Automatica 50 (2014) 2591–2596

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Extremum seeking on submanifolds in the Euclidian space ☆

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ARTICLE INFO

Article history:
Received 4 December 2012
Received in revised form
29 April 2014
Accepted 21 May 2014
Available online 12 September 2014

Keywords:
Extremum seeking
Lie brackets
Manifolds
Synchronization

ABSTRACT

Extremum seeking is a powerful control method to steer a dynamical system to an extremum of a partially unknown function. In this paper, we introduce extremum seeking systems on submanifolds in the Euclidian space. Using a trajectory approximation technique based on Lie brackets, we prove that uniform asymptotic stability of the so-called Lie bracket system on the manifold implies practical uniform asymptotic stability of the corresponding extremum seeking system on the manifold. We illustrate the approach with an example of extremum seeking on a torus.

1. Introduction

Extremum seeking control is used to steer the states of a dynamical system to an extremum of a function which depends on the system states. A characteristic feature of extremum seeking control is that neither a model of the plant nor the function which has to be optimized need not to be known. Only the values of the function must be available (measurable) at the actual system state. The idea of extremum seeking control is to inject periodic signals into the control loop in order to estimate an optimizing direction for the unknown function (Krstić & Ariyur, 2003; Tan, Moase, Manzie, Nesic, & Mareels, 2010). These properties render extremum seeking attractive for many application areas. For example, it has been applied in brake system control (Drakunov, Ozguner, Dix, & Ashrafi, 1995), in real-time optimization of bioprocesses (Guay, Dochain, & Perrier, 2004), in flow control (Becker, King, Petz, & Nitsche, 2007; King et al., 2006) and in multi-agent systems (Binetti, Ariyur, Krstić, & Bernelli, 2003; Stanković, Johansson, & Stipanović, 2012) to mention only a few. Classical extremum seeking can be interpreted as a method for unconstrained optimization problems. In recent years, several extensions have been established for constrained optimization problems. One can distinguish between three main approaches, the Lagrangian approach, the barrier function approach and the approach for optimization on manifolds. The former two were addressed in Coito, Lemos, and Alves (2005) and DeHaan and Guay (2005) in the context of extremum seeking. Furthermore, in e.g. Poveda and Quijano (2012), a special class of manifolds, defined using linear constraints, were considered and an extremum seeking system was constructed that leaves this subspace invariant. In this paper, we construct an extremum seeking system that solves optimization problems on submanifolds in the Euclidian space $\mathbb{R}^n$ equipped with the standard scalar product. These are optimization problems with constraints that describe a smooth submanifold. They have many interesting applications in areas such as eigenvalue computation, principle component analysis or consensus problems, see Absil, Mahony, and Sepulchre (2008) and Helmke and Moore (1994) for more details.

The contribution of this paper is threefold. First, we introduce an extremum seeking feedback that steers a dynamical system to the solution of an optimization problem with constraints given by a submanifold in the Euclidian space. Hereby, we do not require the knowledge of the gradient of the objective function. The class of submanifolds considered in this paper contains many important manifolds such as spheres, tori, matrix Lie groups and the isospectral manifold (see e.g. Absil et al., 2008 and Helmke & Moore, 1994) which appear in many applications. We formulate the dynamics of the extremum seeking feedback using the coordinates of the ambient space, which has the advantage that no local coordinate charts

☆ The material in this paper was partially presented at the 19th IFAC World Congress, August 24–29, 2014 Cape Town, Africa and the 51st Annual Allerton Conference on Communication, Control, and Computing, October 2–4, 2013, Monticello, Illinois, USA. This paper was recommended for publication in revised form by Associate Editor Raul Ordonez under the direction of Editor Miroslav Krstic.

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http://dx.doi.org/10.1016/j.automatica.2014.08.019

of the manifold are required and therefore singularities of these charts are avoided.

Second, we prove non-local stability properties of the extremum seeking feedback and calculate a system whose trajectories approximate those of the extremum seeking system. In particular, we generalize the Lie bracket approximation method for extremum seeking systems introduced in Dürr, Stanković, Ebenbauer, and Johansson (2013). We show in this paper, that the results in Dürr et al. (2013) also hold for extremum seeking systems on manifolds.

Third, we show a novel example for extremum seeking on manifolds. The example is motivated by synchronization problems of oscillators based only on distance measurements. This leads to extremum seeking on the torus. In particular, we derive an extremum seeking feedback for two oscillators that yield synchronization using only distance measurements. We show that there is a close relationship between the extremum seeking system and the well-known Kuramoto model.

The remainder of this paper is structured as follows. In Section 2, we introduce the necessary mathematical preliminaries. In Section 3, we present the main idea and the stability results for extremum seeking systems on submanifolds in the Euclidian space. In Section 4, we present the numerical example. In Section 5, we summarize the results and give an outlook on future research topics.

