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A Simple Self-triggered Sampler for Nonlinear Systems

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Abstract: Self-triggered control is a promising aperiodic control paradigm that allows the design of resource-efficient control schemes. Several results were recently developed for self-triggered control of linear systems, but self-triggered control for nonlinear systems is still not well understood. Existing techniques are mainly limited to homogeneous or polynomial closed-loop systems that are input-to-state stable with respect to measurement errors. These assumptions obviously limit the class of applicable cases. In this paper, a new simple self-triggered controller, based on the assumption of local asymptotic stability of the closed-loop system, is presented. The controller ensures local ultimate uniform boundedness of the state trajectories in the sense that the self-triggered controller guarantees the trajectories to be confined into an arbitrary small region. In the case with measurement time delays, it is shown how the size of the region depends on the maximum delay. The effectiveness of the proposed method is illustrated by simulations.

Keywords: Event-based Control, Self-triggered Control, Nonlinear Systems

1. INTRODUCTION

In the design of digital control schemes it is often assumed that the flow of information from the sensors to the controllers is time-periodic, and that the controller updates are performed at the same rate. If the data exchange between the sensors and the controller is performed over a network, or if the controller is implemented on a processor where other control tasks are running, time-periodic implementations may require high demand of resources even when it is not needed. For example, in the case of Networked Control Systems (NCSs), where the processes, controllers and actuators are spatially distributed and exchange data through a communication network, a time-periodic flow of information from the sensors to the controllers may waste bandwidth if the system is in steady-state. In the same way, when a processor schedules control tasks periodically, a conservative utilization of the processor resources may arise. These are two typical examples of shared-resources systems: in the first case the shared-resource is represented by the network, in the second case it is represented by the processor used by several tasks.

A promising control technique to efficiently design shared resources systems is event-based control, e.g. Tabuada [2007], Wang and Lemmon [2011], Dimarogonas and Johansson [2009], Lunze and Lehmann [2010]. Event-triggered control aims at reducing the conservativeness provided by time-periodic implementations, by taking actions only when relevant information is available. Such actions may be the transmission of a packet from a sensor to a controller for NCS, or the re-instantiation of a control task in the case of several tasks running on the same processor. Within the event-triggered paradigm, the output of the system is continuously monitored, and a new action is taken only when a function of the output crosses a certain threshold. Such a triggering function is designed to ensure a desired behavior of the closed-loop system. While event-triggered control *reacts* to the detection of an event, self-triggered control *predicts* its occurrence based on a system model and the current measurement, see Anta and Tabuada [2009, 2010], Lemmon et al. [2007], Mazo et al. [2010], Wang and Lemmon [2009], Velasco et al. [2003], Millán Gata et al. [2011], Mazo et al. [2009].

In the last few years, self-triggered control provided fertile ground for research. However, most of the existing work addressed linear systems, while nonlinear systems are still not much investigated. While a preliminaty attempt to design a self-triggered sampler can be found in Tiberi [2011], to the best of our knowledge the only contribution that addresses self-triggered nonlinear control are due to Anta and Tabuada [2009, 2010]. In Anta and Tabuada [2009, 2010], a self-triggered sampler for nonlinear systems is developed under the assumption that the closed-loop system with continuous-time control is input-to-state stable (ISS) with respect to measurement errors. Moreover, the approach applies only to polynomial or homogeneous closed-loop systems. Despite the interesting results, the assumptions limit the number of applicable cases. For instance, the problem of designing a controller that renders the closed-loop system ISS with respect to measurement errors received particular attention in the nonlinear control community. In Freeman [1993] its has been proved the existence of such a controller for single input systems in strict feedback form, and for systems that are ISS with respect to actuator errors, when the control law is globally Lipschitz. Freeman provided an example for which there exists a controller that asymptotically stabilize the closed-loop system, but it does not render the closed-loop system ISS with respect to small measurement errors, see Freeman [1995].

In this paper we propose a simple self-triggered sampler that can be used with a wider class of controllers than in Anta and Tabuada [2009, 2010]. The self-triggered sampler we propose is designed under the assumption of asymptotic stabilizability of the system with a continuous-time control, and we assume that the continuous-time control law is locally differentiable. The proposed self-triggered sampler aims at achieving uniformly ultimately boundedness (UUB) of the trajectories. We address both the cases with time delays and without time delays: in the former case, we show how the trajectories can be confined into an arbitrary small region, while in the latter case we show the existence of a minimum ultimate bound guarantee. We wish to remark that our self-triggered sampler is designed without exploiting any particular property of the closed-loop system,

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so the inter-sampling times are more conservative than the approach proposed in Anta and Tabuada [2009, 2010] where homogeneity properties are exploited. However, the proposed self-triggered sampler can be used with a wider class of controllers.

