A Heterogeneous Routing Game

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Abstract— Most literature on routing games make the assumption that drivers or vehicles are of the same type and, hence, experience the same latency or cost when traveling along the edges of the network. In contrast, in this article, we propose a heterogeneous routing game in which each driver or vehicle belongs to a certain type. The type determines the cost of traveling along an edge as a function of the flow of all types of drivers or vehicles over that edge. We examine the existence of a Nash equilibrium in this heterogeneous routing game. We study the conditions for which the problem of finding a Nash equilibrium can be posed as a convex optimization problem and is therefore numerically tractable. Numerical simulations are presented to validate the results.

I. INTRODUCTION

The problem of determining Nash equilibria in routing games and bounding their inefficiency has been extensively studied [1]–[5]. However, most of these studies assume that drivers or vehicles are of the same type and, hence, they experience the same latency or cost when using an edge in the network. This is primarily motivated by transportation networks for which the drivers only worry about the travel time (and indeed under the assumption that all the drivers are equally sensitive to the latency) or packet routing in communication networks where all the packets that are using a particular link experience the same delay. However, in more general traffic networks, this assumption might not be realistic. For instance, due to fuel consumption, heavy-duty vehicles and cars might experience different costs for using the road even if their travel times are equal. In [6], this phenomenon has been studied in atomic congestion games in which the heavy-duty vehicles experience an increased efficiency when a higher number of heavy-duty vehicles are present on the same road (because of a higher platooning possibility and, therefore, a higher fuel efficiency) while this may not be true for cars.

The problem of characterizing Nash equilibria in heterogeneous routing games has been studied previously [7]–[12]. For instance, the problem of routing a finite number of customers with a given supply rate was considered in [7],

[8]. The authors in [9] studied the case in which the classes of drivers react differently to the imposed tolls for the road. In [10], the sensitivity of the agent to the latency was adjusted by a multiplicative weight depending on the class to which the driver belongs. However, to our knowledge, these studies adjust either the sensitivity of the agents to the observed latencies or the tolls through a multiplicative weight and do not address more general classes of cost functions.

In this article, we propose a general heterogeneous routing game in which the drivers or the vehicles might belong to more than one type. The type of each vehicle determines the mapping that calculates its cost for using an edge based on the flow of all types of drivers or vehicles over that edge. We prove the existence of a Nash equilibrium under mild conditions for general heterogeneous routing games. To do so, we prove that the introduced routing game is equivalent to an abstract game with finite number of players in which each player corresponds to one of the types. For the case in which only two types of users are participating in the game, we characterize necessary and sufficient conditions for finding a potential function for this abstract game under which the problem of finding a Nash equilibrium for the game is equivalent to solving an optimization problem. We also present a set of tolls that can satisfy these conditions.

The rest of the article is organized as follows. We formulate the heterogeneous routing game in Section II. In Section III, we prove that a Nash equilibrium may indeed exist in this routing game. We present a set of necessary and sufficient conditions to guarantee the existence of a potential function for this game in Section IV. In Section V, a set of tolls is presented to satisfy the aforementioned conditions. Finally, we conclude the article and present directions for future research in Section VI.

II. A HETEROGENEOUS ROUTING GAME

A. Notation

Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real and integer numbers, respectively. Furthermore, define $\mathbb{Z}_{\geq a} = \{n \in \mathbb{Z} \,|\, n \geq a\}$ and $\mathbb{R}_{\geq a} = \{x \in \mathbb{R} \,|\, x \geq a\}$. We use the notation $[\![N]\!]$ to denote $\{1, \dots, N\}$. All the other sets are denoted by calligraphic letters such as $\mathcal{R}$. Specifically, $\mathcal{C}^k$ consists of all k-times continuously differentiable functions.

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a set such that $0 \in \mathcal{X}$. A mapping $f : \mathcal{X} \to \mathbb{R}$ is called positive definite if $f(x) \geq 0$ for all $x \in \mathcal{X}$ and $f(x) = 0$ implies that $x = 0$.

We use the notation $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to denote a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each entry $(i, j) \in \mathcal{E}$ denotes an edge from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$. A directed path of length z from vertex i to vertex j is a

set of edges $\{(i_0, i_1), (i_1, i_2), \ldots, (i_{z-1}, i_z)\} \subseteq \mathcal{E}$ such that $i_0 = i$ and $i_z = j$.

B. Problem Formulation

Here, we propose an extension of the routing game introduced in [1] to admit more than one type of players[1]. To be specific, we assume types θ belong to a finite set Θ.

Let us assume that a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is given which models the transportation network. We also have a set of source–destination pairs $\{(s_k, t_k)\}_{k=0}^K$ for some constant $K \in \mathbb{Z}_{\geq 1}$. Each pair (s_k, t_k) is called a commodity. We use the notation $\mathcal{P}_k$ to denote the set of all admissible paths over the graph $\mathcal{G}$ that connect vertex $s_k \in \mathcal{V}$ (i.e., the source of this commodity) to vertex $t_k \in \mathcal{V}$ (i.e., the destination of this commodity). Let $\mathcal{P} = \cup_{k=1}^K \mathcal{P}_k$. We assume that each commodity $k \in [\![K]\!]$ needs to transfer a flow equal to $(F_k^\theta)_{\theta \in \Theta} \in \mathbb{R}_{\geq 0}^{|\Theta|}$.

We use the notation $f_p^\theta \in \mathbb{R}_{\geq 0}$ to denote the flow of players of type $\theta \in \Theta$ that use a given path $p \in \mathcal{P}$. We use the notation $f = (f_p^\theta)_{p \in \mathcal{P}, \theta \in \Theta} \in \mathbb{R}^{|\mathcal{P}| \cdot |\Theta|}$ to denote the aggregate vector of flows[2]. A flow vector $f \in \mathbb{R}^{|\mathcal{P}| \cdot |\Theta|}$ is feasible if $\sum_{p \in \mathcal{P}_k} f_p^\theta = F_k^\theta$ for all $k \in [\![K]\!]$ and $\theta \in \Theta$. We use the notation $\mathcal{F}$ to denote the set of all feasible flows. To ensure that the set of feasible flows is not an empty set, we assume that $\mathcal{P}_k \neq \emptyset$ if $F_k^\theta \neq 0$ for any $\theta \in \Theta$. Notice that the constraints associated with each type are independent of the rest. Therefore, the flows of a specific type can be changed without breaking the feasibility of the flows associated with the rest of the types.

