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An algorithm for computing explicit expressions for orthogonal projections onto finite-game subspaces

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Abstract— **The space of finite games can be decomposed into three orthogonal subspaces, which are the subspaces of pure potential games, nonstrategic games, and pure harmonic games as shown in a paper by Candogan et al. [2]. This decomposition provides a systematic characterization for the space of finite games. Explicit expressions for the orthogonal projections onto the subspaces are helpful in analyzing general properties of finite games in the subspaces and the relationships of finite games in different subspaces. In the work by Candogan et al., for the two-player case, explicit expressions for the orthogonal projections onto the subspaces are given. In the current paper, we give an algorithm for computing explicit expressions for the n-player case by developing our framework in the semitensor product of matrices and the group inverses of matrices. Specifically, using the algorithm, once we know the number of players, no matter whether we know their number of strategies or their payoff functions, we can obtain explicit expressions for the orthogonal projections. These projections can then be used to analyse the dynamical behaviors of games belonging to these subspaces.**

I. Introduction

Rosenthal initiated the concept of *potential* games, and proved that every potential game has a pure Nash equilibrium in 1973 [12]. Monderer and Shapley [11] systematically investigate potential games, give a method to verify whether a given game is potential, and prove that every potential game is isomorphic to a congestion game. Intuitively speaking, a potential game is a game with a function that reflects the deviations of the payoffs of all players caused by the strategy deviation of one player. Partially due to the fact that there is one common function describing the deviation of every player's payoff, potential games have been applied to many problems, e.g., traffic networks [10], [14], [13], cooperative control [9], optimization of distributed coverage of graphs [17], etc.

Although potential games possess so good properties and wide applications, there are other types of games that are not potential but still have good properties and applications. For example, the Rock-Paper-Scissors game is not potential, but has the uniformly mixed strategy profile as a mixed Nash equilibrium [2]. It is desirable to give a systematic characterization for finite games to investigate properties of other types of games and find their practical applications. When the number of players and the numbers of their strategies are fixed, Candogan et al. [2] identify the set of finite games with a finite-dimensional Euclidean space, and decompose this space into three orthogonal subspaces as

$$\underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{\text{Potential games}}\!\!\overbrace{\quad \oplus \quad \mathcal{H}}^{\text{Harmonic games}}, \tag{1}$$

where these subspaces are the pure potential subspace $\mathcal{P}$, the *nonstrategic* subspace $\mathcal{N}$, and the pure *harmonic* subspace $\mathcal{H}$. It is also demonstrated that the pure potential subspace plus the nonstrategic subspace is the potential subspace, denoted as $\mathcal{G}_P = \mathcal{P} \oplus \mathcal{N}$; and the pure harmonic subspace plus the nonstrategic subspace is the harmonic subspace, denoted as $\mathcal{G}_H = \mathcal{H} \oplus \mathcal{N}$. Nonstrategic games are such that every strategy profile is a pure Nash equilibrium. Harmonic games generically do not have pure Nash equilibria, but always have the uniformly mixed strategy profiles as mixed Nash equilibria.

Explicit expressions for the orthogonal projections onto these subspaces for the two-player case have been given in [2, Subsection 4.3]. It is important to obtain explicit expressions for the orthogonal projections, because they are helpful to analyse general properties of finite games. However, it is not easy to find explicit expressions for the case with more than two players. As shown in [2], in order to obtain them, one needs to find the explicit expressions of $\delta_0^\dagger$ and $D^\dagger$ in [2, Theorem 4.1], where $\delta_0 = \sum_{i=1}^M D_i$, $D = [D_1^*, ..., D_M^*]^*$, $(\cdot)^*$ denotes the adjoint operator of $\cdot$, $(\cdot)^\dagger$ stands for the *Moore-Penrose* inverse [1] of $\cdot$, $D_1, ..., D_M$ are corresponding linear operators. The explicit expression for $D^\dagger$ has been given in [2, Lemma 4.4], while the explicit expression for $\delta_0^\dagger$ are difficult to obtain because of its complexity structure.

Due to the importance and difficulty of obtaining the explicit expressions for the orthogonal projections onto finite-game subspaces, in this paper we aim at looking a different way to solving the problem. The main contribution of the paper is an algorithm for computing explicit expressions for the orthogonal projections for n-player games. Specifically, using the algorithm, once we know the number of players, no matter whether we know their number of strategies or their payoff functions, we can obtain explicit expressions for the orthogonal projections onto these subspaces. In a companion paper [5], bases for these subspaces are given, which can be used to compute the orthogonal projections. However, bases do not help in obtaining the explicit expressions. The inner product considered in [2] is the same as the one considered in the current paper but is not the conventional inner product. In [15], when the conventional inner product is considered,

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we show that even though the pure potential games and the nonstrategic games are the same as those considered in [2], the corresponding pure harmonic games are different. Our results are given in the framework of the *semitensor product (STP) of matrices* built by Cheng [4], in which a linear equation (called potential equation) is defined such that a finite game is potential if and only if the potential equation has a solution. It is also proved that if the potential equation has a solution, then the potential function of the corresponding game can be computed from any solution. The STP of matrices was for the first time proposed by Cheng [3] in 2001. STP is a natural generalization of the conventional matrix product, and has been applied to many problems, e.g., control problems of Boolean control networks [6], Morgen's problem [3], symmetry of dynamical systems [7], differential geometry [8], etc. In this paper, under the STP framework, we use the *group inverse* as a key tool to obtain our main results. Group inverses are a class of generalized inverses of matrices, which have wide applications in singular differential and difference equations, Markov chains, iterative methods, cryptography, etc. [1].