The reminder of the paper is as follows: in the next Section some notation and preliminaries are introduced. In Section 3 the problem formulation is stated. In Section 4 an event-triggering rule to achieve UUB of the closed loop system is proposed, while in Section 5 the self-triggered implementation of such event-triggering rule is proposed. The theoretical results are validated by simulation in Section 6. A discussion is provided in Section 7, and an appendix is finally reported in Section 8.

2. NOTATION AND PRELIMINARIES

We indicate with $\mathbb{R}_{\geq 0}$ the set of the nonegative real numbers and with $\|x\|$ the Euclidean norm of an element $x \in \mathbb{R}^n$. We denote with $\mathcal{B}_r$ the closed ball center at the origin and radius r, i.e. $\mathcal{B}_r = \{x : \|x\| \leq r\}$. Given a square matrix M we denote $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ the smallest and the largest eigenvalue respectively. A function $f : \mathcal{D} \to \mathbb{R}^n$ is said to be of class $\mathcal{C}^0(\mathcal{D})$ if it is continuous over $\mathcal{D}$, and it is said to be $\mathcal{C}^r(\mathcal{D}), r > 0$ if its derivatives are of class $\mathcal{C}^{r-1}(\mathcal{D})$. A continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}, a > 0$, is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. Given a signal $x : \mathbb{R}^+ \to \mathbb{R}^n$, we denote x_k its realization at time $t = t_k$, i.e. $x_k := x(t_k)$, and with $\|x\|_{\mathcal{L}_\infty,k} := \sup_{t \geq t_k} \|x(t)\|$. Given a system $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, $x(t_0) = x_0$, $f : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^n$, where f is Lipschitz with respect to x and piecewise continuous with respect to t, and where $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin, we say that the solutions are *Ultimately Uniformly Bounded* (UUB) if there exists three constants $a, b, T > 0$ independent of t_0 such that for all $\|x_0\| \leq a$ it holds $\|x(t)\| \leq b$ for all $t \geq t_0 + T$, and *Globally Ultimately Uniformly Bounded* (GUUB) if $\|x(t)\| \leq b$ for all $t \geq t_0 + T$ and for arbitrarily large a. The value of b is referred as *ultimate bound*.

3. PROBLEM FORMULATION

We consider a system of the form

$$\dot{x} = f(x, u)\,, \tag{1}$$

where $x \in \mathcal{D}_x \subset \mathbb{R}^n$, $u \in \mathcal{D}_u \subset \mathbb{R}^p$. The self-triggered sampler is designed under the following assumption.

Assumption 3.1. Consider the system (1), and consider a differentiable state feedback law $\kappa\colon \mathcal{D}_x \to \mathcal{D}_u$ so that the closed loop system

$$\dot{x} = f(x, \kappa(x))\,, \tag{2}$$

satisfies: i. the origin of (2) is locally asymptotically stable over $\mathcal{D}_x$, ii. $f(x, \kappa(x)) \in \mathcal{C}^1(\mathcal{D}_x \times \mathcal{D}_u)$ with Lipschitz continuous derivatives over the set $\mathcal{D}_x \times \mathcal{D}_u$ ◇

Suppose now to sample and hold the measurement at time $t = t_k$ and to use the constant control $u = \kappa(x_k)$ for $t \in [t_k, t_{k+1})$. The dynamics of (2) become, for $t \in [t_k, t_{k+1})$, as

$$\dot{x} = f(x, \kappa(x_k)) \tag{3}$$

The problem we address is to determine a self-triggered sampler so that the system (3) is UUB over $\mathcal{D}_x$. We solve the problem by first proposing an event-triggered sampling rule that ensures UUB of the closed loop system. Then we design the self-triggered implementation of such a sampling rule, by addressing both the cases without time delays and with time delays. In the case with time delays we assume that they are bounded by a maximum delay $\tau_{\max}$.

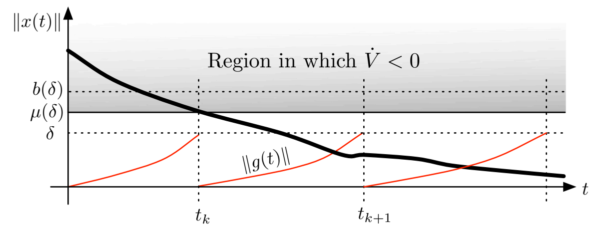

Fig. 1. Proposed event-triggered sampler. Every time the error function $\|g(t)\|$ hits the threshold δ, then the system is sampler again, and the error is reset.

4. EVENT-TRIGGERED SAMPLER

The first contribution of the paper is given by the following result.