A driver or vehicle of type $\theta \in \Theta$ that travels along an edge $e \in \mathcal{E}$ experiences a cost equal to $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$, where for any $\theta \in \Theta$, ϕ_e^θ denotes the total flow of drivers of type θ that are using this specific edge, i.e., $\phi_e^\theta = \sum_{p \in \mathcal{P}: e \in p} f_p^\theta$. This cost can for example encompass aggregates of the latency, fuel consumption, etc. For notational convenience, we assume that we can change the order with which the edge flows $\phi_e^{\theta'}$ can appear as arguments of the cost function $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$. A driver of type $\theta \in \Theta$ from commodity $k \in [\![K]\!]$ that uses path $p \in \mathcal{P}_k$ (for connecting s_k to t_k) experiences a total cost of $\ell_p^\theta(f) = \sum_{e \in p} \tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$.

Here, each player is an infinitesimal part of the flow that tries to minimize its own cost (i.e., each player is inclined to choose the path that has the least cost). Now, based on this fact, we can define the Nash equilibrium.

DEFINITION 2.1: (NASH EQUILIBRIUM IN HETEROGENEOUS ROUTING GAME) A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium if for all $k \in [\![K]\!]$ and $\theta \in \Theta$, $f_p^\theta > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p^\theta(f) \leq \ell_{p'}^\theta(f)$ for all $p' \in \mathcal{P}_k$.

This definition implies that for a commodity $k \in [\![K]\!]$ and type $\theta \in \Theta$, all paths with a nonzero flow for drivers of

type θ have equal costs and the rest (i.e., paths with a zero flow for drivers of type θ) have larger than or equal costs.

EXAMPLE 1: (ROUTING GAME WITH PLATOONING INCENTIVES) For this example, we fix the set of types as $\Theta = \{c, t\}$, where t denotes trucks (or, equivalently, heavy-duty vehicles) and c denotes cars (or, equivalently, light vehicles). Let the edge cost functions be characterized as

$$\tilde{\ell}_e^c(\phi_e^c, \phi_e^t) = \xi_e(\phi_e^c + \phi_e^t),$$
$$\tilde{\ell}_e^t(\phi_e^c, \phi_e^t) = \xi_e(\phi_e^c + \phi_e^t) + \zeta_e(\phi_e^c + \phi_e^t)\gamma_e(\phi_e^t),$$

where mappings $\xi_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\zeta_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $\gamma_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote the latency for using edge e as a function of the total flow of vehicles over that edge, the fuel consumption of trucks as a function of the total flow, and the inverse of the fuel efficiency of the trucks as a function of the flow of trucks, respectively. These costs actually imply that cars only observe the latency $\xi_e(\phi_e^c + \phi_e^t)$ when using the roads (which is only a function of the total flow over that edge and not the individual flows of each type). However, the cost associated with trucks encompasses an additional term which models their fuel consumption. Following this interpretation, $\gamma_e(\phi_e^t)$ is a decreasing function since by having a higher flow of trucks over a given road (i.e., larger ϕ_e^t) each truck gets a higher probability for platooning (and as a result, a higher chance of decreasing its fuel consumption). ◁

We make the following standing assumption regarding the edge latency functions for all the types.

ASSUMPTION 2.1: For all $\theta \in \Theta$ and $e \in \mathcal{E}$, the edge cost function $\tilde{\ell}_e^\theta$ satisfies the following properties:

(i) $\tilde{\ell}_e^\theta \in \mathcal{C}^1$;

(ii) $\tilde{\ell}_e^\theta$ is positive definite;

(iii) $\int_0^{\phi_e^\theta} \tilde{\ell}_e^\theta(u, (\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}}) \mathrm{d}u$ is a convex function in ϕ_e^θ;

Assumption 2.1 (iii) can be replaced with the assumption that $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$ is an increasing function in ϕ_e^θ (see [2] for the case in which $\Theta = 1$). We start by proving the existence of a Nash equilibrium and, then, study the computational complexity of finding an equilibrium.

III. EXISTENCE OF THE NASH EQUILIBRIUM

To prove the existence of a Nash equilibrium, we first show that the problem of characterizing a Nash equilibrium of the heterogeneous routing game is equivalent to characterizing a pure strategy Nash equilibrium of an abstract game with finite number of players. For the sake of simplicity of presentation and without loss of generality (since Θ is finite), we can assume that $\Theta = \{\theta_1, \ldots, \theta_N\}$ where $N = |\Theta|$. Now, let us define the abstract game.

DEFINITION 3.1: Consider a game with N players in which player $i \in [\![N]\!]$ corresponds to type $\theta_i \in \Theta$ in the heterogeneous routing game. The action of player i is denoted by $a_i = (f_{p'}^{\theta_i})_{p' \in \mathcal{P}}$ which belongs to the action set

$$\mathcal{A}_i = \left\{ (f_{p'}^{\theta_i})_{p' \in \mathcal{P}} \in \mathbb{R}^{|\mathcal{P}|} \,\Big|\, \sum_{p' \in \mathcal{P}_k} f_{p'}^{\theta_i} = F_k^{\theta_i} \right\}.$$

Additionally, the utility of player i is defined as

$$U_i(a_i, a_{-i}) = \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u, \quad (1)$$

where a_{-i} represents the actions of the rest of players $(a_j)_{j \in [\![N]\!] \setminus \{i\}}$ and $\phi_e^{\theta_i} = \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_i}$ for each $i \in [\![N]\!]$. Clearly, an action profile $a \in \times_{j=1}^N \mathcal{A}_j$ is a pure strategy Nash equilibrium of this abstract game if for all $i \in [\![N]\!]$,

$$U_i(a_i, a_{-i}) \le U_i(\bar{a}_i, a_{-i}), \ \forall \bar{a}_i \in \mathcal{A}_i.$$

The following result establishes an interesting relationship between the introduced high level game and the underlying heterogeneous routing game.

LEMMA 3.2: A flow vector $(f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium of the heterogeneous routing game if and only if $((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, \dots, (f_{p'}^{\theta_N})_{p' \in \mathcal{P}})$ is a pure strategy Nash equilibrium of the abstract game introduced in Definition 3.1.

