Fifty-first Annual Allerton Conference
Allerton House, UIUC, Illinois, USA
October 2 - 3, 2013

Examples of Distance-based Synchronization: An Extremum Seeking Approach

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***Abstract*— In this paper we consider two examples of synchronization problems, i.e., a network of oscillators and a network of rigid bodies. We propose a controller that requires only the knowledge of the relative distances among the neighboring systems in the network. The controller is based on an extremum seeking controller, that steers the overall system to the minimum of an optimization problem on a manifold. Using a Lie bracket approximation for extremum seeking systems, we show that the controller leads to a synchronization of the overall network in both examples.**

I. Introduction

The problem of synchronizing dynamical systems (agents) has attained more and more attraction in the last decades. It arises in many fields, such as biological systems, power networks, social networks and many more. For an overview we refer to e.g. [2], [5], [13], [19] and the references therein.

In order to reach synchronization in a network, the controller usually requires the knowledge of the position mismatches of the agents. One can distinguish between relative position measurements (*vector valued* information) and distances measurements (*scalar valued* information), see e.g. [21]. In many applications the relative positions are unavailable or difficult to obtain. In such cases, however, the measurements of the distances among the agents may be available and can be obtained, for example, with ultrasonic sensors or infrared sensors.

The motivation of this paper is to demonstrate that extremum seeking controllers on manifolds could be applied to synchronization problems with distance measurements. The objective is to consider two specific synchronization problems and to construct a controller that requires only the knowledge of a scalar distance measurement.

First, we consider the problem of synchronizing a family of coupled harmonic oscillator, each one evolving on the unit-circle in the plane.

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Second, we consider the problem of synchronizing a family of rigid bodies. Again, the rigid bodies can only measure the distances to their neighbors' attitude.

The main result of this paper is to propose for both problems a control law, that achieves synchronization among the agents and uses hereby only measurements of the distances between neighboring agents. We formulate the problems as optimization problems on manifolds, where the synchronized state of the agents coincides with the minimum of a scalar distance function, see e.g. [1], [17]. The design is based on an extremum seeking scheme on manifolds, introduced in [6].

The remainder of the paper is structured as follows. In Section II we introduce both problems and state the assumptions on the desired controller. In Section III we recall the necessary results of [7] and in Section IV, we extend these results and outline the proposed solution. In Section V we summarize the results.

A. Notation

We use the following notation. $\mathbb{Q}$ ($\mathbb{Q}_{++}$) are the (positive) rational numbers. $\mathbb{R}^n$ is equipped with the standard scalar product (standard metric) $\langle x, y\rangle = x^T y$. The Euclidian norm of a vector $v \in \mathbb{R}^n$ is denoted by $\|v\| = \sqrt{\langle v, v\rangle}$. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of $\mathbb{R}^n$. The tangent space of M at x is denoted by T_xM. We denote by C^n with $n \in \mathbb{N}$ the set of n times continuously differentiable functions. The Lie bracket $[.,.] : T_xM \times T_xM \to T_xM$ between two vector fields $g_1, g_2 \in C^1 : M \to T_xM$ is defined as: $[g_1(x), g_2(x)] = \frac{\partial g_2(x)}{\partial x} g_1(x) - \frac{\partial g_1(x)}{\partial x} g_2(x)$. Note that the vector field $[g_1(x), g_2(x)]$ is again a vector field on M (see Corollary 8.28 in [14]). Let $U \subseteq \mathbb{R}$ be open, $M \subseteq U$ and $f : U \to \mathbb{R}$. We denote with $f|_M : M \to \mathbb{R}$ the restriction of f to M. The gradient vector field of f on the Riemannian manifold $(\mathbb{R}^n, \langle ., .\rangle)$ is denoted by $\nabla f(x) = [\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_n}]^T$ and the gradient vector field of f on the manifold M is denoted by $\mathrm{grad} f|_M$. Let $X_1, \ldots, X_N \in \mathbb{R}^{n\times n}$ and let $x_j = \mathrm{vec}(X_j)$, $j \in 1, \ldots, N$ be their vectorizations [9], i.e., the stacked vector of columns of X_j. We define for the block matrix

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \in \mathbb{R}^{nN\times n} \tag{1}$$

the vectorization as follows

$$\mathrm{vec}(X) = \begin{bmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_N) \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n^2 N}. \tag{2}$$

We also introduce the scalar product for matrices $\langle X, Y \rangle := \mathrm{trace}(X^\top Y) = \mathrm{vec}(X)^\top \mathrm{vec}(Y)$ and see that if $X, Y \in \mathbb{R}^{nN \times n}$ we obtain that $\langle X, Y \rangle = \sum_{i=1}^{N} \mathrm{vec}(X_i)^\top \mathrm{vec}(Y_i)$. We define for a function $f \in C^1 : \mathbb{R}^{nN \times n} \to \mathbb{R}$

$$\frac{\partial f(X)}{\partial X_i} = \begin{bmatrix} \frac{\partial f(X)}{\partial X_{i,11}} & \dots & \frac{\partial f(X)}{\partial X_{i,1n}} \\ \vdots & & \vdots \\ \frac{\partial f(X)}{\partial X_{i,n1}} & \dots & \frac{\partial f(X)}{\partial X_{i,nn}} \end{bmatrix} \tag{3}$$

and

$$\frac{\partial f(X)}{\partial X} = \begin{bmatrix} \frac{\partial f(X)}{\partial X_1} \\ \vdots \\ \frac{\partial f(X)}{\partial X_N} \end{bmatrix}. \tag{4}$$

II. Problem Formulation

In this section, we consider two synchronization problems, that can be formulated as optimization problems on manifolds.