2. Preliminaries

We use the following notation. $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Q}$ ($\mathbb{Q}_{++}$) are the (positive) rational numbers. We use $s \in \mathbb{C}$ for the complex variable of the Laplace transformation. The Euclidian space, considered in this paper, is $\mathbb{R}^n$ equipped with the standard scalar product (standard metric) $\langle x_1, x_2\rangle := x_1^\top x_2$, we denote it by $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. $e_1 = [1, 0, \dots, 0]^\top, \dots, e_n = [0, \dots, 0, 1]^\top$ are the standard basis vectors in $\mathbb{R}^n$. We identify the tangent space $T_x\mathbb{R}^n$ of $\mathbb{R}^n$ at x with $\mathbb{R}^n$ and, with a slight abuse of notation, we also denote the constant vector field $[1, 0, \dots, 0]^\top$ by e_1 etc. The Euclidian norm of a vector $v \in \mathbb{R}^n$ is denoted by $\|v\| = \sqrt{\langle v, v\rangle}$. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. The tangent space of M at x is denoted by T_xM. The a-neighborhood of a set $S \subseteq M$ with $a > 0$ is denoted by $U_a^S := \{x \in M : \inf_{y\in S} \|x - y\| < a\}$. We denote by C^n with $n \in \mathbb{N}_0$ the set of n times continuously differentiable functions. The Lie bracket $[.,.] : T_xM \times T_xM \to T_xM$ between two vector fields $g_1, g_2 \in C^1 : M \to T_xM$ is defined as: $[g_1(x), g_2(x)] = \frac{\partial g_2(x)}{\partial x} g_1(x) - \frac{\partial g_1(x)}{\partial x} g_2(x)$. The vector field $[g_1(x), g_2(x)]$ is again a vector field on M (see Corollary 8.28 in Lee, 2003). Let $U \subseteq \mathbb{R}$ be open, $M \subseteq U$ and $f : U \to \mathbb{R}$. We denote with $f|_M : M \to \mathbb{R}$ the restriction of a function $f : U \to \mathbb{R}$ to M. The gradient vector field of f on the Riemannian manifold $(\mathbb{R}^n, \langle .,. \rangle)$ is denoted by $\nabla f = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]^T$ and the gradient vector field of f on the manifold M is denoted by $\mathrm{grad} f|_M$.

An optimization problem on a manifold is an optimization problem of the form

$$\min f(x) \quad \text{s.t. } x \in M \tag{1}$$

where M is a manifold and $f : M \to \mathbb{R}$ a continuously differentiable function. A necessary condition for a local minimum of (1) is that the gradient must necessarily vanish at that point, i.e., if x^* is a local minimum of (1) then $\mathrm{grad} f|_M(x^*) = 0$ (see e.g., p. 284 in Bertsekas, 1995). Typically extremum seeking tries to solve (1) with $M = \mathbb{R}^n$ by seeking points where the gradient vanishes. In this work, we consider the case where M is a submanifold in $\mathbb{R}^n$.

In order to characterize the gradient on M in terms of ∇f, we need the following lemma (see e.g. p. 48 in Absil et al., 2008):

Lemma 1. *Let $U \subseteq \mathbb{R}^n$ be open, $M \subseteq U$ and ∇f be the gradient vector field of $f : U \to \mathbb{R}^n$, defined by the standard scalar product $\langle .,. \rangle$ in $\mathbb{R}^n$. Then the induced gradient vector field* $\mathrm{grad} f|_M : M \to T_xM$ *is given by*

$$\mathrm{grad} f|_M(x) = \mathrm{P}(\nabla f(x)), \tag{2}$$

where $\mathrm{P}(y)$ denotes the orthogonal projection of $y \in \mathbb{R}^n$ onto T_xM.

Next, we review some results from Dürr et al. (2013). As it is shown in Dürr et al. (2013), certain extremum seeking systems can be written as input-affine systems

$$\dot{x} = b_0(x) + \sum_{i=1}^{m} b_i(x)\sqrt{\omega}u_i(\omega t) \tag{3}$$

with $x(t_0) = x_0 \in \mathbb{R}^n$, $\omega > 0$ and $m \in \mathbb{N} \cup \{0\}$ the number of vector fields.

We impose the following assumptions on b_i and u_i:

A1 $b_i \in C^2 : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, \dots, m$;
A2 $u_i \in C^0 : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m$ and for every $i = 1, \dots, m$ there exist constants $M_i > 0$ such that $\sup_{\theta\in\mathbb{R}} |u_i(\theta)| \le M_i$;
A3 $u_i(\cdot)$ is T-periodic, i.e., $u_i(\theta + T) = u_i(\theta)$ and has zero average, i.e., $\int_0^T u_i(\tau)d\tau = 0$, with $T > 0$ for all $\theta \in \mathbb{R}$, $i = 1, \dots, m$.