The remainder of this paper are arranged as follows. Section II introduces necessary background on group inverses, noncooperative finite games and their vector space structure in STP, and finite-dimensional Euclidean spaces. Section III shows orthogonal projections onto subspaces of finite games in the framework of STP. Section IV shows the main contribution of this paper: an algorithm that receives the number of players and returns explicit expressions for the orthogonal projections onto the subspaces of pure potential games, nonstrategic games, and pure harmonic games. Section V ends up with some remarks.

II. Preliminaries

In this section, we introduce necessary basic knowledge. Notations are first shown as below.

A. Notations

- $\emptyset$: the empty set
- 2^S: the power set of set S
- $|S|$: the cardinality of set S
- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}^m$: the set of m-dimensional real column vector space
- $\mathbb{R}^{m\times n}$: the set of $m \times n$ real matrices
- I_n: the $n \times n$ identity matrix
- δ_n^i: the i-th column of the identity matrix I_n
- Δ_n: the set of columns of I_n
- $[1, p]$: the set of the first p positive integers
- $\mathrm{im}(A)$ (resp. $\ker(A)$): the image (resp. kernel) space of matrix A
- A^T: the transpose of matrix A
- $\mathbf{1}_k$: $(\underbrace{1,\dots,1}_{k})^T$
- $\mathbf{1}_{m\times n}$ ($0_{m\times n}$): the $m \times n$ matrix with all entries equal to 1 (0)
- $A^\sharp$: the group inverse of square matrix A
- $A_1 \oplus A_2 \oplus \cdots \oplus A_n$: $\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$, where $A_1, \dots, A_n$ are real matrices

B. Group inverses

In this subsection we introduce necessary basic knowledge on group inverses. The following Propositions 2.1 and 2.2 over the complex field can be found in [1]. Their current version over the real field can be proved similarly by using the singular value decomposition of matrices over the real field. The proof is omitted.

For a matrix $A \in \mathbb{R}^{n\times n}$, a matrix $X \in \mathbb{R}^{n\times n}$ satisfying $AXA = A, XAX = X, AX = XA$ is called the group inverse of A, and is denoted by $X = A^\sharp$.

Proposition 2.1: A matrix $A \in \mathbb{R}^{n\times n}$ has at most one group inverse. The matrix A has a group inverse if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^2)$. If A has a group inverse $A^\sharp$, then $A^\sharp$ is a polynomial of A, and $(rA)^\sharp = \frac{1}{r}A^\sharp$ for each nonzero real number r.

Proposition 2.2: For every matrix $A \in \mathbb{R}^{m\times n}$, $AA^T(AA^T)^\sharp = A(A^TA)^\sharp A^T$.

The following Proposition 2.3 is a special case of our previous result [16, Theorem 4.1], and is the key proposition that will be used to establish the main results of the current paper.

Proposition 2.3 ([16]): A matrix $A \in \mathbb{R}^{n\times n}$ has a group inverse if and only if there is a matrix $X \in \mathbb{R}^{n\times n}$ such that $A^2X = A$. If A has a group inverse, then for every matrix $Y \in \mathbb{R}^{n\times n}$ satisfying $A^2Y = A$, $A^\sharp = AY^2$.

C. Finite games in the framework the semitensor product of matrices

A noncooperative finite game can be described as a triple (N, S, c), where

1) $N = \{1, \dots, n\}$ is the set of players,
2) $S^i = \{1, \dots, k_i\}$ denotes the set of strategies of player i, $i = 1, \dots, n$, $S = \prod_{i=1}^n S^i$ stands for the set of strategy profiles,
3) $c = \{c_1, c_2, \dots, c_n\}$, where function $c_i : S \to \mathbb{R}$ denotes the payoff of player i, $i = 1, \dots, n$.

Hereinafter S^{-i} denotes $\prod_{j=1, j\neq i}^n S^j$, and similarly for a strategy profile $s = (s_1, \dots, s_n) \in S$, s^{-i} denotes $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.

Next we introduce the vector space structure of finite games based on the STP of matrices built in [4]. In this framework, the payoffs of players can be expressed as real vectors.

Definition 2.4: [6] Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, and $\alpha = \mathrm{lcm}(n, p)$ be the least common multiple of n and p. The STP of A and B is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}),$$

where $\otimes$ denotes the Kronecker product.

The STP of matrices is a generalization of conventional matrix product, and preserves many properties of the conventional matrix product valid, e.g., associative law, distributive law, reverse-order law ($(A \ltimes B)^T = B^T \ltimes A^T$). Besides, for all $x \in \mathbb{R}^t$ and $A \in \mathbb{R}^{m\times n}$, one has $x \ltimes A = (I_t \otimes A)x$. Throughout this paper, the default matrix product is STP, so the product of two arbitrary matrices is well defined, and the symbol $\ltimes$ is usually omitted.