In both problems, we consider a group of $N \in \mathbb{N}$ agents. The information topology underlying the overall system is modeled by a connected, undirected graph $G = (V, E)$ with vertices V and edges E (see e.g. Fig. 1). Each vertex represents an agent which is an oscillator or a system with rigid-body-like dynamics. The state of a single agent is denoted by x_j, which evolves on a manifold M. The state vector of the overall system is denoted by $x^\top = [x_1^\top, \dots, x_N^\top]$. The dynamics of each agent j is given by an input affine system

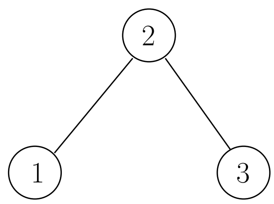

Fig. 1: Network of Oscillators

$$\dot{x}_j = g_{j,0}(x) + \sum_{i=1}^{p} g_i(x) u_{j,i} \tag{5}$$

where $p \in \mathbb{N} \cup \{0\}$ denotes the number of inputs. In (5) we assume that the control vector fields g_i are the same for every agent. The class of optimization problems we consider is given by

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } x \in M \times \ldots \times M =: M^N. \end{aligned} \tag{6}$$

The goal is to construct controller inputs $u_{j,i}$ that steer the agents to the minimum of f, i.e., to the optimum of (6). The function f is designed in such a way that its minimum is attained if and only if the agents synchronize (see e.g. [10], [17]). Furthermore, the control input $u_{j,i}$ of agent j should in particular only depend on the distances to its neighbors. In that sense, the controller we construct is distributed.

A. Synchronization on the Circle

The respective dynamics of agent $j \in V$ is given by

$$\begin{aligned} \dot{x}_j = \begin{bmatrix} \dot{\xi}_j \\ \dot{\eta}_j \end{bmatrix} &= (\nu + u_{j,1}) \begin{bmatrix} -\eta_j \\ \xi_j \end{bmatrix} \\ &= \begin{bmatrix} 0 & (\nu + u_{j,1}) \\ -(\nu + u_{j,1}) & 0 \end{bmatrix} x_j, \end{aligned} \tag{7}$$

where $\nu \in \mathbb{R}\backslash\{0\}$ denotes the natural frequency (see [4]) of the oscillators and is, in this case, the same for all agents. Note that, each of the oscillators evolves on the circle $S^1 = \{[\xi_j, \eta_j]^\top \in \mathbb{R}^2 : \xi_j^2 + \eta_j^2 = 1\}$. The goal is to construct control inputs $u_{j,1}$ that steer the overall system to the solution of the following optimization problem

$$\begin{aligned} &\min\ f(x) = \frac{1}{2} \sum_{(j,k) \in E} \|x_j - x_k\|^2 \\ &\text{s.t. } x \in S^1 \times \ldots \times S^1 =: T^N. \end{aligned} \tag{8}$$

The sum $\sum_{(j,k) \in E} \|x_j - x_k\|^2$ represents the distances among the agents of the network. For agent j, the control input $u_{j,1}$ should depend only on the distances to its direct neighbors.

B. Synchronization of Rigid Bodies

Consider now a network of $N \in \mathbb{N}$ rigid bodies. The attitude of a rigid body is described by the set of special orthogonal matrix (rotation matrix) $SO(3) = \{X \in \mathbb{R}^{3 \times 3} : X^\top X = I, \det(X) = +1\}$. The dynamics of agent $j \in V$ is given by

$$\dot{X}_j = \sum_{i=1}^{3} (\nu_i + u_{j,i}) X_j \Omega_i, \tag{9}$$

where $X_j \in SO(3)$ and

$$\begin{aligned} &\Omega_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \Omega_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \\ &\Omega_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

The parameters $\nu_i \in \mathbb{R}\backslash\{0\}$, $i = 1, 2, 3$ correspond to the natural frequency ν of the foregoing problem and determine the persistent oscillations of the rigid bodies in the respective direction determined by the vector fields. Again, the goal is to construct control inputs $u_{j,i}$ that steer the overall system to the solution of the following optimization problem

$$\begin{aligned} \min f(X) &= \frac{1}{2} \sum_{(j,k) \in E} \mathrm{trace}((X_j - X_k)^\top (X_j - X_k)) \\ \text{s.t. } &X \in SO(3) \times \ldots \times SO(3) =: SO(3)^N. \end{aligned} \tag{11}$$

The function f is the sum of squares of the distances in attitudes of the rigid bodies in the network. In other words, it measures the distances of the body axes between neighboring agents.

This problem setup is in a similar spirit to the one presented in [11] where explicitly only relative information is available to the controller. However, the controller in that paper requires relative position information, while we will require only distance information.

III. Optimization on Manifolds and Extremum Seeking

In this section, we review one possible method to construct a controller that solves the previously introduced problems. The approach is presented in [7], where extremum seeking controller for optimization problems on manifolds are considered.

