Towards a Geometric Theory of Hybrid Systems*

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Abstract. The main purpose of this paper is to introduce a new framework for a global, geometric study of hybrid systems, and demonstrate its usefulness through its application to the analysis of the Zeno phenomenon and stability of hybrid equilibria.

1 Introduction

In this paper we present a unifying approach for treatment of hybrid systems. We define the notions of the *hybrid manifold* (or *hybrifold*) and *hybrid flow*, which enable us to study the hybrid system "in one piece", that is, as a single, generally non-smooth *dynamical system*.

Having established a reasonable framework for the geometric study of hybrid systems as dynamical systems, we focus particularly on the *Zeno phenomenon*, which does not occur in smooth dynamical systems. We study its causes, ways of removing it from the system, and classify it topologically in dimension two.

The last part of the paper deals with stability of isolated hybrid equilibria. We prove a theorem which explains, among others, examples in which a *stable hybrid* equilibrium is composed of *unstable classical* equilibria. Proofs of all statements in the paper can be found in [SJSL].

2 Preliminaries

2.1 Definitions and examples

Definition 1. *An n-dimensional hybrid system is a 6-tuple* $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$, *where:*

- $Q = \{1, \ldots, k\}$ *is the collection of* (discrete) states *of* $\mathbf{H}$, *where* $k \geq 1$ *is an integer;*
- $E \subset Q \times Q$ *is the collection of* edges*;*

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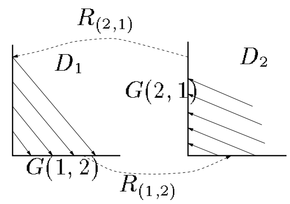

Fig. 1. The water tank example.

- $\mathcal{D} = \{D_i : i \in Q\}$ *is the collection of* domains[1] *of* $\mathbf{H}$*, where* $D_i \subset \{i\} \times \mathbb{R}^n$ *for all* $i \in Q$*;*
- $\mathcal{X} = \{X_i : i \in Q\}$ *is the collection of vector fields such that* X_i *is Lipschitz on* D_i *for all* $i \in Q$*; we denote the local flow of* X_i *by* $\{\phi_t^i\}$*.*
- $\mathcal{G} = \{G(e) : e \in E\}$ *is the collection of* guards*, where for each* $e = (i,j) \in E$*,* $G(e) \subset D_i$*;*
- $\mathcal{R} = \{R_e : e \in E\}$ *is the collection of* resets*, where for each* $e = (i,j) \in E$*,* R_e *is a relation between elements of* $G(e)$ *and elements of* D_j*, i.e.* $R_e \subset G(e) \times D_j$*.*

Remark. If a reset relation R_e is actually a map $G(e) \to D_j$, with $e = (i,j) \in E$, instead of $(x,y) \in R_e$ we write $y = R_e(x)$. Observe that domains D_i lie in distinct copies of $\mathbb{R}^n$. However, we will sometimes abuse the notation and consider the domains as subsets of a single copy of $\mathbb{R}^n$. We also set $D = \bigcup_{i\in Q} D_i$, and call this set the *total domain* of $\mathbf{H}$, and $G = \bigcup_{e\in E} G(e)$, $R = \bigcup_{e\in E} R_e(G(e))$, $\overline{\mathcal{G}} = \{\overline{G(e)} : e \in E\}$, $\overline{\mathcal{R}} = \{\overline{R_e(G(e))} : e \in E\}$.
Given $\mathbf{H}$, the basic idea is that starting from a point in some domain D_i we flow according to X_i until (and if) we reach some guard $G(i,j)$, then switch via the reset $R_{(i,j)}$, continue flowing in D_j according to X_j and so on.

Example 1 (Water Tank WT). Here $n = 2$, $k = 2$, $E = \{(1,2),(2,1)\}$, $D_1 = \{1\} \times C$, $D_2 = \{2\} \times C$, where $C = [l_1, \infty) \times [l_2, \infty)$, $X_1 = (w - v_1, -v_2)^T$, $X_2 = (-v_1, w - v_2)^T$, $G(1,2) = \{(1, x_1, x_2) \in D_1 : x_2 = l_2\}$, $G(2,1) = \{(2, x_1, x_2) \in D_2 : x_1 = l_1\}$, and $R_{(1,2)}(1, x_1, l_2) = (2, x_1, , l_2)$, $R_{(2,1)}(2, l_1, x_2) = (1, l_1, x_2)$.