Proposition 4.1. Consider the system (1). Assume that there exists a state feedback control $\kappa\colon \mathcal{D}_x \to \mathcal{D}_u$ so that the closed loop system is locally asymptotically stable over the domain $\mathcal{D}_x$. Then, by updating the control signal every time the triggering condition

$$\|g(t)\| := \|f(x(t), \kappa(x_k)) - f(x(t), \kappa(x(t)))\| \leq \delta\,, \quad \delta > 0\,, \tag{4}$$

is violated, the closed-loop system is UUB over $\mathcal{D}_x$. Moreover, it holds $t_{k+1} - t_k > 0$ for all k. ◁

An intuitive idea of the event-triggered sampler (4) working principle is given in Figure 1. Every time that the function $\|g(t)\|$ hits the threshold δ, a new measurement is picked, the control law is updated and $\|g(t)\|$ is reset to zero. By doing that, we enforce the Lyapunov function derivative along the trajectories of the system to be strictly negative for all $t \geq t_0$ in the region $\|x(t)\| > \mu$, and then the trajectories must necessarily converge into a bounded invariant set, that is the ultimate bound. According to the sampling rule (4), the ultimate bound is a strictly increasing function $b(\delta)$ of δ, and $b(0) = 0$. Hence, because the value of δ can be arbitrary small, then the trajectories can be ultimately bounded into an arbitrary small region. On the other hand, as we decrease the value of δ, as we are shrinking the inters-sampling times.

Notice that the previous proposition does not require $f \in \mathcal{C}^1$ or the controller to be differentiable, in contrast with Assumtpion 3.1. However, to design a self-triggered sampler we need to entire fulfill Assumption 3.1, as discussed in the next section.

Remark 4.1. We remark that in Anta and Tabuada [2010], the triggering condition is given by

$$\frac{\|x_k - x(t)\|}{\|x(t)\|} \leq \frac{a}{c}\sigma\,,$$

where $a, c > 0$, $0 < \sigma < 1$ are appropriate scalar that depends on the ISS property of the closed loop system, see Anta and Tabuada [2010]. Such a sampling rule ensures asymptotic stability of the closed-loop system, while the sampling-rule (4) ensure UUB into an arbitrary small region. However, in the design of a self-triggered sampler, the triggering rule (4) permits the utilization of a wider class of controller, as we discuss in the next Section. ◁

5. SELF-TRIGGERED SAMPLER

In this section we present the self-triggered implementation of the sampling rule (4). We first analyze the case without time delays, and then we show how to include time delays in the analysis. In both case the design is performed by exploiting an upper bound of $\|g(t)\|$.

5.1 Without time-delay

Lemma 5.1. Consider the system (3) and let

$$\varphi(x(t), \kappa(x(s))) := -\frac{d}{ds} f(x(t), \kappa(x(s))) .$$

Then, $\|g(t)\|$ is upper-bounded with

$$\|g(t)\| \leq \frac{\|\varphi(x^*, \kappa(x_k))\|}{L_{\varphi,u}} \left(e^{L_{\varphi,u}(t-t_k)} - 1\right) := \hat{g}(x_k, t - t_k) , \quad (5)$$

where $L_{\varphi,u}$ is the Lipschitz constant with respect to u of the function $\varphi(x, u)$, and

$$x^* := \arg \max_{y \in \mathcal{D}_x} \|\varphi(y, \kappa(x_k))\| . \quad (6)$$

$\triangleleft$

We are now in position to design a self-triggered sampler that ensure UUB over the domain $\mathcal{D}_x$ of the closed loop-system.

Theorem 5.1. Consider the system (3) and assume that assumption 3.1 holds. Let $\delta > 0$ and $\vartheta \in (0, 1)$. Then, the self-triggered sampler

$$t_{k+1} = t_k + \frac{1}{L_{\varphi,u}} \ln \left(1 + \frac{\delta L_{\varphi,u}}{\|\varphi(x^*, \kappa(x_k))\|}\right) , \quad (7)$$

ensures UUB of the closed-loop system over the set $\mathcal{D}_x$. Moreover, the self-triggered sampler (7) ensures $t_{k+1} - t_k > 0$ for all k. $\triangleleft$

According to the self-triggered sampling, at time $t = t_k$ it is possible to decide the next t_{k+1} by which the system must be sampled to ensure UUB. In the next section we discuss how it is possible to extend to the case of measurement time delays.

Remark 5.1. The proposed self-triggered sampler can be designed provided that $L_{\varphi,u}$ and x^* are computed, and their computation is performed by considering the whole set $\mathcal{D}_x$. However, since the trajectories of the system are upper-bounded, for all $t \geq t_0$, with $\|\dot{x}\| \leq \|f(x, \kappa(x))\| + \delta$, at every sampling time it is possible to re-compute $L_{\varphi,u}$ and x^* over the region $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$. In addition, since $\dot{x} = f(x, \kappa(x_k))$ is UUB, it follows that $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}} \subset \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$ outside the ultimate bound region. Then, the sequence of sets $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}}$ is decreasing outside the ultimate bound region, and $L_{\varphi,u}$ and x^* computed over such a sequence of sets is also decreasing, see Fig. 2. By doing this adaptation, we dynamically subtract conservativeness to the computation of $L_{\varphi,u}$ and x^*, and the proposed self-triggered sampler would provides larger inter-sampling times. $\triangleleft$