One main result in Dürr et al. (2013) is to approximate the trajectories of (3) by the trajectories of a so-called *Lie bracket system*

$$\dot{z} = b_0(z) + \sum_{\substack{i=1\\ j=i+1}}^{m} [b_i(z), b_j(z)]\nu_{ji} \tag{4}$$

where $\nu_{ji} = \frac{1}{T}\int_0^T u_j(\theta)\int_0^\theta u_i(\tau)d\tau d\theta$. Furthermore, we introduce a set B of initial conditions for (4) which have bounded solutions, i.e., there exists an $A > 0$ such that

$$z(0) \in B \Rightarrow z(t) \in U_A^0, \quad t \ge 0. \tag{5}$$

B is used in the proof of the main theorems to assure the existence of trajectories.

Finally, we cite two results from Dürr et al. (2013). The following lemma states that trajectories of (3) are approximated by trajectories of (4).

Lemma 2. *Let Assumptions* A1–A3 *be satisfied. Then for every bounded set $K \subseteq B$ with B as in (5), for every $D > 0$ and for every $t_f > 0$, there exists an $\omega_0 > 0$ such that for every $\omega > \omega_0$, for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathcal{K}$ there exist solutions $x, z : \mathbb{R} \to \mathbb{R}^n$ of (3) and (4) through $x(t_0) = z(t_0) = x_0$ which satisfy $\|x(t) - z(t)\| < D$, $t_0 \le t \le t_0 + t_f$.*

For systems like (3) we need a notion of stability which is closely related to Lyapunov stability and applies to systems depending on a parameter.

Definition 1. A compact set $E \subseteq M$ is said to be *practically uniformly stable* for (3) if for every $\epsilon > 0$ there exist a $\delta > 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega > \omega_0$ $x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E$, $t \ge t_0$.

Definition 2. A compact set $E \subseteq M$ is said to be *practically uniformly attractive* for (3) if there exists a $\delta > 0$ such that for every $\epsilon > 0$ there exist a $t_f \ge 0$ and an $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0 x(t_0) \in U_\delta^E \Rightarrow x(t) \in U_\epsilon^E$, $t \ge t_0 + t_f$.

Definition 3. A compact set $E \subseteq M$ is said to be *practically uniformly asymptotically stable* for (3) if it is practically uniformly stable and it is practically uniformly attractive.

Using Lemma 2 it is shown in Dürr et al. (2013) that the stability properties of systems (3) and (4) are linked. This is captured in the next lemma, where practical stability is shown for (3).

Lemma 3. *Let Assumptions* A1–A3 *be satisfied and suppose that a compact set* $E \subseteq M$ *is asymptotically stable for* (4). *Then* $E \subseteq M$ *is practically uniformly asymptotically stable for* (3).

For systems like (4) which are independent of "ω" and "t" we drop the terms "practically" and "uniformly" in the definitions above.

3. Main results

This section consists of two parts. First, we recall extremum seeking in $\mathbb{R}^n$. Second, we propose extremum seeking systems on submanifolds in the Euclidian space.

3.1. Extremum seeking in $\mathbb{R}^n$

Consider an extremum seeking system as e.g. in Dürr et al. (2013) and Krstić and Ariyur (2003), given by the differential equations

$$\begin{aligned}\dot{x}_1 &= cf(x)\sqrt{\omega_1}\cos(\omega_1 t) + \alpha\sqrt{\omega_1}\sin(\omega_1 t)\\ &\vdots \\ \dot{x}_n &= cf(x)\sqrt{\omega_n}\cos(\omega_n t) + \alpha\sqrt{\omega_n}\sin(\omega_n t),\end{aligned} \tag{6}$$

where $f \in C^2 : \mathbb{R}^n \to \mathbb{R}$, $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ and $\alpha, c > 0$ and $\omega_i = a_i\omega$, $a_i \neq a_j$, $i \neq j$, $a_i \in \mathbb{Q}_{++}$, $\omega > 0$, $i, j = 1, \dots, p$. Identifying $u_i^1(\omega_i t) = \sqrt{\omega_i}\cos(\omega_i t)$, $u_i^2(\omega_i t) = \sqrt{\omega_i}\sin(\omega_i t)$ as inputs and calculating the Lie bracket system associated to (6), we obtain

$$\dot{z} = -\frac{\alpha c}{2}\nabla f(z). \tag{7}$$

We see that the Lie bracket system (7) reveals the optimizing behavior of the extremum seeking system (6). With Lemma 2 the trajectories of the Lie bracket system (7) can be uniformly approximated by trajectories of the extremum seeking system (8) for sufficiently large ω and for pairwise distinct $\omega_i, \omega_j \in \mathbb{Q}$. Moreover, due to Lemma 3, stability properties of (7) turn into practical stability properties of (8), see Dürr et al. (2013) for details.