Proof: Notice that $((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, \dots, (f_{p'}^{\theta_N})_{p' \in \mathcal{P}})$ being a pure strategy Nash equilibrium of the abstract game is equivalent to the fact that for all $i \in [\![N]\!]$, $a_i = (f_{p'}^{\theta_i})_{p' \in \mathcal{P}}$ is the best response of player i to the tuple of actions $a_{-i} = ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}$ or, equivalently,

$$a_i \in \underset{(f_{p'}^{\theta_i})_{p' \in \mathcal{P}}}{\arg\min} \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u,$$

$$\text{s.t.} \ \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_i} = \phi_e^{\theta_i}, \ \forall e \in \mathcal{E},$$

$$\sum_{p \in \mathcal{P}_k} f_p^{\theta_i} = F_k^{\theta_i}, \ \forall k \in [\![K]\!],$$

$$f_p^{\theta_i} \ge 0, \ \forall p \in \mathcal{P}.$$

where $\phi_e^{\theta_j} = \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_j}$ for all $j \in [\![N]\!] \setminus \{i\}$. Notice that due to Assumption 2.1 *(iii)*, this problem is indeed a convex optimization problem. Let us define the Lagrangian as

$$L_i((\phi_{e'}^{\theta_i})_{e' \in \mathcal{E}}, (f_{p'}^{\theta_i})_{p' \in \mathcal{P}}) = \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u$$

$$+ \sum_{e \in E} v_e^i \left(\sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_i} - \phi_e^{\theta_i} \right)$$

$$- \sum_{k=1}^K w_k^i \left(\sum_{p \in \mathcal{P}_k} f_p^{\theta_i} - F_k^{\theta_i} \right)$$

$$- \sum_{p \in \mathcal{P}} \lambda_p^i f_p^{\theta_i},$$

where $(v_e^i)_{e \in \mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$, $(w_k^i)_{k \in [\![K]\!]} \in \mathbb{R}^K$, and $(\lambda_p^i)_{p \in \mathcal{P}} \in \mathbb{R}_{\ge 0}^{|\mathcal{P}|}$ are Lagrange multipliers. Now, using Karush–Kuhn–Tucker theorem [13, p. 244], optimality conditions are

$$\frac{\partial}{\partial \phi_e^{\theta_i}} L_i((\phi_{e'}^{\theta_i})_{e' \in \mathcal{E}}, (f_{p'}^{\theta_i})_{p' \in \mathcal{P}})$$

$$= \tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_i}, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) - v_e^i = 0, \ \forall e \in \mathcal{E}, \quad (2)$$

and

$$\frac{\partial}{\partial f_p^{\theta_i}} L_i((\phi_{e'}^{\theta_i})_{e' \in \mathcal{E}}, (f_{p'}^{\theta_i})_{p' \in \mathcal{P}})$$

$$= \left(\sum_{e \in p} v_e^i \right) - w_k^i - \lambda_p^i = 0, \ \forall p \in \mathcal{P}_k, \forall k \in [\![K]\!]. \quad (3)$$

Additionally, the complimentary slackness conditions (for inequality constraints) results in $\lambda_p^i f_p^i = 0$ for all $p \in \mathcal{P}$. Hence, for all k and $p \in \mathcal{P}_k$, we have

$$\ell_p^{\theta_i}((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}) = \sum_{e \in p} \tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_i}, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}})$$

$$= \sum_{e \in p} v_e^i \qquad \text{by (2)}$$

$$= w_k^i + \lambda_p^i. \qquad \text{by (3)}$$

Therefore, for any $p_1, p_2 \in \mathcal{P}_k$, if $f_{p_1}^{\theta_i}, f_{p_2}^{\theta_i} > 0$, we have $\lambda_{p_1}^{\theta_i} = \lambda_{p_2}^{\theta_i} = 0$ (because of the complimentary slackness conditions), which results in

$$\ell_{p_1}^{\theta_i}((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}) = w_k^i$$

$$= \ell_{p_2}^{\theta_i}((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}).$$

Furthermore, for any $p_3 \in \mathcal{P}_k$ such that $f_{p_3}^{\theta_i} = 0$, we get $\lambda_{p_3}^{\theta_i} \ge 0$ (because of dual feasibility, i.e., the Lagrange multipliers associated with inequality constraints must be positive), which results in

$$\ell_{p_3}^{\theta_i}((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}) = w_k^i + \lambda_{p_3}^{\theta_i}$$

$$\ge w_k^i$$

$$= \ell_{p_1}^{\theta_i}((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}).$$

This completes the proof. ∎

THEOREM 3.3: The heterogeneous routing game admits at least one Nash equilibrium.

Proof: Following the result of Lemma 3.2, proving the statement of this theorem is equivalent of showing the fact that the abstract game introduced in Definition 3.1 admits at least one pure strategy Nash equilibrium. First, notice that $\mathcal{A}_i$, $i \in [\![N]\!]$, is a non-empty, convex, and compact subset of the Euclidean space $\mathbb{R}^{|\mathcal{P}|}$. Second, $U_i(a_i, a_{-i})$ is continuous in all its arguments (because it is defined as an integral of a real-valued measurable function). Finally, because of Assumption 2.1 *(iii)*, $U_i(a_i, a_{-i})$ is a convex function in a_i. Now, we can use the celebrated result of [14] for abstract economies (a generalization of a game) to show that the abstract game introduced in Definition 3.1 admits at least one pure strategy Nash equilibrium. ∎

IV. FINDING A NASH EQUILIBRIUM

A family of games that are relatively easy to analyze are potential games. In this section, we give conditions for when the introduced abstract game is a potential game.

DEFINITION 4.1: (POTENTIAL GAME [15]) The abstract game introduced in Definition 3.1 is a potential game and

admits a potential function $V : \times_{i=1}^{N} \mathcal{A}_i \to \mathbb{R}$ if for all $i \in [\![N]\!]$,

$$V(a_i, a_{-i}) - V(\bar{a}_i, a_{-i}) = U_i(a_i, a_{-i}) - U_i(\bar{a}_i, a_{-i}),$$
$$\forall a_i, \bar{a}_i \in \mathcal{A}_i \text{ and } a_{-i} \in \times_{j \in [\![N]\!] \setminus \{i\}} \mathcal{A}_j.$$

The next lemma provides a necessary condition for the existence of a potential function in $\mathcal{C}^2$.