For a finite game $G = (N, S, c)$ with n players, for each player i, we identify his/her strategy j with $\delta_{k_i}^j$, denoted as $j \sim \delta_{k_i}^j$, $j = 1, \dots, k_i$, then S^i is identified with Δ_{k_i}, $i = 1, \dots, n$. It follows that the payoffs can be expressed as

$$c_i(x_1, \dots, x_n) = V_i^c \ltimes_{j=1}^n \tilde{x}_j, \quad i = 1, \dots, n, \tag{2}$$

where $x_j \in S^j$, $\tilde{x}_j \in \Delta_{k_j}$, and $x_j \sim \tilde{x}_j$, $j = 1, \dots, n$; then $(V_i^c)^T \in \mathbb{R}^k$ is uniquely determined by c_i, and called the structure vector of c_i, hereinafter $k := \prod_{i=1}^n k_i$. Define the structure vector of a game G by

$$(V_G^c)^T = (V_1^c, V_2^c, \dots, V_n^c)^T \in \mathbb{R}^{nk}, \tag{3}$$

where $(V_i^c)^T$ is the structure vector of the ith player's payoff, $i = 1, \dots, n$. Then it is clear that the set $\mathcal{G}_{[n;k_1,\dots,k_n]}$ of finite games such that each game of $\mathcal{G}_{[n;k_1,\dots,k_n]}$ has n players, and the ith players of every two games of $\mathcal{G}_{[n;k_1,\dots,k_n]}$ share the same strategy set of cardinality k_i, $i = 1, \dots, n$, has a natural vector space structure as

$$\mathcal{G}_{[n;k_1,\dots,k_n]} \sim \mathbb{R}^{nk}. \tag{4}$$

That is, games of $\mathcal{G}_{[n;k_1,\dots,k_n]}$ correspond to vectors of $\mathbb{R}^{nk}$.

D. Euclidean spaces and orthogonality

Consider the Euclidean space $\mathbb{R}^{nk}$ with the weighted inner product: for all $x, y \in \mathbb{R}^{nk}$, $\langle x, y\rangle_Q := x^T Q y$, where the weight

$$Q = k_1 I_k \oplus \dots \oplus k_n I_k \tag{5}$$

is a positive definite symmetric matrix, k_i is the number of strategies of player i ($i \in [1, n]$) in a noncooperative n-player game with $n \in \mathbb{Z}_+$, $k = \prod_{i=1}^n k_i$.

It is not difficult to obtain that for each matrix A in $\mathbb{R}^{nk\times p}$, where p is a positive integer, the orthogonal projection of $\mathbb{R}^{nk}$ onto $\mathrm{im}(A)$ is

$$A(A^T QA)^\sharp A^T Q, \tag{6}$$

where the projection comes from $(A(A^T QA)^\sharp A^T Q)^2 = A(A^T QA)^\sharp A^T Q$, and the orthogonality comes from that for all $x \in \mathbb{R}^{nk}$, $\langle A(A^T QA)^\sharp A^T Qx, x - A(A^T QA)^\sharp A^T Qx\rangle_Q = 0$.

Remark 2.1: In [15], the conventional inner product (i.e., when the weight is the identity matrix I_{nk}) is considered.

III. Preliminary results: orthogonal projections onto subspaces of finite games

In this section, we show basic knowledge on subspaces of finite games, and necessary preliminary results.

A. Subspace of nonstrategic games

Let us define some notations. Part of these notations for the first time appear in [4].

Define

$$k^{[p,q]} := \begin{cases} \prod_{j=p}^q k_j, & \text{if } q \geq p, \\ 1, & \text{if } q < p, \end{cases} \tag{7}$$

$$\begin{aligned} E_i &:= I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}} \in \mathbb{R}^{k\times \frac{k}{k_i}}, \\ \mathbf{e}_i &:= E_i E_i^T = I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i\times k_i} \otimes I_{k^{[i+1,n]}} \in \mathbb{R}^{k\times k}, \\ & \quad i \in [1, n]. \end{aligned} \tag{8}$$

Proposition 3.1: For each $i \in [1, n]$,

$$(\mathbf{e}_i)^2 = k_i \mathbf{e}_i, \quad (\mathbf{e}_i)^\sharp = \frac{1}{k_i^2}\mathbf{e}_i.$$

Define

$$B_N := E_1 \oplus \dots \oplus E_n \in \mathbb{R}^{(nk)\times \left(\sum_{i=1}^n \frac{k}{k_i}\right)}. \tag{9}$$

It directly follows that each E_i is of full column rank, hence so is B_N, i.e.,

$$\mathrm{rank}(B_N) = \sum_{i=1}^n \frac{k}{k_i}. \tag{10}$$

From the results in [2], nonstrategic games are exactly the games such that the payoff of each player does not depend on the strategy played by the player himself/herself. Then the formal definition is obtained as below.

Definition 3.2: The nonstrategic games are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying that

$$\begin{aligned} &\forall i \in [1, n], \forall y \in S^i, \forall s \in S^{-i}, \\ &\frac{1}{k_i}\sum_{x\in S^i} c_i(x, s) - c_i(y, s) = 0. \end{aligned} \tag{11}$$

In the framework of STP, by (6), Definition 3.2 can be represented as the following Theorem 3.3.

Theorem 3.3: Consider the finite game space $\mathcal{G}_{[n;k_1,\dots,k_n]}$. The nonstrategic subspace is

$$\begin{aligned} \mathcal{N} &= \mathrm{im}(B_N) = \mathrm{im}(E_1 \oplus \dots \oplus E_n) \\ &= \mathrm{im}\left(\frac{1}{k_1}\mathbf{e}_1 \oplus \dots \oplus \frac{1}{k_n}\mathbf{e}_n\right). \end{aligned} \tag{12}$$