3.2. Extremum seeking on submanifolds in $\mathbb{R}^n$

The main idea is based on the following observations. First, note that (6) can be more compactly written as follows:

$$\dot{x} = \sum_{i=1}^{n} cf(x)e_i\sqrt{\omega_i}\cos(\omega_i t) + \alpha e_i\sqrt{\omega_i}\sin(\omega_i t). \tag{8}$$

Using the identity

$$[f(z)e_i, e_i] = -\langle \nabla f(z), e_i\rangle e_i, \tag{9}$$

we see that the Lie bracket system associated to (8) can be written as

$$\dot{z} = \frac{\alpha c}{2}\sum_{i=1}^{n}[f(z)e_i, e_i] = -\frac{\alpha c}{2}\nabla f(z). \tag{10}$$

Second, we recall that the gradient vector field $\mathrm{grad} f|_M$ of a submanifold M in $\mathbb{R}^n$, which is induced by the standard scalar product in $\mathbb{R}^n$, is the orthogonal projection of ∇f onto the tangent space of M (see Lemma 1). Note that (9) also holds if we replace e_i by (tangent) vector fields $g_i = g_i(x)$ on the submanifold M, i.e.,

$$[f(z)g_i(z), g_i(z)] = -\langle \nabla f(z), g_i(z)\rangle g_i(z). \tag{11}$$

The idea of this paper is now as follows. Note that (11) can be interpreted as a projection of ∇f onto the vector field g_i. If the vector fields g_i form an orthonormal basis of the tangent space T_xM of M, we can see that

$$\mathrm{grad} f|_M(z) = \sum_{i=1}^{n}\langle \nabla f(z), g_i(z)\rangle g_i(z). \tag{12}$$

Consequently using (9) and by replacing the e_i's in (8) and (10) by g_i's, the Lie bracket system (10) turns into a gradient flow on the manifold M. Based on this idea we propose an approach for extremum seeking on submanifolds in the Euclidian space.
Consider the following assumption:

Assumption 1. (1) $M \subseteq \mathbb{R}^n$ is a smooth, m-dimensional Riemannian submanifold without boundary. The metric $\langle .,.\rangle_M : T_xM \times T_xM \to \mathbb{R}$ on M is the metric $\langle .,.\rangle$ induced by the inner product of the ambient space $\mathbb{R}^n$, i.e., $\langle x_1, x_2\rangle_M := \langle x_1, x_2\rangle$, with $x_1, x_2 \in T_xM$;
(2) there are $p \geq m$ vector fields $g_i \in C^2 : M \to T_xM$, $i = 1, \dots, p$, on M such that

$$\mathrm{span}\{g_1(x), \dots, g_p(x)\} = T_xM \quad \text{for all } x \in M, \tag{13}$$

i.e., for each point x on M, the tangent vectors $g_i(x)$, $i = 1, \dots, m$, span the tangent space T_xM and for $p = m$ the tangent vectors $g_i(x)$ form a basis of T_xM;
(3) let $U \subseteq \mathbb{R}^n$ be open, $M \subseteq U$ and $f \in C^2 : U \to \mathbb{R}$. The set of local minima E of $f|_M$ is nonempty and we denote with $E_c \subseteq E$ a compact connected component of E.

We introduce the *extremum seeking system on the manifold M* as follows (which differs from (8) by replacing the unit vectors e_i by the tangent vector fields g_i):

$$\dot{x} = \sum_{i=1}^{p} c_i f(x)g_i(x)\sqrt{\omega_i}\cos(\omega_i t) + \alpha_i g_i(x)\sqrt{\omega_i}\sin(\omega_i t) \tag{14}$$

where $\alpha_i, c_i > 0$ and $\omega_i = a_i\omega$, $a_i \neq a_j$, $i \neq j$, $a_i \in \mathbb{Q}_{++}$, $\omega > 0$, $i, j = 1, \dots, p$. Since $g_i(x) \in T_xM$ for all $x \in M$, the right hand side of the extremum seeking system (14) defines a vector field on M, i.e., solutions initialized on M are uniformly invariant on M. More explicitly, $x(t_0) \in M$ implies that $x(t) \in M$ for all $t_0 \leq t < t_0 + t_{\max}$, where $t_{\max}$ is the maximal interval of existence. As indicated above, we may exploit the identity in (11), which yields the *Lie bracket system on M*:

$$\dot{z} = -\frac{1}{2}\sum_{i=1}^{p}\alpha_i c_i\langle \nabla f(z), g_i(z)\rangle g_i(z). \tag{15}$$

Clearly, the right hand side of (15) is a vector field on M. Observe also that when all α_i and c_i have the same value α and c and if the tangent vectors $g_i(x)$, $i = 1, \dots, p$ with $p = m$ form an orthonormal basis of T_xM for all $x \in M$, then the right hand side of (15) is exactly $-\frac{\alpha c}{2}\mathrm{grad} f|_M(x)$.