LEMMA 4.2: If the abstract game introduced in Definition 3.1 admits a potential function $V \in \mathcal{C}^2$, then

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta}) - \frac{\partial}{\partial \phi_e^{\theta_j}} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}) \right] = 0,$$

for all $i, j \in [\![N]\!]$ and $p_1, p_2 \in \mathcal{P}$.

Proof: Since $V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, \ldots, (f_{p'}^{\theta_N})_{p' \in \mathcal{P}})$ is a potential function for the abstract game, it satisfies

$$V((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}})$$
$$- V((\bar{f}_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}})$$
$$= U_i((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}})$$
$$- U_i((\bar{f}_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}),$$

which results in the identity in (4), presented on top of the next page, in which δ_{ij} denotes the Kronecker index (or delta) defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Hence, we get

$$\frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i}} = \frac{\partial}{\partial f_{p_1}^{\theta_i}} \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) du$$
$$= \sum_{e \in p_1} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}).$$

Now, because of Clairaut-Schwarz theorem [16, p. 1067], we know that the following equality must hold since $V \in \mathcal{C}^2$,

$$\frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i} \partial f_{p_2}^{\theta_j}} = \frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_2}^{\theta_j} \partial f_{p_1}^{\theta_i}}. \quad (5)$$

Let us calculate

$$\frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i} \partial f_{p_2}^{\theta_j}} = \frac{\partial}{\partial f_{p_1}^{\theta_i}} \left[\frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_2}^{\theta_j}} \right]$$
$$= \frac{\partial}{\partial f_{p_1}^{\theta_i}} \left[\sum_{e \in p_2} \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta}) \right] \quad (6)$$
$$= \sum_{e \in p_1 \cap p_2} \frac{\partial \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta})}{\partial \phi_e^{\theta_i}},$$

and, similarly,

$$\frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_2}^{\theta_j} \partial f_{p_1}^{\theta_i}} = \sum_{e \in p_1 \cap p_2} \frac{\partial \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta})}{\partial \phi_e^{\theta_j}}. \quad (7)$$

Substituting (5) and (6) into (7) results in

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_j}} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}) - \frac{\partial}{\partial \phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta}) \right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$ and $\theta_i, \theta_j \in \Theta$. ∎

Interestingly, we can prove that this condition is also a sufficient condition for the existence of a potential function (that belongs to $\mathcal{C}^2$) for the introduced abstract game whenever only two types of players are participating in the heterogeneous routing game.

LEMMA 4.3: Assume that $|\Theta| = 2$. If

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) - \frac{\partial}{\partial \phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) \right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$, then

$$V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) = \sum_{e \in \mathcal{E}} \left[\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) du_1 \right.$$
$$+ \int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) du_2$$
$$\left. - \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u} \tilde{\ell}_e^{\theta_1}(t, u) dt du \right]$$

is a potential function for the abstract game introduced in Definition 3.1.

Proof: Notice that for all $p \in \mathcal{P}$, we get (8) which is presented on top of the next page. Now, let us define

$$\Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}})$$
$$= \sum_{e \in p} \int_0^{\phi_e^{\theta_2}} \left[\frac{\partial}{\partial \phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u) - \frac{\partial}{\partial u} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, u) \right] du.$$

We have

$$\frac{\partial \Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}})}{\partial f_{\hat{p}}^{\theta_2}}$$
$$= \sum_{e \in p \cap \hat{p}} \left[\frac{\partial}{\partial \phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) - \frac{\partial}{\partial u} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) \right] = 0,$$

for all $\hat{p} \in \mathcal{P}$. Noticing that $\phi_e^{\theta_2} = \sum_{\hat{p} \in \mathcal{P}: e \in \hat{p}} f_{\hat{p}}^{\theta_2}$ for all $e \in \mathcal{E}$, we get

$$\frac{\partial \Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}})}{\partial \phi_e^{\theta_2}} = \sum_{\hat{p} \in \mathcal{P}: e \in \hat{p}} \frac{\partial \Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}})}{\partial f_{\hat{p}}^{\theta_2}}$$
$$= 0, \quad \forall e \in \mathcal{E}.$$

Thus, $\Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}}) = \Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, 0) = 0$. Setting $\Psi((\phi_e^{\theta_1})_{e \in \mathcal{E}}, (\phi_e^{\theta_2})_{e \in \mathcal{E}}) = 0$ (see definition above) inside (8) results in

$$\frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_p^{\theta_1}} = \sum_{e \in p} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$$
$$= \frac{\partial U_1((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}})}{\partial f_p^{\theta_1}}, \quad (9)$$

where the partial derivatives of U_1 can be computed from its definition in (1). Let $(f_{p'}^{\theta_1})_{p' \in \mathcal{P}}$ and $(\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}$ be arbitrary points in set of actions $\mathcal{A}_i$. Furthermore, let $r : [0,1] \to \mathcal{A}_i$ be a continuously differentiable mapping (i.e., $r \in \mathcal{C}^1$) such that $r(0) = (\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}$ and $r(1) = (f_{p'}^{\theta_1})_{p' \in \mathcal{P}}$ which remains inside $\mathcal{A}_i \subseteq \mathbb{R}^{|\mathcal{P}|}$ for all $t \in (0,1)$. We define graph(r) as the collection of all ordered pairs $(t, r(t))$ for all $t \in [0,1]$,

$$\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_{p_1}^{\theta_i}} = \lim_{\epsilon\to 0} \frac{V((f_{p_i}^{\theta_i}+\epsilon\delta_{p_1 p'})_{p'\in\mathcal{P}},((f_{p'}^{\theta_j})_{p'\in\mathcal{P}})_{\theta_j\in\Theta\setminus\{\theta_i\}}) - V((f_{p'}^{\theta_i})_{p'\in\mathcal{P}},((f_{p'}^{\theta_j})_{p'\in\mathcal{P}})_{\theta_j\in\Theta\setminus\{\theta_i\}})}{\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{U_i((f_{p'}^{\theta_i}+\epsilon\delta_{p_1 p'})_{p'\in\mathcal{P}},((f_{p'}^{\theta_j})_{p'\in\mathcal{P}})_{\theta_j\in\Theta\setminus\{\theta_i\}}) - U_i((f_{p'}^{\theta_i})_{p'\in\mathcal{P}},((f_{p'}^{\theta_j})_{p'\in\mathcal{P}})_{\theta_j\in\Theta\setminus\{\theta_i\}})}{\epsilon} \quad (4)$$