Proof By [5, Definition 3.4], one has $\mathcal{N} = \mathrm{im}(B_N)$. Then by (6), Propositions 2.1, 2.2, and 3.1, we have

$$\begin{aligned} &B_N(B_N^T QB_N)^\sharp B_N^T Q \\ =&(E_1 \oplus \dots \oplus E_n)(k_1 E_1^T E_1 \oplus \dots \oplus k_n E_n^T E_n)^\sharp \\ &(k_1 E_1^T \oplus \dots \oplus k_n E_n^T) \\ =&(E_1(k_1 E_1^T E_1)^\sharp k_1 E_1^T) \oplus \dots \oplus (E_n(k_n E_n^T E_n)^\sharp k_n E_n^T) \\ =&(E_1(E_1^T E_1)^\sharp E_1^T) \oplus \dots \oplus (E_n(E_n^T E_n)^\sharp E_n^T) \\ =&(E_1 E_1^T (E_1 E_1^T)^\sharp) \oplus \dots \oplus (E_n E_n^T (E_n E_n^T)^\sharp) \\ =&\frac{1}{k_1}\mathbf{e}_1 \oplus \dots \oplus \frac{1}{k_n}\mathbf{e}_n, \end{aligned}$$

i.e., (12) holds, and $\left(\frac{1}{k_1}\mathbf{e}_1 \oplus \dots \oplus \frac{1}{k_n}\mathbf{e}_n\right)$ is the explicit expression for the orthogonal projection onto $\mathcal{N}$. □

The following result follows from Theorem 3.3.

Theorem 3.4: The projection of a game $(N,S,c) \in \mathcal{G}_{[n;k_1,\dots,k_n]}$ onto the nonstrategic subspace $\mathcal{N}$ is (N,S,c'), where $c=\{c_1,\dots,c_n\}$, $c'=\{c'_1,\dots,c'_n\}$, for all $i\in[1,n]$, all $x\in S^i$, all $y\in S^{-i}$, $c'_i(x,y)=\frac{1}{k_i}\sum_{z\in S^i}c_i(z,y)$.

Remark 3.1: Note that although we consider a different inner product in [15], the explicit expression for the orthogonal projection onto the nonstrategic subspace shown in Theorem 3.3 is the same as the explicit expression for the nonstrategic games considered in [15].

B. Subspace of potential games

In [11], potential games are defined as the games (N,S,c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying

$$\begin{aligned}&\exists\phi:S\to\mathbb{R},\forall i\in[1,n],\forall x,y\in S^i,\forall z\in S^{-i},\\&c_i(x,z)-c_i(y,z)=\phi(x,z)-\phi(y,z),\end{aligned}\tag{13}$$

where ϕ is called potential function. The result of [11] shows that the difference of two potential functions of a potential game is a constant function.

From this definition, nonstrategic games are exactly the potential games that have constant potential functions.

Necessary notations are given as follows. Regard $2^{[1,n]}$ as an index set, for all $N_s\subset[1,n]$,

$$\mathbf{e}_{N_s}:=\begin{cases}\prod_{i\in N_s}\mathbf{e}_i, & \text{if } N_s\neq\emptyset,\\ I_k, & \text{otherwise,}\end{cases}\tag{14}$$

where $\mathbf{e}_i$'s are as in (8).

Then

$$\mathbf{e}_{N_s}=A_1\otimes A_2\otimes\cdots\otimes A_n,\tag{15}$$

where

$$A_i=\begin{cases}I_{k_i}, & \text{if } i\notin N_s,\\ \mathbf{1}_{k_i\times k_i}, & \text{if } i\in N_s.\end{cases}\tag{16}$$

Define

$$B_P:=\begin{bmatrix}I_k & E_1 & 0 & 0 & \cdots & 0\\ I_k & 0 & E_2 & 0 & \cdots & 0\\ I_k & 0 & 0 & E_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ I_k & 0 & 0 & 0 & \cdots & E_n\end{bmatrix}\in\mathbb{R}^{(nk)\times\left(k+\sum_{i=1}^n\frac{k}{k_i}\right)}.\tag{17}$$

The following theorem follows from [5, Theorem 2.2].

Theorem 3.5: Consider the finite game space $\mathcal{G}_{[n;k_1,\dots,k_n]}$. The potential subspace is $\mathcal{G}_P=\operatorname{im}(B_P)$.

By Theorem 3.5 and (6), $B_P(B_P^TQB_P)^\sharp B_P^TQ$ is the orthogonal projection onto $\mathcal{G}_P$, where Q is as shown in (5). Later on we will design an algorithm for returning the explicit expression of $(B_P^TQB_P)^\sharp$ in terms of k_i's and $\mathbf{e}_{N_s}$'s as shown in (14).

C. Subspace of pure potential games

Define

$$P_N:=\begin{bmatrix}I_k-\frac{1}{k_1}\mathbf{e}_1\\ \vdots\\ I_k-\frac{1}{k_n}\mathbf{e}_n\end{bmatrix}\in\mathbb{R}^{(nk)\times k},\tag{18}$$

where $\mathbf{e}_1,\dots,\mathbf{e}_n$ are defined in (8).

It follows that

$$P_N^TQB_N=0,\tag{19}$$

$$[P_N,B_N]=B_P\begin{bmatrix}I_k & & & \\ -\frac{1}{k_1}E_1^T & I_{\frac{k}{k_1}} & & \\ \vdots & & \ddots & \\ -\frac{1}{k_n}E_n^T & & & I_{\frac{k}{k_n}}\end{bmatrix},\tag{20}$$

where B_N is defined in (9), Q is shown in (5), and $E_1,\dots,E_n$ are defined in (8).

In view of (19) and (20), the following theorem holds.

Theorem 3.6: Consider the finite game space $\mathcal{G}_{[n;k_1,\dots,k_n]}$. The pure potential subspace is $\mathcal{P}=\operatorname{im}(P_N)=\operatorname{im}(P_N(P_N^TQP_N)^\sharp P_N^TQ)$, where $P_N(P_N^TQP_N)^\sharp P_N^TQ$ is the orthogonal projection onto $\mathcal{P}$.