5.2 With time-delay

In this section we show how to design a self-triggered sampler in the case with time-delays, provided that they are smaller than the inter-sampling times. In this case, the input applied to the system is piecewise constant, and satisfies

$$\begin{cases} u = \kappa(x_{k-1}) & \text{for } t \in [t_k, t_k + \tau_k) , \\ u = \kappa(x_k) & \text{for } t \in [t_k + \tau_k, t_{k+1}) , \end{cases} \quad (8)$$

where τ_k is the elapsed time between the instant when the measurement x_k is picked and the instant in which the actuator is updated. The dynamics of (3) can be split into

$$\dot{x} = f(x, \kappa(x)) + f(x, \kappa(x_{k-1})) - f(x, \kappa(x)) , \quad (9)$$

for $t \in [t_k, t_k + \tau_k)$ and

$$\dot{x} = f(x, \kappa(x)) + f(x, \kappa(x_k)) - f(x, \kappa(x)) , \quad (10)$$

for $t \in [t_k + \tau_k, t_{k+1})$. In this case, the perturbation due to the sampling for $t \in [t_k, t_{k+1})$ is composed by two terms: the first term depends on the measurement x_{k-1} and it acts for $t \in [t_k, t_k + \tau_k)$; the second term depends on the measurement x_k and it acts for $t \in [t_k + \tau_k, t_{k+1})$. Nonetheless, to easily design a self-triggered sampler to accomodate time delays, it is convenient to consider time intervals of the form $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$. Hence, by following the same arguments as in Lemma 5.1, an upper bound for the perturbation due to the sampling and acting on (9)–(10), for $t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1})$, is given by

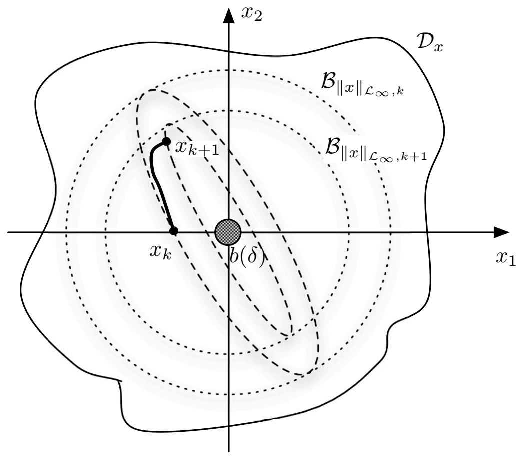

Fig. 2. Adaptation of the operating region. The dashed-line represent the Lyapunov surface levels, the dotted-line the adapted operating region and $b(\delta)$ denotes the ultimate bound. For $t \in [t_k, t_{k+1})$, the trajectories cannot escape from $\mathcal{B}_{\mathcal{L}_\infty,k}$, while for $t \in [t_{k+1}, t_{k+2})$ the trajectories cannot escape from $\mathcal{B}_{\mathcal{L}_\infty,k+1}$. Since the Lyapunov function is negative outside the region delimited by b, we have $\mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k+1}} \subset \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,k}}$.

$$\|g(t)\| \leq \hat{g}(x_{k-1}, \tau_k) e^{L_{\varphi,u}(t-t_k-\tau_k)} + \frac{\|\varphi(x^*, \kappa(x_k))\|}{L_{\varphi,u}} \left(e^{L_{\varphi,u}(t-t_k-\tau_k)} - 1\right) . \quad (11)$$

Given the upper bound (11), it is possible to design a self-triggered sampler to ensure UUB of (3) by including time delays, as stated in the next result.

Corollary 5.1. Consider the assumption of Theorem 5.1, and consider the self-triggered sampler

$$t_{k+1} = t_k + \tau_k - \tau_{\max} + \frac{1}{L_{\varphi,u}} \cdot \ln \left(\frac{\delta L_{\varphi,u} + \|\varphi(x^*, \kappa(x_k))\|}{L_{\varphi,u} \hat{g}(x_{k-1}, \tau_k) + \|\varphi(x^*, \kappa(x_k))\|}\right) \quad (12)$$

Then, the closed loop system (9)–(10) is UUB. $\triangleleft$

Since in the case with time delays the design of the self-triggered sampler is based on the the prediction of the time it takes for the upper-bound of $\|\hat{g}(t)\|$ to go from $\|\hat{g}(t_k + \tau_k)\| \neq 0$ to $\|\hat{g}(t_{k+1} + \tau_{k+1})\| = \delta$, it may happen that $\|\hat{g}(t_k + \tau_k)\| \geq \delta$ and then we would have $t_{k+1} - t_k \leq 0$. However, it is possible to achieve $t_{k+1} - t_k > 0$ for all k just by increasing the value of δ, but such increasing also increases the size of the ultimate bound $b(\delta)$.