We consider optimization problems on manifolds, which can be written as

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } x \in M. \end{aligned} \tag{12}$$

A necessary condition for a local minimum of (12) is that the gradient vector field must necessarily vanish at that point, i.e., if x^* is a local minimum of (12) then $\mathrm{grad} f|_M(x^*) = 0$ (see e.g., p. 284 in [3]). Typically extremum seeking tries to solve (12) with $M = \mathbb{R}^n$ by seeking points where the gradient vanishes (see e.g. (see e.g. [6], [12]).

In order to characterize the gradient on M in terms of ∇f, we need the following lemma (see e.g. p. 48 in [1]):

Lemma 1: Let $U \subseteq \mathbb{R}^n$ be open, $M \subseteq U$ and ∇f be the gradient vector field of $f : U \to \mathbb{R}^n$, defined by the standard scalar product $\langle ., . \rangle$ in $\mathbb{R}^n$. Then the induced gradient vector field $\mathrm{grad} f|_M : M \to T_xM$ is given by

$$\mathrm{grad} f|_M(x) = \mathrm{P}(\nabla f(x)), \tag{13}$$

where $\mathrm{P}(y)$ denotes the orthogonal projection of $y \in \mathbb{R}^n$ onto T_xM.

We restrict our investigations on submanifolds where the ambient space is the Euclidian space $\mathbb{R}^n$ with standard scalar product. In view of optimization on manifolds, it was shown in [7] that the idea of approximating the trajectories of the extremum seeking by the associated Lie bracket system translates to systems evolving on manifolds. This is due to the property of Lie brackets which leave the tangent space of a manifold invariant (see Corollary 8.28 in [14]). The following assumptions have to be satisfied:

- A1 $M \subseteq \mathbb{R}^n$ is a smooth, m-dimensional Riemannian submanifold without boundary. The metric $\langle ., . \rangle_M : T_xM \times T_xM \to \mathbb{R}$ on M is the metric $\langle ., . \rangle$ induced by ambient space $\mathbb{R}^n$, i.e., $\langle x, y \rangle_M := \langle x, y \rangle$.
- A2 There are $p \geq m$ vector fields $g_i \in C^2 : M \to T_xM$, $i = 1, ..., p$, on M such that
$$\mathrm{span}\{g_1(x), ..., g_p(x)\} = T_xM \text{ for all } x \in M, \tag{14}$$
i.e., for each point x on M, the tangent vectors $g_i(x)$, $i = 1, ..., m$, span the tangent space T_xM and for $p = m$ the tangent vectors $g_i(x)$ form a basis of T_xM.
- A3 Let $U \subseteq \mathbb{R}^n$ be open, $M \subseteq U$ and $f \in C^2 : U \to \mathbb{R}$. The set of local minima E of $f|_M$ is nonempty and we denote with $E_c \subseteq E$ a compact connected component of E.

We introduce the *extremum seeking system on the manifold* M as follows:

$$\dot{x} = \sum_{i=1}^{p} c_i f(x) g_i(x) \sqrt{\omega_i} \cos(\omega_i t) + \alpha_i g_i(x) \sqrt{\omega_i} \sin(\omega_i t) \tag{15}$$

with

$$\begin{aligned} &\alpha_i, c_i > 0 \text{ and} \\ &\omega_i = a_i \omega, a_i \neq a_j, i \neq j, a_i \in \mathbb{Q}_{++}, \omega > 0, \end{aligned} \tag{16}$$

$i, j = 1, ..., p$. The parameter ω plays a crucial role in the definitions of stability (see the appendix). Since $g_i(x) \in T_xM$ for all $x \in M$, the right hand side of the extremum seeking system (20) defines a vector field on M, i.e., solutions initialized on M are uniformly invariant on M. More explicitly, $x(t_0) \in M$ implies that $x(t) \in M$ for all $t_0 \leq t < t_0 + t_{max}$, where t_{max} is the maximal interval of existence. Similar as in [6], we use a Lie bracket approximation in order to determine the qualitative behavior of the extremum seeking system (15). Identifying $u_i^1(\omega_i t) = \sqrt{\omega_i} \cos(\omega_i t)$, $u_i^2(\omega_i t) = \sqrt{\omega_i} \sin(\omega_i t)$ as inputs, the corresponding Lie bracket system of (20) on M is given by

$$\dot{z} = \frac{1}{2} \sum_{i=1}^{p} \alpha_i c_i [f(z) g_i(z), g_i(z)]. \tag{17}$$

An elementary but important calculation shows that

$$[f(z) g_i(z), g_i(z)] = -\langle \nabla f(z), g_i(z) \rangle g_i(z). \tag{18}$$

Thus, the *Lie bracket system* (23) *on* M can be written as

$$\dot{z} = -\frac{1}{2} \sum_{i=1}^{p} \alpha_i c_i \langle \nabla f(z), g_i(z) \rangle g_i(z). \tag{19}$$

Clearly, the right hand side of (19) is a vector field on M. Observe also that when all α_i and c_i have the same value α and c and if the tangent vectors $g_i(x)$, $i = 1, ..., p$ with $p = m$ form an orthonormal basis of T_xM for all $x \in M$, then the right hand side of (19) is exactly $-\frac{\alpha c}{2} \mathrm{grad}\ f|_M(x)$.

Using the methodology developed in [6], the following theorems have been established in [7]. For the stability definitions used in the following theorems, we refer to the appendix.