Remark 1. Compared to the extremum seeking setup in Krstić and Ariyur (2003) and Tan et al. (2010), where the dynamics of the extremum seeking feedback and the dynamic plant are separated, we consider a slightly different setup. In this paper, the dynamics in (14) can be interpreted in two ways. First, the dynamics of the plant are assumed to be in quasi-steady state and thus appears static to the dynamics of the extremum seeking feedback, i.e., the dynamics in (14) contain only the extremum seeking dynamics. Second, the dynamics in (14) may include the dynamics of a plant in terms of a drift term f_0, i.e.,

$$\begin{aligned}\dot{x} = f_0(x) + \sum_{i=1}^{p} & c_i f(x)g_i(x)\sqrt{\omega_i}\cos(\omega_i t)\\ &+ \alpha_i g_i(x)\sqrt{\omega_i}\sin(\omega_i t),\end{aligned} \tag{16}$$

where the extremum seeking feedback is *static* and consists only of the sinusoidal perturbations. Note that, the results herein can also be extended to extremum seeking systems of the form (16).

Using Lemmas 2 and 3, i.e., the methodology developed in Dürr et al. (2013), we can establish the main results.

Theorem 1. *Consider the Lie bracket system (15) and let Assumption 1 be satisfied. Let $W \subseteq M$ be an open set and let E_c be a compact connected set of minima of $f|_M$ which is contained in W. Assume that the gradient of $f|_M$ vanishes in W only at points in E_c, i.e., $\operatorname{grad} f|_M(z) = 0$ if and only if $z \in E_c$ for all $z \in W$. Then the set E_c of equilibria is asymptotically stable. Moreover, E_c is practically uniformly asymptotically stable with respect to the extremum seeking system (14).*

In other words, the extremum seeking system locally converges arbitrary close to the set of local minima of $f|_M$ for sufficiently large ω_i, $i = 1, \dots, p$. The existence of a set W around a compact connected component of local minima usually exists except in certain pathological cases, see for example Whitney (1935).

Proof. First, define the Lyapunov function candidate $V(x) = f(x)$. Since $\operatorname{grad} f|_M(z) = 0$ if and only if $z \in E_c$ for all $z \in W$ and since E_c is a compact connected component of (local) minima of f on M, we know that $V(x) > V(\bar{x})$ for all pairs $(x, \bar{x}) \in (W \setminus E_c) \times E_c$. Since E_c is compact, we also know that there exists a constant ϵ such that $\{x \in W : V(x) \le \epsilon\}$ is a compact subset of W. Thus V is a suitable Lyapunov function. The Lie derivative along the Lie bracket system (15) is negative semidefinite: $\dot{V} = -\frac{1}{2}\sum_{i=1}^{k} \alpha_i c_i \langle \nabla f(z), g_i(z)\rangle^2 \le 0$. From Assumption 1 $\operatorname{span}\{g_1(z), \dots, g_p(z)\} = T_z M$ for all $z \in M$ so we see that $\dot{V}$ vanishes either when $\nabla f(z) = 0$ or when $\nabla f(z)$ is orthogonal to $T_z M$. By Lemma 1, this is equivalent to $\operatorname{grad} f|_M(z) = 0$ and by assumption this is the case if and only if $z \in E_c$. Thus, we conclude that the set E_c is asymptotically stable and E_c is a set of equilibria.
Second, the extremum seeking system (14) fulfills Assumptions A1–A3. Thus, all assumptions of Lemma 3 are satisfied and we conclude that the set E_c of minima of $f|_M$ is practically uniformly asymptotically stable for the extremum seeking system (14). □

Theorem 1 provides a local stability result of the extremum seeking system based on the Lie bracket system. The next theorem provides a nonlocal result. This requires the introduction of a modified version of practical stability, that also captures the region of attraction.

Definition 4. Let $S \subseteq M$. A compact set $E \subseteq M$ is said to be *S-practically uniformly asymptotically stable* for (3) if it is practically uniformly asymptotically stable and for every $\delta, \epsilon > 0$ there exist a $t_f \ge 0$, a $c > 0$ and $\omega_0 > 0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega > \omega_0 x(t_0) \in S \cap U_\delta^E \Rightarrow x(t) \in U_\epsilon^E, t \ge t_0 + t_f$ and $x(t) \in U_c^E, t \ge t_0$.