Proposition 5.1. Consider the self-triggered sampler (12). If

$$\delta > \hat{g}(\bar{x}, \tau_{\max}) , \quad (13)$$

where $\bar{x} = \arg \max_{y \in \mathcal{B}_{\|x\|_{\mathcal{L}_\infty,0}}} \hat{g}(y, \tau_{\max})$, then $t_{k+1} - t_k > 0$ for all k and the trajectories of the closed loop system (9)–(10) are ultimately bounded with $b(\delta) > 0$. $\triangleleft$

Remark 5.2. Proposition 5.1 gives a tradeoff among the minimum inter-sampling time, the size of the ultimate bound region and the maximum allowable time delay. For example, given a δ that satisfies (13), it is possible to check what is the maximum allowable time delay $\tau_{\max}$ to ensure a minimum inter-sampling time guaranteed, and what is the ultimate bound $b(\delta)$. $\triangleleft$

Remark 5.3. Assumption 3.1 it requires that $f(x, \kappa(x)) \in \mathcal{C}^1(\mathcal{D}_x \times \mathcal{D}_u)$. However, if $f(x, \kappa(x))$ is not of class $\mathcal{C}^1$ but it is only Lipschitz continuous over $\mathcal{D}_x \times \mathcal{D}_u$ with Lipschitz constant with respect to the second argument $L_{f,u}$, then it is possible to design a self-triggered sampler to ensure UUB by considering $L_{f,u}\|f(x(t), \kappa(x_k)) - f(x(t), \kappa(x(s)))\|$ instead of $\|\varphi(x(t), \kappa(x_k)) - \varphi(x(t), \kappa(x(s)))\|$ in the derivation of the upper bound of $\|g(t)\|$. Then, given this new upper-bound of $\|g(t)\|$, the design of the self-triggered sampler can be performed as in the case in which $f \in \mathcal{C}^1$. $\triangleleft$

6. SIMULATION RESULTS

In the first example we consider the control of the rigid body, showing how our method results more conservative with respect to the one proposed in Anta and Tabuada [2010]. In the second example we take the system provided in Freeman [1995], for which it is used a controller that renders the closed-loop system asymptotically stable but it cannot achieve ISS with respect to the measurement errors.

6.1 Example 1: Rigid Body Control

In this section we perform a comparison between our approach and the one proposed in Anta and Tabuada [2010], by considering a rigid body control. The state space model of the rigid body is given by

$$\dot{x}_1 = u_1 \tag{14}$$

$$\dot{x}_2 = u_2 \tag{15}$$

$$\dot{x}_3 = x_1 x_2\,. \tag{16}$$

Byrnes and Isidori [1989] proved that the control law

$$u_1 = -x_1x_2 - 2x_2x_3 - x_1 - x_3 \tag{17}$$

$$u_2 = 2x_1x_2x_3 + 3x_3^2 - x_2 \tag{18}$$

renders the closed-loop system globally asymptotically stable with Lyapunov function

$$V = \frac{1}{2}(x_1 + x_3)^2 + \frac{1}{2}(x_2 - x_3^2)^2 + x_3^2\,. \tag{19}$$

The previous Lyapunov function can be rewritten as $V = 0.5x^T P(x_3)x$, where

$$P(x_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -x_3 \\ 1 & -x_3 & 3 + x_3^2 \end{pmatrix}, \tag{20}$$

and then $0.5\lambda_{\min}(P(x_3))\|x\|^2 \leq V(x) \leq 0.5\lambda_{\max}(P(x_3))\|x\|^2$, We consider the ball $\mathcal{B}_r$ with $r = 1$ as operating region, and we use a period of $4.5 \cdot 10^{-5}$ s for the periodic implementation. We set $\delta = 0.016$ and $x(t_0) = [0.1 \quad -0.1 \quad 0.1]^T$. With this setting, we get $0.22\|x\|^2 \leq V(x) \leq 2.28\|x\|^2$ and $\alpha_4(\|x\|) = 9.12\|x\|$ over $\mathcal{B}_r$. By considering the perturbation due to the sampling and by exploiting the comparison functions, an upper-bound of the trajectories of the sampled system are given, for all $t \geq t_0$, by

$$\|x\| \leq \sqrt{\frac{V(t_0)}{c_1} + \frac{c_2}{2c_1}c_4 r\delta} \leq 0.96 = \|x\|_{\mathcal{L}_\infty,0} < 1\,. \tag{21}$$

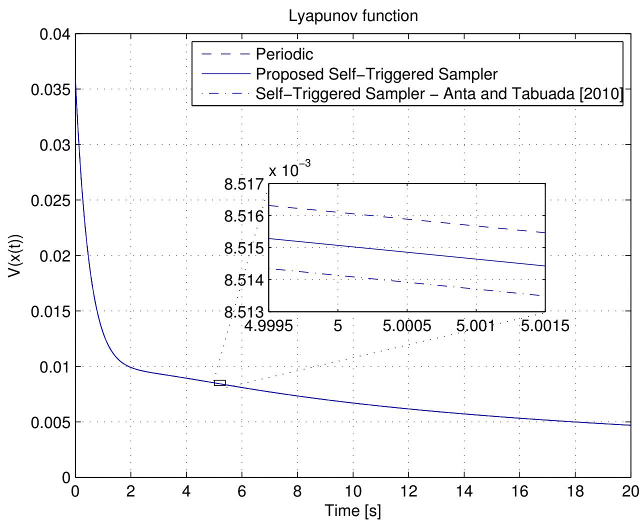

Fig. 3. Lyapunov functions evolution with the self-triggered and with the periodic implementation of the controller.