Proof Eqn. (19) implies that $\operatorname{im}(P_N)\perp\mathcal{N}$. Eqn. (20) implies that $\operatorname{im}(P_N)\oplus\mathcal{N}=\operatorname{im}(B_P)=\mathcal{G}_P$. Then $\operatorname{im}(P_N)=\mathcal{P}$. □

Theorem 3.7: The pure potential games are exactly the games (N,S,c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying (13) and

$$\forall i\in[1,n],\forall y\in S^{-i},\quad\sum_{x\in S^i}c_i(x,y)=0.\tag{21}$$

Proof Let $(V_G^c)^T=(V_1^c,\dots,V_n^c)^T\in\mathbb{R}^{nk}$ be an arbitrary pure potential game. Then $V_G^c\in\mathcal{N}^\perp$, which by Theorem 3.3 is equivalent to

$$V_i^c\mathbf{e}_i=0,\quad\forall i\in[1,n],$$

which is also equivalent to

$$\begin{aligned}&\forall i\in[1,n],\forall j_1\in[1,k^{[1,i-1]}],\forall j_2\in[1,k^{[i+1,n]}],\\&V_i^c\delta_{k^{[1,i-1]}}^{j_1}\ltimes\left(\sum_{j=1}^{k_i}\delta_{k_i}^j\right)\ltimes\delta_{k^{[i+1,n]}}^{j_2}=0,\end{aligned}$$

which is equivalent to (21).

It is evident that game V_G^c is pure potential if and only if V_G^c is potential and $V_G^c\in\mathcal{N}^\perp$, which completes the proof. □

Remark 3.2: Finite games satisfying (21) are called *normalized* in [2], and the subspace of normalized games is the orthogonal complement of the nonstrategic subspace for both the conventional inner product and the weighted inner product considered in the current paper.

Remark 3.3: Note that although the subspace of pure potential games here is the same as those considered in [15], the orthogonal projection onto the pure potential subspace considered in [15] is $P_N(P_N^TP_N)^\sharp P_N^T$, which is different from the one in the current paper (see Theorem 3.6), because a different inner product is considered in [15].

D. Subspace of harmonic games

Theorem 3.8: The harmonic games are exactly the games (N,S,c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying that

$$\forall s\in S,\quad\sum_{i=1}^n\sum_{x\in S^i}\left(c_i(x,s^{-i})-c_i(s)\right)=0.\tag{22}$$

The harmonic subspace is

$$\mathcal{G}_H = \mathcal{N} \oplus \mathcal{H} = \ker\left(\begin{bmatrix} k_1 I_k - \mathbf{e}_1 & \cdots & k_n I_k - \mathbf{e}_n \end{bmatrix}\right). \tag{23}$$

Proof Note that Eqn. (22) has already appeared in [2]. Here we give an alternative proof in the framework of STP. Let $(V_G^c)^T = (V_1^c, \dots, V_n^c)^T \in \mathbb{R}^{nk}$ be an arbitrary harmonic game.

It is evident that

$$\operatorname{im}(P_N) \oplus \ker(P_N^T Q) = \mathbb{R}^{nk},$$

where P_N is shown in (18), and $\mathcal{P} = \operatorname{im}(P_N)$. Hence $\ker(P_N^T Q)$ is the harmonic subspace, i.e., (23) holds.

Let $(V_G^c)^T = (V_1^c, \dots, V_n^c)^T \in \mathbb{R}^{nk}$ be an arbitrary harmonic game. Then $(V_G^c)^T \in \ker(P_N^T Q)$, which is equivalent to

$$\sum_{i=1}^{n} (k_i V_i^c - V_i^c \mathbf{e}_i) = 0, \tag{24}$$

which is also equivalent to

$$\sum_{i=1}^{n} \left(k_i c_i(s) - \sum_{x \in S^i} c_i(x, s^{-i}) \right) = 0, \quad \forall s \in S, \tag{25}$$

and

$$\sum_{i=1}^{n} \sum_{x \in S^i} \left(c_i(s) - c_i(x, s^{-i}) \right) = 0, \quad \forall s \in S, \tag{26}$$

which is the same as (22). □

Remark 3.4: The harmonic games considered in [15] are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying that

$$\forall s \in S, \quad \sum_{i=1}^{n} \left(\frac{1}{k_i} \sum_{x \in S^i} c_i(x, s^{-i}) - c_i(s) \right) = 0, \tag{27}$$

and the harmonic subspace is

$$\mathcal{G}_H = \mathcal{N} \oplus \mathcal{H} = \ker\left(\begin{bmatrix} I_k - \frac{1}{k_1}\mathbf{e}_1 & \cdots & I_k - \frac{1}{k_n}\mathbf{e}_n \end{bmatrix}\right), \tag{28}$$

which are different from those in the current paper.

E. Subspace of pure harmonic games

A finite game in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ is pure harmonic if and only if the game is orthogonal to both $\mathcal{P}$ and $\mathcal{N}$.

Theorem 3.9: Consider the finite game space $\mathcal{G}_{[n;k_1,\dots,k_n]}$. The pure harmonic subspace is $\mathcal{H} = \operatorname{im}(B_P)^\perp$.

Theorem 3.10: The pure harmonic games are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying

$$\forall s \in S, \quad \sum_{i=1}^{n} k_i c_i(s) = 0, \tag{29}$$

and (21).

