\alpha t^*) + \gamma\sqrt{1+\gamma^2}\|L\|(k_2 \exp(-\alpha t^*) + k_3 \exp(\mathrm{Re}(\lambda_3(\Gamma))t^*))$. Denote this bound by $C(t^*)$ since it depends on t^*. The measurement error is bounded by $\|[e_{\xi,i}(t), \gamma e_{\zeta,i}(t)]^T\| \leq \int_{t^*}^t \|\dot{e}(s)\| ds \leq (t - t^*)C(t^*)$. The next event will not be triggered before (26) crosses zero, i.e., before $(t - t^*)C(t^*) = c_1 \exp(-\alpha t)$. This leads to the implicit equation $c_1 \exp(-\alpha\tau) = \gamma\sqrt{1+\gamma^2}\|L\|(k_2 + k_3 \exp((\mathrm{Re}(\lambda_3(\Gamma)) + \alpha)t^*))\tau + \left(1/\gamma + \gamma\sqrt{2}\|L\|\right)\sqrt{N}c_1\tau$ for $\tau = t - t^*$. Note that $\mathrm{Re}(\lambda_3(\Gamma)) + \alpha < 0$ by assumption. By the same graphical argument as in the proof of Theorem 3, it can be concluded that a lower bound on the inter-event times is given by the positive constant τ, which solves the implicit equation for $t^* = 0$.

Since there is a positive lower bound on the inter-event times in both cases, Zeno behavior of the closed-loop system is excluded and the proof is complete.

6. CONCLUSION

We proposed a novel event-based control strategy for consensus problems of both single- and double-integrator multi-agent systems. The main advantage of this approach with respect to our previous work is that neighboring agents do not have to exchange information continuously, but only at specific instances of time which are determined by events. Each agent decides itself, based on local information, when it has to send a new measurement value over the network. Nevertheless, desired convergence properties are preserved. The results were illustrated in simulations.

Extensions to time-delayed communication and switching network topologies are currently investigated. Future work will also address distributed estimation of $\lambda_2(G)$, as proposed by Yang et al. (2010), such that a-priori knowledge of $\lambda_2(G)$ is not necessary for using trigger functions (17).

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