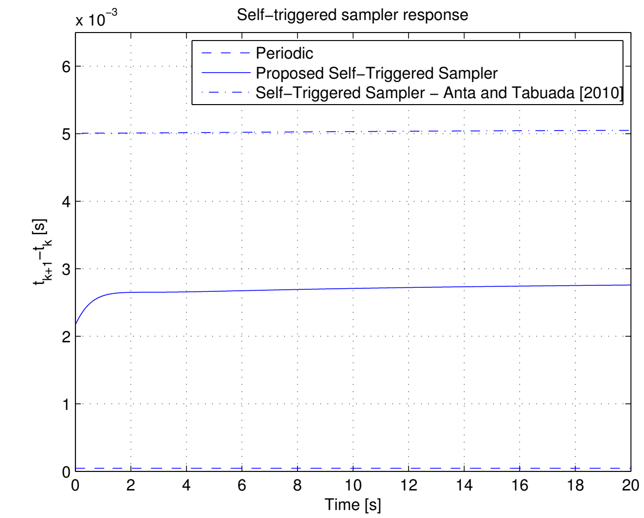

Fig. 4. Inter-sampling times obtained with the self-triggered and with the periodic implementation of the controller.

This means that the considered operating region $\mathcal{B}_r$, is an invariant set. Then, by computing $L_{\varphi,u}$ over $\mathcal{B}_r$, we get $L_{\varphi,u} = 2.47$.

The simulation results are depicted in Figures 3 and 4. It is interesting to see how the sampling intervals enlarges as the Lyapunov function decreases. However, it is not surprising to see how the method proposed in Anta and Tabuada [2010] provides larger inter-sampling times, since such a self-triggered sampler explicitly exploits the polynomial structure of the closed-loop system, while the proposed method, not exploiting such a property, adds conservativeness to the computation of the inter-sampling times. Finally, it is interesting to see how the closed-loop system behavior with the periodic and the self-triggered implementations of the controller looks very similar.

6.2 Example 2: Non ISS closed-loop system

In this second example we consider the system used in Freeman [1995] for which it it is used a controller that render the closed-loop system globally asymptotically stable, but does not render the closed-loop ISS with respect to measurement errors, and

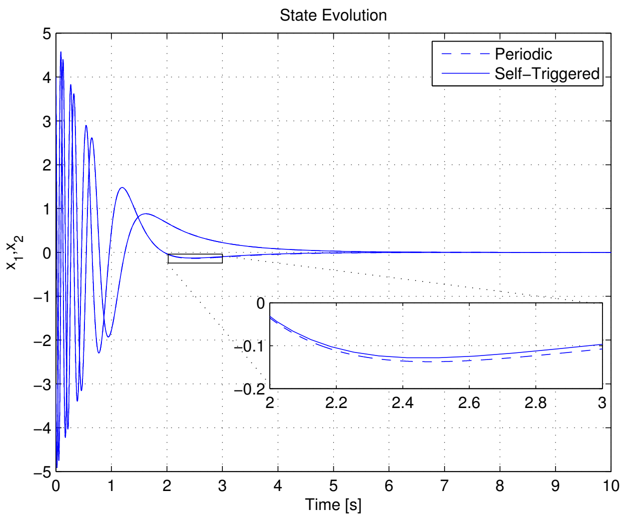

Fig. 5. State evolution with the self-triggered and with the periodic implementation of the controller.

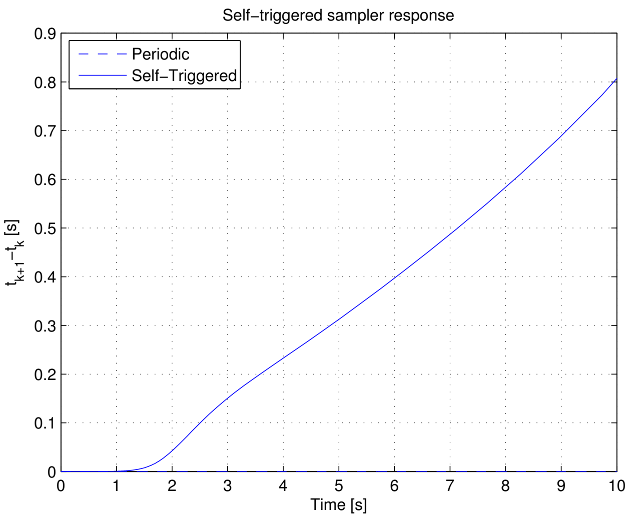

Fig. 6. Inter-sampling times obtained with the self-triggered and with the periodic implementation of the controller.