Theorem 1: Consider the Lie bracket system (19) and let Assumptions A1 – A3 be satisfied. Let $W \subseteq M$ be an open set and let E_c be a compact connected set of minima of $f|_M$ which is contained in W. Assume that the gradient of $f|_M$ vanishes in W only at points in E_c, i.e., $\mathrm{grad} f|_M(z) = 0$ if and only if $z \in E_c$ for all $z \in W$. Then the set E_c is asymptotically stable. Moreover, E_c is practically uniformly asymptotically stable with respect to the extremum seeking system (15).

In other words, the extremum seeking system locally converges arbitrary close to the set of local minima of $f|_M$ for sufficiently large ω_i, $i = 1, ..., p$. Theorem 1 provides a local stability result of the extremum seeking system based on the Lie bracket system. The next theorem provides a nonlocal result.

Theorem 2: Consider the Lie bracket system (19) and let Assumption A1 – A3 be satisfied. Let $S \subseteq M$ and assume a compact connected set E_c of local minima of $f|_M$ is S-asymptotically stable. Then, E_c is S-practically uniformly asymptotically stable with respect to the extremum seeking system (15).

In other words, this theorem states if S is a subset of the region of attraction of E_c for the Lie bracket system, then S is also a subset of the 'practical' region of attraction of E_c for the extremum seeking system. This notion is similar as to the notion of semi-global practical stability (e.g. [6], [16])

IV. Main Results

In this section we modify the results of the previous section such that they suit the problem formulation in Section II, i.e., constructing inputs $u_{j,i}$ in (5) such that the overall systems converges to the solution of (6). Specifically, we incorporate dynamical systems with drift vector fields and consider the case of individual cost functions. Since in these problems, we consider multiple dynamical systems, each one evolving on the same manifold, we introduce a modification of Assumption A3:

A3' Let $U \subseteq \mathbb{R}^n$ be open, $M^N \subseteq U$ and $f \in C^1 : U \to \mathbb{R}$. The set of local minima E of $f|_{M^N}$ is nonempty and we denote with $E_c \subseteq E$ a compact connected component of E.

Next, we introduce the extremum seeking agents on the manifold M with dynamics as follows:

$$\begin{aligned} \dot{x}_j = \; & g_{j,0}(x_j) + \sum_{i=1}^{p} c_i f_j(x) g_i(x_j) \sqrt{\omega_{j,i}} \cos(\omega_{j,i} t) \\ & + \alpha_i g_i(x_j) \sqrt{\omega_{j,i}} \sin(\omega_{j,i} t), j \in V \end{aligned} \tag{20}$$

with

$$f_j \in C^2 : M^N \to \mathbb{R} \tag{21}$$

satisfying

$$\nabla_{x_j} f_j(x) = \nabla_{x_j} f(x), j \in V, \tag{22}$$

with f in (6). The constants satisfy $\alpha_i, c_i > 0$ and all $\omega_{j,i}$ are rational multiples of some $\omega > 0$ and pairwise distinct. The idea of requiring that each of the individual functions f_j satisfies Eq. (22) originates in Potential Games, see [15]. It allows us to use f as a Lyapunov function for the Lie bracket system of the overall system (see also [6], where a similar approach has been used). Note that by this choice of vector fields, the manifold M is invariant for each agent $j \in V$. Identifying again $u_i^1(\omega_i t) = \sqrt{\omega_i} \cos(\omega_i t)$, $u_i^2(\omega_i t) = \sqrt{\omega_i} \sin(\omega_i t)$ as inputs, the corresponding Lie bracket system for agent j (see Eq. (20)) on M is given in this case by

$$\dot{z}_j = g_{j,0}(z_j) + \frac{1}{2} \sum_{i=1}^{p} \alpha_i c_i [f_j(z) g_i(z_j), g_i(z_j)]. \tag{23}$$

We also observe that

$$[f_j(z) g_i(z_j), g_i(z_j)] = -\langle \nabla_{z_j} f_j(z), g_i(z_j) \rangle g_i(z_j). \tag{24}$$

Together with (22), we obtain

$$[f_j(z) g_i(z_j), g_i(z_j)] = -\langle \nabla_{z_j} f(z), g_i(z_j) \rangle g_i(z_j). \tag{25}$$

The Lie bracket system for the overall system is then obtained by defining $z^\top = [z_1^\top, \ldots, z_N^\top]$, which then yields

$$\dot{z} = G_0(z) - \frac{1}{2} \sum_{j \in V} \sum_{i=1}^{p} \alpha_i c_i \langle \nabla f(z), G_{j,i}(z) \rangle G_{j,i}(z), \tag{26}$$

with

$$G_0(x) = \begin{bmatrix} g_{1,0}(x_1) \\ \vdots \\ g_{N,0}(x_N) \end{bmatrix}, \tag{27}$$

$$G_{1,1}(x) = \begin{bmatrix} g_1(x_1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, G_{N,p}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g_p(x_N) \end{bmatrix}. \tag{28}$$

Note that the tangent space $T_x M^N$ of M^N is the product space of the tangent spaces $T_x M$ of M, i.e., $T_x M^N = T_x M \times \ldots \times T_x M$ (see Chapter 1.2, Exercise 9 in [8]). Therefore, the right hand side of (23) is a vector field on $T_x M^N$. Observe also that when all α_i and c_i have the same value α and c and if the tangent vectors $g_i(x)$, $i = 1, ..., p$ with $p = m$ form an orthonormal basis of $T_x M^N$ for all $x \in M^N$, then the right hand side of (26) is exactly $-\frac{\alpha c}{2} \text{grad } f|_{M^N}(x)$.