$$= \frac{\partial U_i((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_{p_1}^{\theta_i}}.$$

$$\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_p^{\theta_1}} = \frac{\partial}{\partial f_p^{\theta_1}}\left(\sum_{e\in\mathcal{E}}\left[\int_0^{\phi_e^{\theta_1}}\tilde\ell_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}}\int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial u}\tilde\ell_e^{\theta_1}(t,u)\mathrm{d}t\mathrm{d}u\right]\right)$$

$$= \sum_{e\in p}\left[\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}}\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}}\frac{\partial}{\partial u}\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},u)\mathrm{d}u\right] \quad (8)$$

$$= \sum_{e\in p}\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \sum_{e\in p}\int_0^{\phi_e^{\theta_2}}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},u) - \frac{\partial}{\partial u}\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},u)\right]\mathrm{d}u.$$

which denotes a continuous path that connects $(f_{p'}^{\theta_1})_{p'\in\mathcal{P}}$ and $(\bar f_{p'}^{\theta_1})_{p'\in\mathcal{P}}$. We know at least one such mapping exists because $\mathcal{A}_i$ is a simply connected set for all $i\in[\![N]\!]$. Hence, we have

$$\int_{\mathrm{graph}(r)}\left[\left.\frac{\partial V(a_1,a_2)}{\partial a_1}\right|_{a_1=r}\right]^\top\mathrm{d}r$$

$$= \int_0^1\left[\left.\frac{\partial V(a_1,a_2)}{\partial a_1}\right|_{a_1=r(t)}\right]^\top\frac{\partial r(t)}{\partial t}\mathrm{d}t$$

$$= \int_0^1\left[\frac{\mathrm{d}}{\mathrm{d}t}V(r(t),a_2)\right]\mathrm{d}t$$

$$= V(r(1),a_2) - V(r(0),a_2)$$

$$= V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) - V((\bar f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}),$$

where the second to last equality is a direct consequence of the fundamental theorem of calculus [16, p. 1257]. Note that this equality holds irrespective of the selected path. Therefore,

$$V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) - V((\bar f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$$

$$= \int_{\mathrm{graph}(r)}\left[\left.\frac{\partial V(a_1,a_2)}{\partial a_1}\right|_{a_1=r}\right]^\top\mathrm{d}r$$

$$= \int_{\mathrm{graph}(r)}\left[\left.\frac{\partial U_1(a_1,a_2)}{\partial a_1}\right|_{a_1=r}\right]^\top\mathrm{d}r \qquad \text{by (9)}$$

$$= U_1((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) - U_1((\bar f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}),$$

Similarly, we can also prove (10), presented on top of the next page, which results in

$$\frac{\partial V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})}{\partial f_p^{\theta_2}} = \sum_{e\in p}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2})$$

$$= \frac{\partial U_2((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})}{\partial f_p^{\theta_2}},$$

and, consequently,

$$V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) - V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(\bar f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$$

$$= U_2((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) - U_2((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(\bar f_{p'}^{\theta_2})_{p'\in\mathcal{P}}).$$

This concludes the proof. ∎

Now, combing the previous two lemmas results in the main result of this section.

THEOREM 4.4: Assume that $|\Theta|=2$. The abstract game admits a potential function $V\in\mathcal{C}^2$ if and only if

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \frac{\partial}{\partial\phi_e^{\theta_2}}\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right] = 0,$$

for all $p_1,p_2\in\mathcal{P}$.

Proof: The proof easily follows from Lemmas 4.2 and 4.3. Note that the potential function presented in Lemma 4.3 belongs to $\mathcal{C}^2$ due to Assumption 2.1 (*i*). ∎

Following a basic property of potential games, it is easy to prove the following corollary which shows that the process of finding a Nash equilibrium of the heterogeneous routing game is equivalent to solving an optimization problem.

COROLLARY 4.5: Assume that $|\Theta|=2$. Furthermore, let

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde\ell_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \frac{\partial}{\partial\phi_e^{\theta_2}}\tilde\ell_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right] = 0,$$

for all $p_1,p_2\in\mathcal{P}$. If $f=(f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ is a solution of the optimization problem

$$\min\ V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}),$$

$$\text{s.t.}\ \sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_1}=\phi_e^{\theta_1}\ \text{and}\ \sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_2}=\phi_e^{\theta_2},\ \forall e\in\mathcal{E},$$

$$\sum_{p\in\mathcal{P}_k}f_p^{\theta_1}=F_k^{\theta_1}\ \text{and}\ \sum_{p\in\mathcal{P}_k}f_p^{\theta_2}=F_k^{\theta_2},\ \forall k\in[\![K]\!],$$

$$f_p^{\theta_1},f_p^{\theta_2}\in\mathbb{R}_{\geq 0},\ \forall p\in\mathcal{P},$$

where $V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$ is defined in Lemma 4.3, then $f=(f_p^\theta)_{p\in\mathcal{P},\theta\in\Theta}$ is a Nash equilibrium of the heterogeneous routing game.

$$\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_p^{\theta_2}} = \frac{\partial}{\partial f_p^{\theta_2}}\left(\sum_{e\in\mathcal{E}}\left[\int_0^{\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}}\int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(t,u)\mathrm{d}t\mathrm{d}u\right]\right)$$

$$= \sum_{e\in p}\left[\int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(t,\phi_e^{\theta_2})\mathrm{d}t\right],$$
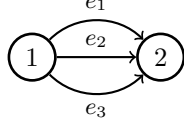
(10)

Fig. 1. An example of graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$.

TABLE I

PARAMETERS OF THE HETEROGENEOUS ROUTING GAME.