then the assumption in Anta and Tabuada [2010] does not hold. Moreover, we compare our self-triggered sampler with a periodic implementation of the controller of period $h = 1.5$ ms. The dynamics of the system are given by

$$\dot{x} = \left(I + 2\Theta\left(\frac{\pi}{2}\right) xx^T\right) \Theta(x^T x) \cdot \left(\begin{bmatrix} -1 & 0 \\ 0 & x^T x \end{bmatrix} \Theta(-x^T x) x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) , \tag{22}$$

where

$$\Theta(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

By considering the coordinate transformation $z = \Theta(-x^T x)x$, the system (22) in the z coordinate becomes

$$\begin{aligned} \dot{z}_1 &= -z_1 , \\ \dot{z}_2 &= (z_1^2 + z_2^2) z_2 + u . \end{aligned} \tag{23}$$

By using this coordinate transform, a stabilizing control for (22) is given by

$$u = \Gamma(\Theta(-x^T x)x) , \tag{24}$$

where the function $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\Gamma(z) = -(1 + z_1^2 + z_2^2) z_2 . \tag{25}$$

Since the coordinate transformation is a diffeomorphism, and since $\|x\| = \|z\|$, if we achieve UUB for the system in the new coordinate, then the original system is UUB. Hence, we design the self-triggered sampler to ensure UUB of (23) to determine when to update the control (24) for the system (22). A Lyapunov function for the system (23) with the control (24), is given by $V(z) = 0.5 z^T z$. Then, we get $\alpha_4(\|z\|) = \|z\|$, and $\|z(t)\|_{\mathcal{L}\infty,0} = \|z_0\|$. By considering an operating region $\mathcal{B}_r$, $r = 5$, we get $L_{\varphi,u} = 28.24$. The simulation is performed by considering $x_0 = [4 \ \ -3]^T$ as initial condition.

The simulation result is depicted in Fig. 5–6. At the beginning the periodic implementation provides larger inter-sampling times, but, after $t \simeq 1.8$ s, the self-triggered sampler starts to give larger inter-sampling times with respect to the periodic implementation. After 10 s of simulation time, we experienced 66666 number of updates for the periodic implementation versus the 56338 number of controller updates of the self-triggered implementation. Finally, it can be appreciated from Fig. 5 how the behavior of the closed-loop system under the periodic and the self-triggered implementation is very similar.

7. CONCLUSIONS

In this paper we presented a simple self-triggered sampler for nonlinear systems. The proposed self-triggered sampler allows the utilization of a wider class of controllers with respect to previous work, Anta and Tabuada [2009, 2010]. However, because the proposed self-triggered sampler does not use additional information of the closed-loop system, the obtained inter-sampling times results to be more conservative with respect to Anta and Tabuada [2009, 2010], where the homogeneity property is exploited. Moreover, while in Anta and Tabuada [2009, 2010] it is ensured asymptotic stability of the closed-loop system, here we can achieve UUB in an arbitrary small region. In the case with time delays the size of the ultimate bound region cannot be arbitrary small, but it depends on the maximum time delay. The proposed method has been validated through simulations, where it has been shown how our method is more conservative than the one proposed in Anta and Tabuada [2010], but how it can be used with a wider class of controllers.

Future work include the derivation of less conservative inter-sampling times, and the design of self-triggered samplers under different sampling rules.

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8. APPENDIX

Proof of Proposition 4.1. From Assumption 3.1, converse theorems (Kurzweil [1963], Khalil [2002]) ensure the existence of a Lyapunov function $V(x)$ for the system (1) such that

$$\begin{aligned} \alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)\,, \\ \frac{\partial V(x)}{\partial x} f(x, \kappa(x)) \le -\alpha_3(\|x\|)\,, \\ \left\| \frac{\partial V(x)}{\partial x} \right\| \le \alpha_4(\|x\|)\,, \end{aligned} \tag{26}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are $\mathcal{K}$-class functions. The time derivative of V along the trajectories of the sampled-data system (3), for $t \in (t_k, t_{k+1})$, satisfy

$$\begin{aligned} \dot V &= \frac{\partial V}{\partial x} f(x, \kappa(x)) + \frac{\partial V}{\partial x}\big(f(x, \kappa(x_k)) - f(x, \kappa(x))\big) \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\|g\|\,. \end{aligned}$$

By using the triggering rule (4), and since α_4 is increasing it holds

$$\begin{aligned} \dot V &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\delta \\ &= -(1-\vartheta)\alpha_3(\|x\|) + \vartheta\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\delta\,, \end{aligned}$$

for any $\vartheta \in (0, 1)$. Then, we have $\dot V \le -(1-\vartheta)\alpha_3(\|x\|)$ if

$$\|x\| \ge \alpha_3^{-1}\left(\frac{\alpha_4(\|x\|_{\mathcal{L}_\infty,k})\delta}{\vartheta}\right) := \mu(\delta)\,.$$