Using the methodology above, the following corollaries can be shown in a similar way as Theorems 1 and 2.

Corollary 1: Consider the Lie bracket system (23). Suppose (21), (22) and

$$\langle \nabla f(x), G_0(x) \rangle = 0, x \in M^N \tag{29}$$

hold. Furthermore, let Assumptions A1, A2, A3' be satisfied and let $W \subseteq M^N$ be an open set and let E_c be a compact connected set of minima of $f|_{M^N}$ which is contained in W. Assume that the gradient of $f|_M$ vanishes in W only at points in E_c, i.e., $\text{grad} f|_{M^N}(z) = 0$ if and only if $z \in E_c$ for all $z \in W$. Then the set E_c is asymptotically stable. Moreover, E_c is practically uniformly asymptotically stable with respect to the extremum seeking system (20).

Theorem 1 provides a local stability result of the extremum seeking system based on the Lie bracket system. The next theorem provides a nonlocal result.

Corollary 2: Consider the Lie bracket system (23). Suppose (21), (22) and

$$\langle \nabla f(x), G_0(x) \rangle = 0, x \in M^N \tag{30}$$

hold. Furthermore, let Assumptions A1, A2, A3' be satisfied and let $S \subseteq M$ and assume a compact connected set E_c of local minima of $f|_M$ is S-asymptotically stable. Then, E_c is S-practically uniformly asymptotically stable with respect to the extremum seeking system (20).

In both corollaries, we additionally assume that the drift vector field G_0 is orthogonal to the gradient of the objective function of the optimization problem. The orthogonality condition (29) (see also (30)) assures that persistent dynamics are invariant with respect to the level sets of f. In particular, the condition is meaningful in order to allow persistent oscillations by the drift vector field G_0 once the agents are synchronized. Compared to the extremum seeking based on an averaging analysis (see e.g. [12]), we do not require an additional time-scale separation among the dynamics of the agents and the dynamics of the extremum seeking controller.

V. Problem Solution

In this section, we exploit the results of the previous section to solve the problems introduced in Section II. We exploit the structures of the optimization problems (8) and (11) and construct objective functions f_j that satisfy (21), (22) and consists only of distance measurements among the agents. In this way, we are able to propose for each synchronization problem a distributed extremum seeking controller of the form (20). We then apply the results of the previous section and show that the extremum seeking system converges to a solution of (8) and (11) respectively.

Consider a Graph $G = (V, E)$. In the following, we denote the neighbors of a agent $j \in V$ as

$$\mathcal{N}_j = \{k \in V : (j, k) \in E\}. \tag{31}$$

We consider each of the synchronization problems separately. We propose a controller for synchronization on the circle and show that the overall system is related to the well-known Kuramoto model but requires only distance measurements. Then, we show that a similar controller as for the synchronization on the circle can also be used for synchronization of rigid bodies.

A. Synchronization on the Circle

In the following, we consider the setup introduced in Section II.A. Consider f from (8), we introduce individual objective functions of agent $j \in V$ as

$$f_j(x) = \frac{1}{2} \sum_{k \in \mathcal{N}_j} \|x_j - x_k\|^2. \tag{32}$$

One can immediately verify that the functions f_j satisfy (22). Furthermore, we also observe that each of the functions f_j only depend on the distances among agent $j \in V$ and its neighbors.

Consider furthermore the dynamics of the agents in (7), we note that there is for each agent only one control input. We propose the following controller for agent $j \in V$

$$u_{j,1} = c_j f_j(x)\sqrt{\omega_j}\cos(\omega_j t) + \alpha_j \sqrt{\omega_j}\sin(\omega_j t). \tag{33}$$

We see that the closed loop of agent (7) with (33) can be written in the form (20) with $p = 1$ and

$$g_{j,0}(x_j) = \nu \begin{bmatrix} -\eta_j \\ \xi_j \end{bmatrix},\ g_1(x_j) = \begin{bmatrix} -\eta_j \\ \xi_j \end{bmatrix}, j \in V. \tag{34}$$

Next, we will apply Corollary 1 of the foregoing section. We note that Assumptions A1, A2 and A3' are satisfied. In particular, Assumption A1 is satisfied since S^1 is a smooth one-dimensional submanifold of $\mathbb{R}^2$. Assumption A2 is satisfied, since the vector field g_1 satisifies (14). Assumption A3' is satisfied since $f|_{T^N}(x) \geq 0$ for all $x \in T^N$ and furthermore $f|_{T^N}(x) = 0$ if and only if $x \in E_s := \{x \in T^N : x_j = x_k\}$. Therefore, $E_s \subseteq E$ with E being the set of local minima of f and E is nonempty. Let E_c be a connected component of E_s. Since T^N is compact, E_c is also compact.