Proof This result has already appeared in [2], here we give an alternative proof in the framework of STP. By Theorems 3.7 and 3.8, pure harmonic games are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying (21) and (22). Plugging (21) into (22), we have pure harmonic games are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying (21) and (29). □

Remark 3.5: The pure harmonic games here are not necessarily zero-sum games, but the harmonic games considered in [15] are zero-sum games. The pure harmonic games in [15] are exactly the games (N, S, c) in $\mathcal{G}_{[n;k_1,\dots,k_n]}$ satisfying

$$\forall s \in S, \quad \sum_{i=1}^{n} c_i(s) = 0, \tag{30}$$

and (21).

IV. The Main Result: An Algorithm for Computing Explicit Expressions for the Orthogonal Projections onto Finite-Game Subspaces

In this section, we show the main results.

In Section III, the explicit expression for the orthogonal projection onto the nonstrategic subspace $\mathcal{N}$, i.e., $\left(\frac{1}{k_1}\mathbf{e}_1 \oplus \cdots \oplus \frac{1}{k_n}\mathbf{e}_n\right)$, is given in Theorem 3.3. In what follows, based on the results in Section III, we give an algorithm that receives the number of players and returns explicit expressions for the orthogonal projections onto the pure potential subspace $\mathcal{P}$ and the pure harmonic subspace $\mathcal{H}$.

Theorem 3.6 shows that $P_N(P_N^T Q P_N)^\sharp P_N^T Q$ is the orthogonal projection onto the pure potential subspace $\mathcal{P}$. Hence in order to compute the explicit expression for the orthogonal projection, we must compute the explicit expression for $(P_N^T Q P_N)^\sharp = \left(\sum_{i=1}^n (k_i I_k - \mathbf{e}_i)\right)^\sharp$. Next we design an algorithm to compute this explicit expression for $(P_N^T Q P_N)^\sharp$ in terms of k_i's and $\mathbf{e}_i$'s. The following proposition plays an important role in designing this algorithm.

Proposition 4.1: Consider the finite game space $\mathcal{G}_{[n;k_1,\dots,k_n]}$. Matrices $\mathbf{e}_{N_s}$, $N_s \subset [1, n]$ (defined in (14)), are linearly independent.

Proof Let $c_{N_s} \in \mathbb{R}$, $N_s \subset [1, n]$, and

$$\sum_{N_s \subset [1,n]} c_{N_s} \mathbf{e}_{N_s} = 0. \tag{31}$$

Next we verify that $c_{N_s} = 0$ for all $N_s \subset [1, n]$.

First we consider the $(1, k)$-entry of $\mathbf{e}_{N_s}$, $N_s \subset [1, n]$. It can be seen that $\mathbf{e}_{[1,n]}(1, k) = 1$, and for all $N_s \subsetneq [1, n]$, $\mathbf{e}_{N_s}(1, k) = 0$. Hence $c_{[1,n]} = 0$. Remove $c_{[1,n]}\mathbf{e}_{[1,n]}$ from (31).

Second we consider the $(1, \frac{k}{k_1})$-entry of $\mathbf{e}_{N_s}$, $N_s \subsetneq [1, n]$. It can be seen that $\mathbf{e}_{[2,n]}(1, \frac{k}{k_i}) = 1$, and for all $[2, n] \neq N_s \subsetneq [1, n]$, $\mathbf{e}_{N_s}(1, \frac{k}{k_i}) = 0$. Hence $c_{[2,n]} = 0$. Remove $c_{[2,n]}\mathbf{e}_{[2,n]}$ from (31). Similarly we have for all $i \in [1, n]$, $c_{[1,n]\setminus\{i\}} = 0$. Remove all $c_{N_s}\mathbf{e}_{N_s}$ from (31), where $|N_s| = n - 1$.

Similarly for all $N_s \subset [1, n]$ satisfying $|N_s| = n - 2$, $c_{N_s} = 0$. Remove $c_{N_s}\mathbf{e}_{N_s}$ from (31), where $|N_s| = n - 2$.

Repeat this procedure until $|N_s| = 0$. Finally we have for all $N_s \subset [1, n]$, $c_{N_s} = 0$.

Based on the above analysis, matrices $\mathbf{e}_{N_s}$, $N_s \subset [1, n]$, are linearly independent. □

By Proposition 2.1, $\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp$ is a polynomial of $\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)$. Then by Propositions 3.1 and 4.1, one has $\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp$ is of the form $\sum_{N_s\subset[1,n]}c_{N_s}\mathbf{e}_{N_s}$, where all coefficients c_{N_s}'s belong to $\mathbb{R}$ and are unique.

We construct a linear equation

$$\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^2\left(\sum_{N_s\subset[1,n]}d_{N_s}\mathbf{e}_{N_s}\right)=\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i), \tag{32}$$

where $d_{N_s}\in\mathbb{R}$, $N_s\subset[1,n]$, are variables to be determined. Since $\sum_{N_s\subset[1,n]}c_{N_s}\mathbf{e}_{N_s}=\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp$, $(c_\emptyset,\dots,c_{[1,n]})$ is a solution to Eqn. (32).

On the other hand, the left hand side of Eqn. (32) is of the form $\sum_{N_s\subset[1,n]}e_{N_s}(d_S,S\subset N_s)\mathbf{e}_{N_s}$, where each $e_{N_s}(d_S,S\subset N_s)\in\mathbb{R}$ is a linear combination of $d_S,S\subset N_s$, then from (32) we obtain a linear equation

$$\begin{cases} e_\emptyset(d_\emptyset)=\left(\sum_{i=1}^{n}k_i\right)^2d_\emptyset=\left(\sum_{i=1}^{n}k_i\right),\\ e_{\{i\}}(d_\emptyset,d_{\{i\}})=\left(\sum_{j=1}^{n}k_j-k_i\right)^2d_{\{i\}}+\\ \quad\left(k_i-2\sum_{i=1}^{n}k_i\right)d_\emptyset=-1,\quad i\in[1,n],\\ e_{N_s}(d_S,S\subset N_s)=\left(\sum_{j\in[1,n]\setminus N_s}k_j\right)^2d_{N_s}+\cdots=0,\\ \quad N_s\subset[1,n],|N_s|>1.\end{cases} \tag{33}$$

By Proposition 4.1, the solutions to Eqn. (32) coincide with the solutions to Eqn. (33). It is directly seen that for every two solutions to (33), the d_{N_s}th components of them are equal, since $\sum_{j\in[1,n]\setminus N_s}k_j\neq0$, where $N_s\subsetneq[1,n]$.