As before, we drop the term "practically" and "uniformly" in the definition above for systems like (4). If (4) is S-asymptotically stable then E is asymptotically stable and S belongs to the region of attraction.

Theorem 2. *Consider the Lie bracket system (15) and let Assumption 1 be satisfied. Let $S \subseteq M$ and assume a compact connected set E_c of local minima of $f|_M$ is S-asymptotically stable. Then, E_c is S-practically uniformly asymptotically stable with respect to the extremum seeking system (14).*

In other words, this theorem states if S is a subset of the region of attraction of E_c for the Lie bracket system, then S is also a subset of the 'practical' region of attraction of E_c for the extremum seeking system. This notion is similar as to the notion of semi-global practical stability (e.g. Dürr et al., 2013 and Moreau & Aeyels, 2000).

Proof. By assumption the set E_c is asymptotically stable for the Lie bracket system (15). Since the extremum seeking system (14) satisfies all assumptions of Lemma 3 we conclude that E_c is practically uniformly asymptotically stable.

Now we prove the second claim of the theorem, i.e., for every $\delta, \epsilon > 0$ there exist a $t_f > 0$, a $c > 0$ and an $\omega_0 > 0$ such that for all $\omega > \omega_0$ and all $t_0 \in \mathbb{R}$

$$x(t_0) \in S \cap U_\delta^E \Rightarrow x(t) \in U_\epsilon^{E_c}, \quad t \ge t_0 + t_f \text{ and } x(t) \in U_c^E, \ t \ge t_0. \tag{17}$$

The following proof consists of two steps. First, we exploit the result above, i.e., trajectories initialized in the vicinity of E_c are practically asymptotically stable. Second, we establish practical convergence of the extremum seeking system for all initial conditions in S to E_c. Combining these facts, we obtain that trajectories of the extremum seeking system initialized in S converge to a vicinity of E_c and by practical asymptotic stability they do not leave the vicinity of E_c and practically converge to E_c.
First step: given any $\delta, \epsilon > 0$. Note that by practical uniform asymptotic stability, there exist a $\delta_1 > 0$ and an $\omega_1 > 0$ such that for all $\omega > \omega_1$ and all $t_0 \in \mathbb{R}$ we have that

$$x(t_0) \in U_{\delta_1}^{E_c} \Rightarrow x(t) \in U_\epsilon^{E_c}, \quad t \ge t_0. \tag{18}$$

Note that, this also implies that $\delta_1 \le \epsilon$.
Second step: let $0 < \epsilon_2 < \delta_1$. Since the set E_c is S-asymptotically stable for the Lie bracket system (15), there exist a $t_f > 0$ and a $\tilde{c} > 0$ such that

$$z(0) \in S \cap U_\delta^{E_c} \Rightarrow z(t) \in U_{\epsilon_2}^{E_c}, \quad t \ge t_f \text{ and } z(t) \in U_{\tilde{c}}^{E_c}, \quad t \ge 0. \tag{19}$$

Now, we apply Lemma 2. Let $B := K := S \cap U_\delta^{E_c}$, $D = \delta_1 - \epsilon_2$ and t_f as above and note that due to (19) B satisfies (5). Thus, there exists an $\omega_2 > 0$ such that for every $\omega > \omega_2$, $t_0 \in \mathbb{R}$ and $x_0 \in K$ we have for $x(t_0) = z(t_0)$ that $\|x(t) - z(t)\| < D$, $t_0 \le t \le t_0 + t_f$. Observe that this together with (19) leads for all $\omega > \omega_0 := \max\{\omega_1, \omega_2\}$ and all $t_0 \in \mathbb{R}$ to

$$x(t_0) \in S \cap U_\delta^{E_c} \Rightarrow x(t_0 + t_f) \in U_{\delta_1}^{E_c} \text{ and } x(t) \in U_{\tilde{c}+D}^{E_c}, \quad t_0 \le t \le t_0 + t_f. \tag{20}$$

With this result and since (18) holds uniformly in t_0 we choose $\tilde{t}_0 = t_0 + t_f$ and obtain $x(t_0) \in S \cap U_\delta^{E_c} \Rightarrow x(t) \in U_\epsilon^{E_c}, t \ge t_0 + t_f$. Finally, with (20) and $c = \max\{\tilde{c} + D, \epsilon\}$ we obtain the desired result in (17). We conclude that the set E_c is S-practically uniformly asymptotically stable for the extremum seeking system (14). □

Remark 2. For the sake of simplicity we removed the washout filter (see, e.g., Dürr et al., 2013 and Krstić & Ariyur, 2003) which is usually present in extremum seeking. The role of the filter is to improve the transient behavior and to remove constant offsets in the nonlinear function. Note that, the stability definitions explicitly include the existence of a lower bound ω_0 for the parameter ω in (14). The washout filter does not influence the existence of such a lower bound, but it may change its value.