Next, we show that there are only finitely many connected components of the set of critical points of f. Note that the gradient of f in (8) can be calculated using (13), the projection onto the tangent space of S^1 in [1], i.e., $P(\nabla f(x)) = (I - xx^\top)\nabla f(x)$, and that $T^N = S^1 \times \ldots S^1$, which yields

$$\text{grad}_{T^N} f(x) = \begin{bmatrix} (I - x_1 x_1^\top) \sum_{k \in \mathcal{N}_1}(x_1 - x_k) \\ \vdots \\ (I - x_N x_N^\top) \sum_{k \in \mathcal{N}_N}(x_N - x_k) \end{bmatrix} \tag{35}$$

Since T^N is a zero-level set of polynomials and the gradient of f in (35) is also polynomial, we may use Theorem 3 in [20] which states that the zero set of a polynomial has only a finite number of connected components. Thus, every connected component E_c is isolated from other connected components of the set of critical points of f. Therefore, there exists a neighborhood $W \subseteq T^N$ containing every E_c and satisfying the conditions required in Corollary 1. We may thus apply Corollary 1 and conclude that the solution of (8) is practically uniformly asymptotically stable with respect to the extremum seeking consisting of (7) with controller (33). We also note that

$$\begin{aligned} \langle \nabla f(x), G_0(x) \rangle = \nu \sum_{(j,k) \in E} & \begin{bmatrix} \xi_j - \xi_k \\ \eta_j - \eta_k \end{bmatrix}^\top \begin{bmatrix} -\eta_j \\ \xi_j \end{bmatrix} \\ & - \begin{bmatrix} \xi_j - \xi_k \\ \eta_j - \eta_k \end{bmatrix}^\top \begin{bmatrix} -\eta_k \\ \xi_k \end{bmatrix} \\ = \nu \sum_{(j,k) \in E} & - \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix}^\top \begin{bmatrix} -\eta_j \\ \xi_j \end{bmatrix} \\ & - \begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix}^\top \begin{bmatrix} -\eta_k \\ \xi_k \end{bmatrix} = 0, \end{aligned} \tag{36}$$

for all $[\xi_i, \eta_i]^\top \in S^1$, $i \in V$, which shows that (30) is satisfied.

The following observation may help to get a better intuition about the extremum seeking system for synchronization on the circle. We now show that after transforming the extremum seeking system consisting of (7) with controller (33) to polar coordinates, the corresponding Lie bracket system coincides with the Kuramoto model. We introduce the coordinates for the system in (7) as follows

$$\xi_j = \cos(\theta_j) \tag{37a}$$

$$\eta_j = \sin(\theta_j) \tag{37b}$$

with $0 \leq \theta_j < 2\pi$, $j \in V$. We then obtain the transformed extremum seeking system in polar coordinates as

$$\dot{\theta}_j = \nu + \sqrt{\omega_j} f_j(\theta)\cos(\omega_j t) + \alpha\sqrt{\omega_j}\sin(\omega_j t), \tag{38}$$

where we write for short $\theta^\top = [\theta_1, \ldots, \theta_N]$ and by a slight abuse of notation we also write

$$f_j(\theta) = -\frac{1}{2}\sum_{k\in\mathcal{N}_j}\cos(\theta_j - \theta_k), j \in V. \tag{39}$$

Calculating the Lie bracket system of (38), we obtain

$$\dot{\tilde{\theta}}_j = \nu - \alpha\nabla_{\tilde{\theta}_j} f_j(\tilde{\theta}) = \nu - \alpha\sum_{k\in\mathcal{N}_j}\sin(\tilde{\theta}_j - \tilde{\theta}_k), \tag{40}$$

which is the well-known Kuramoto model. Therefore, the θ-coordinates reveal that the Lie bracket system (40) coincides with the Kuramoto model. This is an interesting fact and means, that the trajectories of extremum seeking system using only distance measurements can be interpreted as a distance-based Kuramoto model for synchronization.

Since the Lie bracket system corresponding to the overall system (7) with controller (33) coincides with the Kuramoto model, we may exploit the results in e.g., [4], [18], to conclude that the minimum of (8), i.e., the set of local minima E_c where $x_j = x_k$, $(j,k) \in E$, is locally asymptotically stable for the Lie bracket system. In terms of the stability definitions used in here, this means that there exists a set $S \subseteq T^N$ such that E_c is S-asymptotically stable for the Lie bracket system. With Corollary 2 we may then conclude that the set E_c is S-practically uniformly asymptotically stable with respect to the overall system (7) with controller (33).

B. Synchronization of Rigid Bodies

Consider the objective function (11). We observe that the function f can be written as

$$f(X) = \frac{1}{2}\sum_{(j,k)\in E}\|X_j - X_k\|_F^2,$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. Similar as in the problem before, we introduce the individual objective functions for agent $j \in V$ as

$$\begin{aligned} f_j(X) &= \frac{1}{2}\sum_{k\in\mathcal{N}_j}\|X_j - X_k\|_F^2 \\ &= \frac{1}{2}\sum_{k\in\mathcal{N}_j}\operatorname{trace}((X_k - X_j)^\top(X_k - X_j)). \end{aligned} \tag{41}$$

Again, one can immediately verify that the functions f_j, $j \in V$ satisfy (22). As in the problem above, we observe that each of the functions f_j only depend on the distances among agent $j \in V$ and its neighbors. In contrast to the dynamics on the circle (7), which have only one control input, the dynamics on $SO(3)$ have three inputs. We propose the following controller for agent $j \in V$ and control input $i = 1, 2, 3$ as follows

$$u_{j,i} = c_j f_j(X)\sqrt{\omega_{j,i}}\cos(\omega_{j,i}t) + \alpha_j\sqrt{\omega_{j,i}}\sin(\omega_{j,i}t). \tag{42}$$