Algorithm 4.2:
1) Find a solution $\{d^0_{N_s}\in\mathbb{R}|N_s\subset[1,n]\}$ to Eqn. (33) according to the following steps: first find the unique $d^0_\emptyset=\frac{1}{\sum_{j=1}^{n}k_j}$; second find the unique $d^0_{\{i\}}=\frac{1}{(\sum_{j=1}^{n}k_j)(\sum_{j=1}^{n}k_j-k_i)}$, $i\in[1,n]$; …; find the unique $d^0_{N_s}$ satisfying that $N_s\subset[1,n]$ and $|N_s|=n-1$; finally find an arbitrary $d^0_{[1,n]}$.
2) Use Proposition 2.3 to compute $\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp=\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)\left(\sum_{N_s\subset[1,n]}d^0_{N_s}\mathbf{e}_{N_s}\right)^2=:\left(\sum_{N_s\subset[1,n]}e^0_{N_s}\mathbf{e}_{N_s}\right)$, where $e^0_{N_s}\in\mathbb{R}$ is a polynomial of $d^0_\emptyset,\dots,d^0_{[1,n]}$, $N_s\subset[1,n]$.

Proposition 4.3: In Algorithm 4.2, $d^0_{N_s}=e^0_{N_s}$, where $N_s\subsetneq[1,n]$; $e^0_{N_s}$ are unique, where $N_s\subset[1,n]$.

Proof Since $\sum_{N_s\subset[1,n]}e^0_{N_s}\mathbf{e}_{N_s}=\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp$, $(e^0_\emptyset,\dots,e^0_{[1,n]})$ is a solution to Eqn. (32). By the uniqueness of the group inverse and Proposition 4.1, the conclusion holds. □

By Proposition 4.3, when executing 2) of Algorithm 4.2, one does not need to compute $e^0_{N_s}$, where $N_s\subsetneq[1,n]$. By using Algorithm 4.2, for any given n, one can obtain the explicit expression for $\left(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i)\right)^\sharp$.

Example 4.4: Consider 2-player games, where the players have k_1 and k_2 strategies, respectively. Denote $k:=k_1k_2$. One obtains the corresponding Eqn. (33) as

$$\begin{cases}(k_1+k_2)^2d_\emptyset=k_1+k_2,\\ d_{\{i\}}k_{3-i}^2+d_\emptyset(-k_i-2k_{3-i})=-1,\quad i=1,2,\\ \cdots.\end{cases} \tag{34}$$

By executing 1) of Algorithm 4.2, one obtains the solution to Eqn. (34) as $\left(\frac{1}{k_1+k_2},\frac{1}{k_2(k_1+k_2)},\frac{1}{k_1(k_1+k_2)},*\right)$, where $*$ can be any real number, then by executing 2) of Algorithm 4.2 one has

$$\begin{aligned}&\left(\sum_{i=1}^{2}(k_iI_k-\mathbf{e}_i)\right)^\sharp\\ =&\frac{1}{k_1+k_2}I_k+\frac{1}{k_2(k_1+k_2)}\mathbf{e}_1+\frac{1}{k_1(k_1+k_2)}\mathbf{e}_2-\\ &\frac{k_1^2+k_1k_2+k_2^2}{k_1^2k_2^2(k_1+k_2)}\mathbf{e}_1\mathbf{e}_2.\end{aligned} \tag{35}$$

By using Eqn. (35), one can obtain explicit expressions for the orthogonal projections onto the pure potential subspace $\mathcal{P}$, the nonstrategic subspace $\mathcal{N}$, and the pure harmonic subspace $\mathcal{H}$ as

$$\begin{bmatrix}I_k-\frac{1}{k_1}\mathbf{e}_1\\ I_k-\frac{1}{k_2}\mathbf{e}_2\end{bmatrix}(*)\begin{bmatrix}k_1I_k-\mathbf{e}_1\\ k_2I_k-\mathbf{e}_2\end{bmatrix}^T,$$

$$\left(\frac{1}{k_1}\mathbf{e}_1\oplus\frac{1}{k_2}\mathbf{e}_2\right),$$

and

$$I_{2k}-\begin{bmatrix}I_k-\frac{1}{k_1}\mathbf{e}_1\\ I_k-\frac{1}{k_2}\mathbf{e}_2\end{bmatrix}(*)\begin{bmatrix}k_1I_k-\mathbf{e}_1\\ k_2I_k-\mathbf{e}_2\end{bmatrix}^T-\left(\frac{1}{k_1}\mathbf{e}_1\oplus\frac{1}{k_2}\mathbf{e}_2\right),$$

where $(*)$ is the right hand side of (35), $\mathbf{e}_i$'s are defined in (8).

The above explicit expressions are obtained for the first time to the best of our knowledge, although explicit expressions in different froms have also been obtained in [2, Subsection 4.3].