	$\alpha_{\text{aa}}^{(1)}$	$\alpha_{\text{ab}}^{(1)}$	$\alpha_{\text{bb}}^{(1)}$	$\beta_{\text{a}}^{(1)}$	$\beta_{\text{b}}^{(1)}$
e_1	3.00	1.50	2.50	2.00	4.00
e_2	4.00	2.00	3.50	2.00	4.00
e_3	3.50	1.75	1.00	4.50	1.50

Proof: The proof is consequence of the fact that a minimizer of the potential function is a pure strategy Nash equilibrium of a potential game [15]. ∎

EXAMPLE 2: Let us consider a numerical example with graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ illustrated in Figure 1. Here, we assume that $\Theta = \{\text{a},\text{b}\}$. We also have three paths $p_i = \{e_i\}$ for $i = 1,2,3$. The edge cost functions are taken to be affine functions of the form

$$\ell_{e_i}^{\text{a}}(\phi_{e_i}^{\text{a}},\phi_{e_i}^{\text{b}}) = \alpha_{\text{aa}}^{(i)}\phi_{e_i}^{\text{a}} + \alpha_{\text{ab}}^{(i)}\phi_{e_i}^{\text{b}} + \beta_{\text{a}}^{(i)},$$
$$\ell_{e_i}^{\text{b}}(\phi_{e_i}^{\text{a}},\phi_{e_i}^{\text{b}}) = \alpha_{\text{ba}}^{(i)}\phi_{e_i}^{\text{a}} + \alpha_{\text{bb}}^{(i)}\phi_{e_i}^{\text{b}} + \beta_{\text{b}}^{(i)}.$$

If $\alpha_{\text{ba}}^{(i)} = \alpha_{\text{ab}}^{(i)}$ for all $i = 1,2,3$, the condition of Corollary 4.5 is satisfied. In this case, we can calculate the potential function as

$$V((\phi_{e_i}^{\text{a}})_{i=1}^3,(\phi_{e_i}^{\text{b}})_{i=1}^3) = \sum_{i=1}^3\left[\frac{1}{2}\alpha_{\text{aa}}^{(i)}(\phi_{e_i}^{\text{a}})^2 + (\alpha_{\text{ab}}^{(i)}\phi_{e_i}^{\text{b}} + \beta_{\text{a}}^{(i)})\phi_{e_i}^{\text{a}}\right.$$
$$\left. - \alpha_{\text{ab}}^{(i)}\phi_{e_i}^{\text{a}}\phi_{e_i}^{\text{b}} + \frac{1}{2}\alpha_{\text{bb}}^{(i)}(\phi_{e_i}^{\text{b}})^2 + (\alpha_{\text{ba}}^{(i)}\phi_{e_i}^{\text{a}} + \beta_{\text{b}}^{(i)})\phi_{e_i}^{\text{b}}\right].$$

Noticing that solving a non-convex quadratic programming problem might be numerically intractable in general, we focus on the case in which $V((\phi_{e_i}^{\text{a}})_{i=1}^3,(\phi_{e_i}^{\text{b}})_{i=1}^3)$ is a convex function. Following the argument of [13, p. 71], we know that $V((\phi_{e_i}^{\text{a}})_{i=1}^3,(\phi_{e_i}^{\text{b}})_{i=1}^3)$ is a convex function if and only if

$$\begin{bmatrix}\alpha_{\text{aa}}^{(i)} & \frac{1}{2}\alpha_{\text{ab}}^{(i)} \\ \frac{1}{2}\alpha_{\text{ba}}^{(i)} & \alpha_{\text{bb}}^{(i)}\end{bmatrix} \geq 0, \ \forall i = 1,2,3.$$

Let us pick the parameters for the routing game according to Table I. Furthermore, we choose $F_1^{\text{a}} = 5$ and $F_1^{\text{b}} = 1$. After solving the optimization problem in Corollary 4.5, we get $(f_{p_i}^{\text{a}})_{i=1}^3 = (2.3836, 1.7877, 0.8288)$, $(f_{p_i}^{\text{b}})_{i=1}^3 = (0.0, 0.0, 1.0)$, $(\ell_{p_i}^{\text{a}}(f))_{i=1}^3 = (9.1507, 9.1507, 9.1507)$, and $(\ell_{p_i}^{\text{b}}(f))_{i=1}^3 = (7.5753, 7.5753, 3.9503)$ which hence is a Nash equilibrium. ◁

Notice that so far we have proved that a minimizer of the potential function is a Nash equilibrium but not the other way round. Now, we are ready to prove this whenever the

potential function is convex. However, this result is proved at the price of a more conservative condition.

COROLLARY 4.6: Let $|\Theta| = 2$ and

$$\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) = \frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}),$$

for all $e \in \mathcal{E}$. Furthermore, assume that the potential function $V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$, defined in Lemma 4.3, is a convex function. Then $f = (f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ is a Nash equilibrium of the heterogeneous routing game if and only if it is a solution of the convex optimization problem

$$\min \quad V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}),$$
$$\text{s.t.} \sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_1} = \phi_e^{\theta_1} \text{ and } \sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_2} = \phi_e^{\theta_2}, \ \forall e \in \mathcal{E},$$
$$\sum_{p\in\mathcal{P}_k}f_p^{\theta_1} = F_k^{\theta_1} \text{ and } \sum_{p\in\mathcal{P}_k}f_p^{\theta_2} = F_k^{\theta_2}, \ \forall k \in [\![K]\!],$$
$$f_p^{\theta_1} \geq 0 \text{ and } f_p^{\theta_2} \geq 0, \ \forall p \in \mathcal{P}.$$

Proof: Let us define the Lagrangian as

$$L = V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$$
$$+ \sum_{i=1}^2\sum_{e\in E}v_e^i\left(\sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_i} - \phi_e^{\theta_i}\right)$$
$$- \sum_{i=1}^2\sum_{k=1}^K w_k^i\left(\sum_{p\in\mathcal{P}_k}f_p^{\theta_i} - F_k^{\theta_i}\right) - \sum_{i=1}^2\sum_{p\in\mathcal{P}}\lambda_p^i f_p^i,$$

where $(v_e^1)_{e\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$, $(v_e^2)_{e\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$, $(w_k^1)_{k\in[\![K]\!]} \in \mathbb{R}^K$, $(w_k^2)_{k\in[\![K]\!]} \in \mathbb{R}^K$, $(\lambda_p^1)_{p\in\mathcal{P}} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$, and $(\lambda_p^2)_{p\in\mathcal{P}} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$ are Lagrange multipliers. Using Karush–Kuhn–Tucker conditions [13, p. 244], optimality conditions are

$$\frac{\partial}{\partial\phi_e^{\theta_1}}L = \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}}\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2$$
$$- \int_0^{\phi_e^{\theta_2}}\frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u)\mathrm{d}u - v_e^1$$
$$= \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^1$$
$$+ \int_0^{\phi_e^{\theta_2}}\left(\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u) - \frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u)\right)\mathrm{d}u$$
$$= \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^1 = 0, \ \forall e \in \mathcal{E},$$