The closed loop of agent $j \in V$ can be written in the form (20) with $p = 3$, by vectorizing the matrices and using the conventions of Section I.A, i.e., we introduce for $i = 1, 2, 3$

$$\tilde{g}_{j,0}(X_j) = \sum_{i=1}^{3}\nu_i X_j\Omega_i, \quad \tilde{g}_i(X_j) = X_j\Omega_i \tag{43}$$

and

$$\tilde{G}_0(X) = \begin{bmatrix}\tilde{g}_{1,0}(X_j)\\ \vdots \\ \tilde{g}_{N,0}(X_j)\end{bmatrix} \tag{44}$$

and observe that we obtain (20) with $g_{j,0}(X_j) = \operatorname{vec}(\tilde{g}_{j,0}(X_j))$, $g_i(X_j) = \operatorname{vec}(\tilde{g}_i(X_j))$ and furthermore $G_0(X) = \operatorname{vec}(\tilde{G}_0(X))$ with $G_0(X)$ in (27). Next, we will apply Corollary 1 of the foregoing section. We note that Assumptions A1, A2 and A3' are satisfied. In particular, Assumption A1 is satisfied since $SO(3)$ is a smooth three-dimensional submanifold of $\mathbb{R}^{3\times 3}$ and therefore also a smooth submanifold of $\mathbb{R}^9$. Assumption A2 is satisfied, since the particular choice of matrices Ω_i, $i = 1, 2, 3$ span the set of skew-symmetric matrices and thus the vector fields g_i, $i = 1, 2, 3$ satisify (14). Assumption A3' is satisfied since $f|_{SO(3)^N}(X) \geq 0$ for all $X \in SO(3)^N$ and furthermore $f|_{SO(3)^N}(X) = 0$ if and only if $X \in E_s := \{X \in SO(3)^N : X_j = X_k\}$. Therefore, $E_s \subseteq E$ with E being the set of local minima of f and E is nonempty. Let E_c be a connected component of E_s. Since T^N is compact, E_c is also compact.

By the same arguments as in the previous problem solution one can show that there are only finitely many connected components of the set of critical points of f.

Next, we verify condition (29). We see that

$$\frac{\partial f(X)}{\partial X_j} = \sum_{k\in\mathcal{N}_j} X_j - X_k, j \in V \tag{45}$$

and with the conventions in Section I.A that

$$\nabla f(X) = \begin{bmatrix}\frac{\partial f(X)}{\partial \operatorname{vec}(X_1)}^\top \\ \vdots \\ \frac{\partial f(X)}{\partial \operatorname{vec}(X_N)}^\top\end{bmatrix} = \operatorname{vec}\left(\begin{bmatrix}\frac{\partial f(X)}{\partial X_1}\\ \vdots \\ \frac{\partial f(X)}{\partial X_N}\end{bmatrix}\right). \tag{46}$$

With these observations, we obtain

$$\langle \nabla f(X), G_0(X) \rangle = \left\langle \frac{\partial f(X)}{\partial X}, \tilde{G}_0(X) \right\rangle$$
$$= \sum_{(j,k)\in E} \sum_{i=1}^{3} \mathrm{trace}((X_j^\top - X_k^\top) X_j \Omega_i)\nu_i$$
$$+ \mathrm{trace}(X_k^\top - X_j^\top) X_k \Omega_i)\nu_i$$
$$= - \sum_{(j,k)\in E} \sum_{i=1}^{3} \mathrm{trace}((X_k^\top X_j + X_j^\top X_k)\Omega_i)\nu_i$$
$$= 0, \tag{47}$$

where we have used that the trace of the product of a symmetric and a skew-symmetric matrix is zero.

We may thus apply Corollary 1 and conclude that the solution of (11) is practically uniformly asymptotically stable with respect to the extremum seeking consisting of (9) with controller (42).

C. Discussion

First, one can see that the two synchronization problems are similar in the sense that the minimum of the objective functions in (8) and (11) is attained where $x_j = x_k$ respectively $X_j = X_k$, $(j,k) \in E$. Strictly speaking, Corollaries 1 and 2 do not exclude trivial behavior of the agents on the set E_c. One has to extend them in order to guarantee also practical synchronization of the agents, i.e., asymptotic stability of a periodic solution of the Lie bracket system implies practical uniform asymptotic stability of the extremum seeking system. However, due to (29) and (30) we see that drift vector fields in (34) and (44) are orthogonal to the gradient of the objective functions in (8) and (11). Therefore, one can intuitively expect that the agents admit a non-trivial behavior on the solution sets of (8) and (11) which is determined by the drift vector field. This will be shown in the next section using a numerical simulation.

Second, one could have also required in the problem formulations in Section II that the controller inputs $u_{j,1}$, $j \in V$ in the synchronization problem on the circle and $u_{j,i}$, $i = 1,2,3$ in the synchronization of rigid bodies vanish for agents being at the minimum of the respective objective function. However, we see that the controller inputs (33) and (42) contain periodic excitations which never vanish. Thus, an extremum seeking controller of this kind can not satisfy such a requirement.

VI. Numerical Examples

In this section, we show two examples for the extremum seeking controllers constructed in the section above.

A. Synchronization on the Circle

We show a numerical example for the synchronization on the circle, by considering the network of three agents as in Fig. 1. We choose the constants $\nu = 1$ for the dynamics (7) as well as $c_j = 1$, $\alpha_j = 0.1$ and $\omega_{j,1} = 18 + 2j$ for the controller (33) of agent $j = 1, 2, 3$. In Figs. 2 and 3 we see that for different initial conditions of the agents, the respective components synchronize in the sense that they reach a common non-trivial closed solution.