The proofs of Theorems 1 and 2 goes along similar lines as in Dürr et al. (2013). Therein, local practical uniform asymptotic stability and semi-global practical uniform asymptotic stability were shown using the methodology developed in Moreau and Aeyels (2000). This methodology also allows to approximate the trajectories of the extremum seeking system by the trajectories of the Lie bracket system. This property directly translates to the case of extremum seeking on submanifolds in $\mathbb{R}^n$.
Note that instead of using sinusoids for the perturbations of the extremum seeking feedbacks, the results herein also cover a larger class of periodic perturbation signals (see Dürr et al., 2013).

4. Example

In this section, we illustrate the results on a numerical example of extremum seeking on submanifolds of $\mathbb{R}^n$. It is motivated by synchronization problems in the spirit of Kuramoto's gradient flow model, e.g. Sarlette, Tuna, Blondel, and Sepulchre (2008). Consider two controlled harmonic oscillators given by

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{bmatrix} = (1+u_i) \begin{bmatrix} x_{i2} \\ -x_{i1} \end{bmatrix}, \tag{21}$$

where we denote the state of the oscillators by $x_i = [x_{i1}, x_{i2}]^\top$, $i = 1, 2$, and the overall state by $x = [x_1^\top, x_2^\top]^\top$. The goal is to find inputs u_1, u_2 such that the oscillators synchronize by using only relative distance measurements $f(x) = \|x_1 - x_2\|^2 = \langle x_1 - x_2, x_1 - x_2 \rangle$. Each of the oscillators evolves on the circle S^1. The problem of synchronizing the oscillators can be formulated as the minimization problem

$$\begin{aligned} &\min \frac{1}{2}\|x_1 - x_2\|^2 \\ &\text{s.t. } x_i^\top x_i = 1, \quad i = 1, 2. \end{aligned} \tag{22}$$

The state space of the overall system is a torus because the system evolves on the product of two unit circles $T^2 = S^1 \times S^1 = \{x_1, x_2 \in \mathbb{R}^2 : x_1^\top x_1 = 1, x_2^\top x_2 = 1\}$. The tangent space of T^2 is given by $T_x T^2 = \{[u^\top, v^\top]^\top \in \mathbb{R}^4 : x_1^\top u = 0, x_2^\top v = 0\}$. We choose $g_1(x_1) = [x_{12}, -x_{11}, 0, 0]^\top$ and $g_2(x_2) = [0, 0, x_{22}, -x_{21}]^\top$. One can again easily verify that these vector fields span the tangent space $T_x T^2$ for all $x \in T^2$. The extremum seeking system is given by (21) with $u_i = c_i f(x)\sqrt{\omega_i}\cos(\omega_i t) + \alpha_i \sqrt{\omega_i}\sin(\omega_i t)$. The overall extremum seeking system can be written as

$$\begin{aligned} \dot{x} = &\sum_{i=1}^{2} g_i(x_i) + c_i f(x) g_i(x_i)\sqrt{\omega_i}\cos(\omega_i t) \\ &+ \alpha_i g_i(x_i)\sqrt{\omega_i}\sin(\omega_i t). \end{aligned} \tag{23}$$

Notice that, the system has a drift term $g_1(x_1)+g_2(x_2)$ and thus it is not of the form (14). However, the results in Section 3 can be easily extended to extremum seeking systems with drifts, which directly follow from the results established in Dürr et al. (2013). In contrast to the first example and in view of Remark 1, the drift vector field can be interpreted as the dynamics of the plants, which are, in this case, the oscillators.

One can show that the Lie bracket system associated to (23) is a gradient flow for (22), therefore it coincides with a classical Kuramoto model, e.g., Sarlette et al. (2008). This allows us to interpret the proposed extremum seeking system as a coupled Kuramoto model that requires only distance measurements and which leads to practical synchronization with similar properties as the classical Kuramoto model. This observation is in the following, discussed in more detail. Since $x_i \in S^1$, i.e., $x_{i1}^2 + x_{i2}^2 = 1$, we may introduce the coordinate transformation $x_{i1} = \cos(\eta_i)$, $x_{i2} = \sin(\eta_i)$, $\eta_i = \arctan(\frac{x_{i1}}{x_{i2}})$ and with $\eta = [\eta_1, \eta_2]^\top$ we obtain for $f(x_1, x_2) = \frac{1}{2}\|x_1 - x_2\|^2$ in (22) $f(\eta) = \frac{1}{2}\|\cos(\eta_1) - \cos(\eta_2)\|^2 + \frac{1}{2}\|\sin(\eta_1) - \sin(\eta_2)\|^2$. Furthermore, $\dot{\eta}_i = \frac{x_{i1}\dot{x}_{i2} - \dot{x}_{i1}x_{i2}}{x_{i1}^2 + x_{i2}^2} = x_{i1}\dot{x}_{i2} - \dot{x}_{i1}x_{i2}$, and with (21), this yields

$$\dot{\eta}_i = 1 + f(\eta)\sqrt{\omega_i}\cos(\omega_i t) + \sqrt{\omega_i}\sin(\omega_i t), \tag{24}$$