(11a)

$$\frac{\partial}{\partial\phi_e^{\theta_2}}L = \int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2})$$
$$- \int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(t,\phi_e^{\theta_2})\mathrm{d}t - v_e^2$$
$$= \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^2 = 0, \ \forall e \in \mathcal{E},$$

(11b)

and

$$\frac{\partial}{\partial f_p^{\theta_1}} L = \sum_{e \in p} v_e^1 - w_k^1 - \lambda_p^1 = 0, \quad \forall p \in \mathcal{P}, \qquad (12a)$$

$$\frac{\partial}{\partial f_p^{\theta_2}} L = \sum_{e \in p} v_e^2 - w_k^2 - \lambda_p^2 = 0, \quad \forall p \in \mathcal{P}. \qquad (12b)$$

In addition, the complimentary slackness conditions for inequality constraints result in $\lambda_p^1 f_p^1 = 0$ and $\lambda_p^2 f_p^2 = 0$ for all $p \in \mathcal{P}$. Hence, for all k and $p \in \mathcal{P}_k$, we have

$$\ell_p^{\theta_i}(f) = \sum_{e \in p} \tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$$

$$= \sum_{e \in p} v_e^i \qquad \text{by (11)}$$

$$= w_k^i + \lambda_p^i. \qquad \text{by (12)}$$

Thus, if $f_p^{\theta_i}, f_{p'}^{\theta_i} > 0$, using complimentary slackness, we get $\lambda_p^{\theta_i} = 0$ and $\lambda_{p'}^{\theta_i} = 0$, which results in

$$\ell_p^{\theta_i}(f) = \ell_{p'}^{\theta_i}(f) = w_k^i.$$

Additionally, for all $p'' \in \mathcal{P}_k$, where $f_{p''}^{\theta_i} = 0$, we have $\lambda_p^{\theta_i} \geq 0$ (because of dual feasibility), which results in

$$\ell_{p''}^{\theta_i}(f) = w_k^i + \lambda_{p''}^{\theta_i} \geq w_k^i = \ell_p^{\theta_i}(f).$$

This is the definition of a Nash equilibrium. ∎

EXAMPLE 1 (CONT'D): Let us examine the implications of Corollary 4.6 in studying platooning incentives. We can easily calculate that

$$\frac{\partial \tilde{\ell}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{t}}} = \frac{\mathrm{d}\xi_e(u)}{\mathrm{d}u}\Big|_{u = \phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}}, \qquad (13)$$

$$\frac{\partial \tilde{\ell}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{c}}} = \frac{\mathrm{d}\xi_e(u)}{\mathrm{d}u}\Big|_{u = \phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}} + \frac{\mathrm{d}\zeta_e(u)}{\mathrm{d}u}\Big|_{u = \phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}} \gamma_e(\phi_e^{\mathrm{t}}). \qquad (14)$$

Assuming that $\gamma_e(\phi_e^{\mathrm{t}}) \neq 0$ (since otherwise both types are equivalent), for the condition in Corollary 4.6 to hold, we have $\mathrm{d}\zeta_e(u)/\mathrm{d}u = 0$ for all u or, equivalently, we have $\zeta_e(u) = c$ for all u. Intuitively, this condition translates to the fact that the fuel consumption of the trucks is independent of total flow of vehicles $\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}$ (which might only be valid in free-flow traffic since the total flow is not dictating the velocity of the trucks). ◁

Noting that if the problem of finding a Nash equilibrium in the heterogeneous routing game is numerically intractable, it might be highly unlikely for the drivers to figure out a Nash equilibrium in finite time (let alone an efficient one) and utilize it, which might result in wasting parts of the transportation network resources. Therefore, a natural question that comes to mind is whether it is possible to guarantee the existence of a potential function for a heterogeneous routing game by imposing appropriate tolls.

V. IMPOSING TOLLS TO GUARANTEE THE EXISTENCE OF A POTENTIAL FUNCTION

Let us assume that a driver or vehicle of type $\theta \in \Theta$ must pay a toll $\tilde{\tau}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$ for using an edge $e \in \mathcal{E}$, where (as stated earlier) $\phi_e^{\theta} = \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta}$. Therefore, a driver that

is using path $p \in \mathcal{P}_k$ endures a total cost of $\ell_p^{\theta}(f) + \tau_p^{\theta}(f)$, where $\tau_p(f)$ is the total amount of money that this driver must pay for using path p and can be calculated as $\tau_p^{\theta}(f) = \sum_{e \in p} \tilde{\tau}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$. The definition of a Nash equilibrium should then be slightly modified to account for the tolls. A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium for the routing game with tolls if, for all $k \in [\![K]\!]$ and $\theta \in \Theta$, whenever $f_p^{\theta} > 0$ for some path $p \in \mathcal{P}_k$, then $\ell_p^{\theta}(f) + \tau_p^{\theta}(f) \leq \ell_{p'}^{\theta}(f) + \tau_{p'}^{\theta}(f)$ for all $p' \in \mathcal{P}_k$.

Before stating the main result of this section, note that we can have both distinguishable and indistinguishable types. This characterization is of special interest when considering the implementation of tolls. For distinguishable types, we can impose individual tolls for each type. However, for indistinguishable types, the tolls are independent of the type. To give an example, if $\Theta = \{\mathrm{cars}, \mathrm{trucks}\}$, we can impose different tolls for each group of vehicles while if $\Theta = \{\mathrm{patient\ drivers}, \mathrm{impatient\ drivers}\}$, we cannot. We treat these two cases separately.

PROPOSITION 5.1: (DISTINGUSHABLE TYPES) Assume that $|\Theta| = 2$. The abstract game admits the potential function

$$V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}})$$

$$= \sum_{e \in \mathcal{E}} \Bigg[\int_0^{\phi_e^{\theta_1}} (\tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_1}(u_1, \phi_e^{\theta_2})) \mathrm{d}u_1$$

$$+ \int_0^{\phi_e^{\theta_2}} (\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) + \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, u_2)) \mathrm{d}u_2$$

$$- \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u} (\tilde{\ell}_e^{\theta_1}(t, u) + \tilde{\tau}_e^{\theta_1}(t, u)) \mathrm{d}t \mathrm{d}u \Bigg]$$

if

$$\frac{\partial \tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}}$$

$$= \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

for all $e \in \mathcal{E}$.