Example 4.5: Consider 3-player games, where the players have k_1, k_2 and k_3 strategies, respectively. Denote $k:=$

$k_1k_2k_3$. By using Algorithm 4.2, one has

$$\begin{aligned}
&\left(\sum_{i=1}^{3}(k_iI_k-\mathbf{e}_i)\right)^{\sharp}\\
=&\frac{1}{\sum_{j=1}^{3}k_j}I_k+\frac{1}{(k_2+k_3)\sum_{j=1}^{3}k_j}\mathbf{e}_1+\\
&\frac{1}{(k_1+k_3)\sum_{j=1}^{3}k_j}\mathbf{e}_2+\frac{1}{(k_1+k_2)\sum_{j=1}^{3}k_j}\mathbf{e}_3+\\
&\frac{k_1+k_2+2k_3}{k_3(k_1+k_3)(k_2+k_3)(\sum_{j=1}^{3}k_j)}\mathbf{e}_1\mathbf{e}_2+\\
&\frac{k_1+2k_2+k_3}{k_2(k_1+k_2)(k_2+k_3)(\sum_{j=1}^{3}k_j)}\mathbf{e}_1\mathbf{e}_3+\\
&\frac{2k_1+k_2+k_3}{k_1(k_1+k_2)(k_1+k_3)(\sum_{j=1}^{3}k_j)}\mathbf{e}_2\mathbf{e}_3-\\
&\frac{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}{k_1^2k_2^2k_3^2(k_1+k_2)(k_1+k_3)(k_2+k_3)(\sum_{j=1}^{3}k_j)}(5k_1^2k_2^2k_3^2+\\
&k_1^4k_2k_3+k_1k_2^4k_3+k_1k_2k_3^4+\\
&k_1^4k_2^2+k_1^2k_2^4+k_2^4k_3^2+k_2^2k_3^4+k_1^4k_3^2+k_1^2k_3^4+\\
&2(k_1^3k_2^3+k_1^3k_3^3+k_2^3k_3^3)+4(k_1k_2^2k_3^3+k_1k_2^3k_3^2+\\
&k_1^2k_2k_3^3+k_1^2k_2^3k_3+k_1^3k_2k_3^2+k_1^3k_2^2k_3)).
\end{aligned} \tag{36}$$

By using Eqn. (36), one can obtain explicit expressions for the orthogonal projections onto the pure potential subspace $\mathcal{P}$, the nonstrategic subspace $\mathcal{N}$, and the pure harmonic subspace $\mathcal{H}$ as

$$\begin{bmatrix} I_k-\frac{1}{k_1}\mathbf{e}_1\\ I_k-\frac{1}{k_2}\mathbf{e}_2\\ I_k-\frac{1}{k_3}\mathbf{e}_3\end{bmatrix}(*)\begin{bmatrix} k_1I_k-\mathbf{e}_1\\ k_2I_k-\mathbf{e}_2\\ k_3I_k-\mathbf{e}_3\end{bmatrix}^T,$$

$$\left(\frac{1}{k_1}\mathbf{e}_1\oplus\frac{1}{k_2}\mathbf{e}_2\oplus\frac{1}{k_3}\mathbf{e}_3\right),$$

and

$$I_{3k}-\begin{bmatrix} I_k-\frac{1}{k_1}\mathbf{e}_1\\ I_k-\frac{1}{k_2}\mathbf{e}_2\\ I_k-\frac{1}{k_3}\mathbf{e}_3\end{bmatrix}(*)\begin{bmatrix} k_1I_k-\mathbf{e}_1\\ k_2I_k-\mathbf{e}_2\\ k_3I_k-\mathbf{e}_3\end{bmatrix}^T-\left(\frac{1}{k_1}\mathbf{e}_1\oplus\frac{1}{k_2}\mathbf{e}_2\oplus\frac{1}{k_3}\mathbf{e}_3\right),$$

where $(*)$ is the right hand side of (36), $\mathbf{e}_i$'s are defined in (8).

The above explicit expressions are obtained for the first time to the best of our knowledge.

By using Algorithm 4.2, for n-player games, where n is an arbitrary positive integer, one can compute the explicit expression for $(P_N^TQP_N)^{\sharp}=(\sum_{i=1}^{n}(k_iI_k-\mathbf{e}_i))^{\sharp}$. Hence one can use this algorithm to compute the explicit expression for $P_N(P_N^TQP_N)^{\sharp}P_N^TQ$, the explicit expression for the orthogonal projection onto the pure potential subspace $\mathcal{P}$ in terms of n and $\mathbf{e}_i$'s. On the other hand, we have got that the explicit projection onto the nonstrategic subspace $\mathcal{N}$ is $\left(\frac{1}{k_1}\mathbf{e}_1\oplus\cdots\oplus\frac{1}{k_n}\mathbf{e}_n\right)$. Hence the explicit expression for the explicit projection onto the pure harmonic subspace $\mathcal{H}$, $I_{nk}-P_N(P_N^TQP_N)^{\sharp}P_N^TQ-\left(\frac{1}{k_1}\mathbf{e}_1\oplus\cdots\oplus\frac{1}{k_n}\mathbf{e}_n\right)$, can also be computed in terms of n and $\mathbf{e}_i$'s.

V. Conclusion

In this paper, we gave an algorithm for computing explicit expressions for the orthogonal projections onto the subspaces of pure potential games, nonstrategic games, and pure harmonic games. Specifically, using the algorithm, once we know the number of players, no matter whether we know their number of strategies or payoff functions, we can obtain the explicit expressions for the orthogonal projections. Further works include the study of general properties of finite games belongs to these subspaces based on the explicit expressions and their applications to dynamical behaviors of these types of finite games.

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