By observing that at each sampling time $t = t_k$ it holds $\dot V \le -\alpha_3(\|x_k\|)$, by triggering accordingly to (4) it follows that the system is UUB over the set $\mathcal{D}_x$, Khalil [2002]. An ultimate bound is given by

$$b(\delta) := \alpha_1^{-1}(\alpha_2(\mu(\delta))) \tag{27}$$

Finally, since for $t = t_k$ we have $\|g(t_k)\| = 0$, and since $\delta > 0$, by continuity of $\|g(t)\|$ in every time intervals $[t_k, t_{k+1})$, it holds $t_{k+1} - t_k > 0$. □

Proof of Lemma 5.1. Let $u(s) = \kappa(x(s))$ and $g(s) = f(x(t), u(t_k)) - f(x(t), u(s))$. It holds

$$\begin{cases} \dfrac{d}{ds} g(s) = -\dfrac{d}{ds} f(x(t), u(s)) := \varphi(x(t), u(s))\,, \\ g(s_k) \ \ = 0\,. \end{cases} \tag{28}$$

The solution of the previous differential equation is given by

$$\begin{aligned} g(s) &= \int_{s_k}^{s} \varphi(x(t), u(\sigma)) d\sigma \\ &= \int_{s_k}^{s} \big(\varphi(x(t), u(\sigma)) - \varphi(x(t), u_k)\big) d\sigma \\ &\quad + \int_{s_k}^{s} \varphi(x(t), u_k) d\sigma \end{aligned} \tag{29}$$

By taking the norm of both sides it holds:

$$\begin{aligned} \|g(s)\| &\le \int_{s_k}^{s} \big(\|\varphi(x(t), u(\sigma)) - \varphi(x(t), u_k)\|\big) d\sigma \\ &\quad + \int_{s_k}^{s} \|\varphi(x(t), u_k)\| d\sigma \\ &\le \int_{s_k}^{s} L_{\varphi,u} \|f(x(t), u(\sigma)) - f(x(t), u_k)\| d\sigma \\ &\quad + \int_{s_k}^{s} \|\varphi(x(t), u_k)\| d\sigma \\ &= \int_{s_k}^{s} L_{\varphi,u} \|g(\sigma)\| d\sigma + \int_{s_k}^{s} \|\varphi(x(t), u_k)\| d\sigma\,, \end{aligned} \tag{30}$$

By using the Leibniz Theorem, we get

$$\frac{d}{ds}\|g(s)\| \le L_{\varphi,u}\|g(s)\| + \|\varphi(x(t), u_k)\| \tag{31}$$

and then

$$\|g(s)\| \le \frac{\|\varphi(x(t), u_k)\|}{L_{\varphi,u}}\Big(e^{L_{\varphi,u}(s - s_k)} - 1\Big), \quad \forall x. \tag{32}$$

By replacing $x(t)$ with x^* it follows the upper-bound (5). □

Proof of Theorem 5.1. By considering the Lyapunov function (26) , we have, for $t \in (t_k, t_{k+1})$

$$\begin{aligned} \dot V &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\|g\| \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\hat g(x_k, t - t_k)\,. \end{aligned}$$

Since $\hat g(x_k, t - t_k)$ is continuous and it is strictly increasing with t, then there exists a time t_{k+1} satisfying

$$\hat g(x_k, t_{k+1} - t_k) = \delta\,.$$

Hence, the Lyapunov derivative is further bounded with $\dot V \le -\alpha_3(\|x\|) + \alpha_4(\|x\|_{\mathcal{L}_\infty,k})\delta$. By following the same line as in the proof of Lemma 4.1, is easy to prove UUB over the domain $\mathcal{D}_x$. Hence, the next triggering time is given by the inverse of $\hat g(x_k, t_{k+1} - t_k)$, that gives (7). Finally, since for $t = t_k$ we have $g(x_k, t_k) = 0$ for any x_k, $\delta > 0$ and since $\hat g(x_k, t - t_k)$ is increasing with $t > t_k$, by continuity in the time interval $[t_k, t_{k+1})$ it holds $t_{k+1} - t_k > 0$ for any k. □

Proof of Proposition 5.1. To achieve $t_{k+1} - t_k > 0$, the argument of the logarithm in (12) must be greaten than 1, and this happens if, and only if $\delta > \hat g(x_{k-1}, \tau_k)\,, \forall k$.Since $\hat g(\bar x, \tau_{\max}) > \hat g(\bar x, \tau_k) > \hat g(x_{k-1}, \tau_k)\,, \forall k$, then $t_{k+1} - t_k > 0$ for all k if inequality (13) holds, for which we have $b(\delta) > b(\hat g(\bar x, \tau_{\max})) > 0$. □