Proof: Note that introducing the tolls $\tilde{\tau}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$ has the same impact on the routing game as replacing the edge cost functions in the original heterogeneous routing game from $\tilde{\ell}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$ to $\tilde{\ell}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$. Thanks to Lemma 4.3, the abstract game based upon this new heterogeneous routing game admits the potential function V if

$$\frac{\partial(\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}))}{\partial \phi_e^{\theta_2}}$$

$$- \frac{\partial(\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}))}{\partial \phi_e^{\theta_1}} = 0.$$

With rearranging the terms in this equality, we can extract the condition in the statement of the proposition. ∎

PROPOSITION 5.2: (INDISTINGUSHABLE TYPES) Assume that $|\Theta| = 2$. The abstract game admits the potential function $V \in \mathcal{C}^2$ in Proposition 5.1 with $\tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) =$

$$\tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2}) \text{ if}$$

$$\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}}$$
$$= \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

for all $e \in \mathcal{E}$.

Proof: The proof immediately follows from using Proposition 5.1 with the constraint that the tolls may not depend on the type, i.e., $\tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})$. ∎

EXAMPLE 1 (CONT'D): Let us examine the possibility of finding a set of tolls that satisfies the conditions of Propositions 5.1 and 5.2. The first case is the distinguishable types. Substituting (13) and (14) into the condition of Proposition 5.1 results in

$$\frac{\partial \tilde{\tau}_e^{\rm c}(\phi_e^{\rm c}, \phi_e^{\rm t})}{\partial \phi_e^{\rm t}} - \frac{\partial \tilde{\tau}_e^{\rm t}(\phi_e^{\rm c}, \phi_e^{\rm t})}{\partial \phi_e^{\rm c}} = \frac{{\rm d}\zeta_e(u)}{{\rm d}u}\Big|_{u=\phi_e^{\rm c}+\phi_e^{\rm t}} \gamma_e(\phi_e^{\rm t}). \quad (15)$$

Following simple algebraic calculations, we can check that the tolls $\tilde{\tau}_e^{\rm c}(\phi_e^{\rm c}, \phi_e^{\rm t}) = 0$ and $\tilde{\tau}_e^{\rm t}(\phi_e^{\rm c}, \phi_e^{\rm t}) = \gamma_e(\phi_e^{\rm t})(\kappa - \zeta_e(\phi_e^{\rm c}, \phi_e^{\rm t}))$, for some appropriately chosen constant $\kappa \in \mathbb{R}_{\geq 0}$, satisfy (15). Another example of appropriate tolls is $\tilde{\tau}_e^{\rm t}(\phi_e^{\rm c}, \phi_e^{\rm t}) = 0$ and

$$\tilde{\tau}_e^{\rm c}(\phi_e^{\rm c}, \phi_e^{\rm t}) = \int_0^{\phi_e^{\rm t}} \frac{{\rm d}\zeta_e(u)}{{\rm d}u}\Big|_{u=\phi_e^{\rm c}+q} \gamma_e(q){\rm d}q.$$

These two sets of tolls certainly will have different implications on the behavior of cars and trucks.

The second case that we study is the indistinguishable types. For that case, we study solve the partial differential equation

$$\frac{\partial \tilde{\tau}_e(\phi_e^{\rm c}, \phi_e^{\rm t})}{\partial \phi_e^{\rm t}} - \frac{\partial \tilde{\tau}_e(\phi_e^{\rm c}, \phi_e^{\rm t})}{\partial \phi_e^{\rm c}} = \frac{{\rm d}\zeta_e(u)}{{\rm d}u}\Big|_{u=\phi_e^{\rm c}+\phi_e^{\rm t}} \gamma_e(\phi_e^{\rm t}).$$

Noting the resemblance of this partial differential equation with ones studied in [17, Ch. 4], we can devise the tolls

$$\tilde{\tau}_e(\phi_e^{\rm c}, \phi_e^{\rm t}) = \frac{{\rm d}\zeta_e(u)}{{\rm d}u}\Big|_{u=\phi_e^{\rm c}+\phi_e^{\rm t}} \int_0^{\phi_e^{\rm t}} \gamma_e(q){\rm d}q.$$

◁

In general, we can prove the following corollary concerning the type-independent tolls.

COROLLARY 5.3: (INDISTINGUSHABLE TYPES) Assume that $|\Theta| = 2$. The abstract game admits a potential function $V \in \mathcal{C}^2$ if the imposed tolls are of the following from

$$\tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = c_e + \int_0^{\phi_e^{\theta_2}} f_e(q, \phi_e^{\theta_1} + \phi_e^{\theta_2} - q){\rm d}q + \psi_e(\phi_e^{\theta_1} + \phi_e^{\theta_2}),$$

where $c \in \mathbb{R}_{\geq 0}$, $\psi_e \in \mathcal{C}^1$, and $f_e(x, y) = \partial \tilde{\ell}_e^{\theta_2}(y, x)/\partial y - \partial \tilde{\ell}_e^{\theta_1}(y, x)/\partial x$ for all $e \in \mathcal{E}$.

Proof: The proof is an application of the result of [17, Ch. 4] to Proposition 5.2. ∎

Throughout this subsection, we assumed that all the drivers portray similar sensitivity to the imposed tolls. This is indeed a source of conservatism, specially when dealing with routing games in which the heterogeneity is caused by the fact that the drivers react differently to the imposed tolls. Certainly, an avenue for future research is to develop tolls for a more general setup.

VI. CONCLUSIONS

In this article, we proposed a heterogeneous routing game in which the players may belong to more than one type. The type of each player determines the cost of using an edge as a function of the flow of all types over that edge. We proved that this heterogeneous routing game admits at least one Nash equilibrium. Additionally, we gave a necessary and sufficient condition for the existence of a potential function for the introduced routing game, which indeed implies that we can transform the problem of finding a Nash equilibrium into an optimization problem. Finally, we developed tolls to guarantee the existence of a potential function. Possible future research will focus on bounding the efficiency of a Nash equilibrium.

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