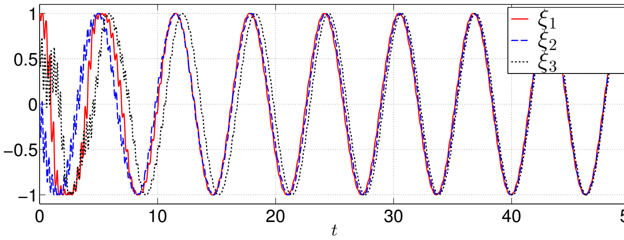

Fig. 2: Synchronization on the Circle, ξ-components of Agents

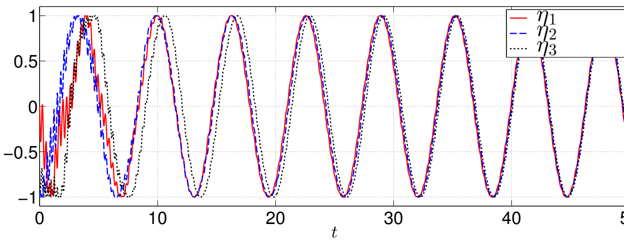

Fig. 3: Synchronization on the Circle, η-components of Agents

B. Synchronization of Rigid Bodies

We show a numerical example for the synchronization of rigid bodies by considering a network of two agents. We choose the constants $\nu_1 = \nu_3 = 0$ and $\nu_2 = 1$ for the dynamics (9) as well as $c_j = 0.4$, $\alpha_j = 0.1$, $\omega_{1,i} = 45 + 5i$, $\omega_{2,i} = 60 + 5i$, $i = 1,2,3$ for the controller (42) of Agents 1 and 2. In Fig. 4 we see that coordinate frames of each agent synchronizing on a rotation tangential to the body frame axis which is induced by the drift vector field.

VII. Summary

We presented two examples of synchronization problems on manifolds using only the distance measurements among neighboring agents. We demonstrated the potential use of extremum seeking on manifolds to solve synchronization problems. The results also reveal that the approximating Lie bracket system for the synchronization on the circle coincides with the well-known Kuramoto model. We showed that the Lie bracket approximation for extremum seeking systems can also be exploited for systems on Lie groups, in particular we showed how to construct a controller that leads to synchronization of rigid bodies.

Appendix

Consider the dynamical system

$$\dot{x} = f_\omega(t, x) \tag{48}$$

with $x(t_0) = x_0$, $t_0 \in \mathbb{R}$ and which depends on a parameter $\omega > 0$.

Definition 1: A compact set $E \subseteq M$ is said to be **practically uniformly stable** for (48) if for every $\epsilon > 0$

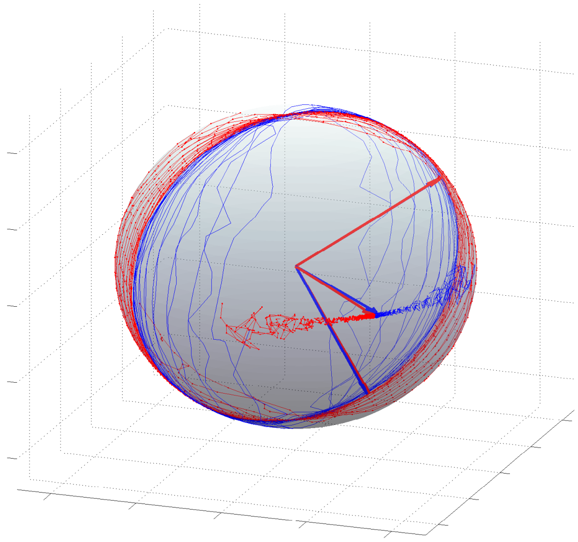

Fig. 4: Synchronization of Rigid Bodies (body-frame of rigid body 1: red, dash-dotted line, body-frame of rigid body 2: blue, solid line)

there exist a $\delta > 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega > \omega_0$

$$x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, \quad t \geq t_0. \tag{49}$$

Definition 2: A compact set $E \subseteq M$ is said to be **practically uniformly attractive** for (48) if there exists a $\delta > 0$ such that for every $\epsilon > 0$ there exist a $t_f \geq 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0$

$$x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, \quad t \geq t_0 + t_f. \tag{50}$$

Definition 3: A compact set $E \subseteq M$ is said to be **practically uniformly asymptotically stable** for (48) if it is practically uniformly stable and it is practically uniformly attractive.

Definition 4: Let $S \subseteq M$. A compact set $E \subseteq M$ is said to be S**-practically uniformly asymptotically stable** for (48) if it is practically uniformly stable and for every $\delta, \epsilon > 0$ there exist a $t_f \geq 0$, a $c > 0$ and $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0$

$$\begin{aligned} x(t_0) \in S \cap U_\delta^E \Rightarrow \ & x(t) \in U_\epsilon^E, \quad t \geq t_0 + t_f \\ & \text{and } x(t) \in U_c^E, \quad t \geq t_0. \end{aligned} \tag{51}$$

For system which are independent of "ω" and "t" we drop the terms "practically" and "uniformly" in the definitions above. That this results in the common notion of Lyapunov stability.

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