$i = 1, 2$. Note that this system is in the form (8) and we can now calculate the corresponding Lie bracket system, which yields

$$\dot{\theta}_1 = 1 - \nabla_{\theta_1} f(\theta) = 1 - \sin(\theta_1 - \theta_2), \tag{25a}$$

$$\dot{\theta}_2 = 1 - \nabla_{\theta_2} f(\theta) = 1 - \sin(\theta_2 - \theta_1). \tag{25b}$$

Note, that (25) is the well-known Kuramoto model for two oscillators, e.g., Kuramoto (1975) and Sarlette et al. (2008). The synchrony seeking system can be generalized to multiple oscillators, for further details we refer to Dürr, Stanković, Johansson, and Ebenbauer (2013). In that paper, we also present an example for synchronization of oscillators on the special orthogonal group $SO(3)$. In Montenbruck, Dürr, Ebenbauer and Allgöwer (2014), we show how extremum seeking on manifolds can be used for obstacle avoidance problems on $SO(3)$.

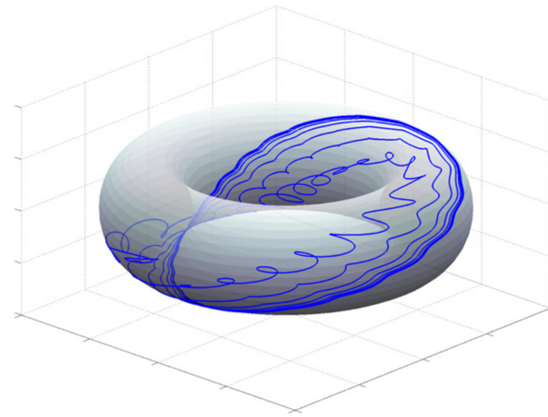

Fig. 1. Simulation result for extremum seeking on T^2.

Next, we show a numerical simulation with parameters $c_1 = c_2 = 0.5, \alpha_1 = \alpha_2 = 0.1, \omega_1 = \omega$ and $\omega_2 = 1.1\omega$ with $\omega = 20$.

The phase system associated to the coupled oscillators can be visualized as a system evolving on a torus depicted in Fig. 1. The phase of the first oscillator evolves in poloidal direction (small radius) and the phase of the second oscillator evolves on the toroidal direction (large radius) on the torus.

We see that the oscillators synchronize, i.e., $x_1 = x_2$ which is a solution of (22). Thus the overall system practically converges according to the results of the previous section.

5. Summary and outlook

In this paper, we introduced extremum seeking systems on submanifolds of the real Euclidian space. By exploiting the Lie bracket approximation analysis introduced in Dürr et al. (2013), we proved practical uniform asymptotic stability of the set of minima of an unknown function on a manifold when only the function values are measurable. We illustrated the results with a numerical example for extremum seeking on submanifolds of the Euclidian space.

Several interesting extensions of this setup are possible.

First, an interesting extension is to consider non-integrable distributions, i.e., the vector fields in (13) do not span the tangent space but the Lie algebra generated by these vector fields do span the tangent space, i.e., extremum seeking for non-holonomic systems.

Second, the introduced tangent vector fields g_i can either be imposed by the proposed feedback or they can be part of the dynamics of the plant, i.e., in the above setup the dynamics evolve exclusively on the manifold and therefore cannot leave the manifold. For the former case, a future research direction is to consider the robustness of the system with respect to the possible presence of disturbances which drive the states away from the manifold. Third, it is also attractive to consider the minimization of dynamic maps, as it was done in the classical extremum seeking (see e.g. Krstić & Ariyur, 2003). This requires a combination of Lie bracket approximation method of extremum seeking systems with a singular perturbation analysis, which allows to consider the dynamics of the map as a quasi-steady map.

Finally, the introduced approach potentially allows to extend many results in the synchronization literature to the case when only distance measurements are available.

Acknowledgments

This work was supported by the Deutsche Forschungsgemeinschaft (Emmy-Noether-Grant, Novel Ways in Control and Computation, EB 425/2-1, and Cluster of Excellence in Simulation Technology, EXC 310/1), the Swedish Research Councils, the Knut and Alice Wallenberg Foundation and the EU Marie Curie CIG, PCIG12-GA-2012-334098.

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