The interpretation is as follows (cf. Fig. 1). For $i \in Q$, x_i denotes the volume of water in tank i, v_i is the constant rate of flow of water out of tank i, and l_i is the desired volume of water in tank i. The constant rate of water flow into the system, dedicated exclusively to one tank at a time, is denoted by w. The control task is to keep the water volume above l_1 and l_2 (assuming the initial volumes are above l_1 and l_2 respectively) by a strategy that switches the inflow to the first tank whenever $x_1 = l_1$ and to the second tank whenever $x_2 = l_2$.

Example 2 (Bouncing Ball BB).

This is a simplified model of an elastic ball that is bouncing and losing a fraction of its energy with each bounce. We denote by x_1 its altitude and by x_2 its vertical speed. Here $n = 2$, $k = 1$, $E = \{(1,1)\}$, $D_1 = \{(x_1, x_2) : x_1 \geq 0\}$, $X_1(x_1, x_2) = (x_2, -g)^T$, $G(1,1) = \{(0, x_2) : x_2 \leq 0\}$, $R_{(1,1)}(0, x_2) = (0, -cx_2)$, where g is the acceleration due to gravity and $0 < c < 1$ (cf. Fig. 2).

Example 3 (Bouncing m-Ball BB(m)).

The only difference between this and the previous example is that we have m different domains in which the ball can bounce and after each bounce the

[1] In the literature also known as "invariants".

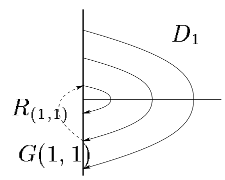

Fig. 2. Bouncing ball.

ball switches to the next domain in a cyclic order. That is, $n = 2$, $k = m > 1$, $E = \{(1,2),(2,3),\ldots,(m-1,m),(m,1)\}$, and for all $i \in Q$, $D_i = \{i\}\times\{(x_1,x_2) : x_1 \geq 0\}$, $G(i,i+1) = \{i\}\times\{(0,x_2) : x_2 \leq 0\}$, $R_{(i,i+1)}(i,0,x_2) = (i+1,0,-cx_2)$, where we conveniently identify $m+1 := 1$. Note that here the domains are just different copies of the closed right half-plane in $\mathbb{R}^2$.

Example 4 (Ball Bouncing on an N-step Staircase $BBS(N)$).
Here a ball is bouncing on an N-step staircase. Assume that step $i = 1,\ldots,N$ has width $w_i > 0$ and height $h_i > 0$, and define $\hat{w}_m = \sum_{i=1}^m w_i$ and $\hat{h}_m = \sum_{i=1}^m h_i$. Assume also that the ball loses a proportional amount of its vertical velocity (x_2) with each bounce and that the ball has constant horizontal speed (x_3). Denote by x_1 its vertical position. Then we have: $Q = \{1,\ldots,N+1\}$, $E = \{(i,i) : 1 \leq i \leq N+1\} \cup \{(1,2),\ldots,(N,N+1)\}$, and for $1 \leq i \leq N+1$: $D_i = \{i\} \times [\hat{h}_i,\infty) \times (-\infty,0] \times (-\infty,\hat{w}_i]$, $G(i,i) = \{(x_1,x_2,x_3) \in D_i : x_1 = \hat{h}_i\}$, $R_{(i,i)}(i,x_1,x_2,x_3) = (i,x_1,-cx_2,x_3)$ and $X_i(x_1,x_2,x_3) = (x_2,-g,v)^T$. Furthermore, for $1 \leq i \leq N$: $G(i,i+1) = \{(x_1,x_2,x_3) \in D_i : x_3 = \hat{w}_i\}$, $R_{(i,i+1)}(i,\mathbf{x}) = (i+1,\mathbf{x})$. For more details see [JLSM].

Example 5 (Two Saddles $S2(\lambda)$).
Here $n = 2$, $k = 2$, $\lambda > 0$, $E = \{(1,2),(2,1)\}$, the domains are two copies of the square $S = [-1,1]\times[-1,1]$, i.e. for $i \in Q$, $D_i = \{i\}\times S$, $X_1(x_1,x_2) = (\lambda x_1,-x_2)^T$, $X_2(x_1,x_2) = (-x_1,\lambda x_2)^T$, $G(1,2) =$ union of the vertical sides of D_1, $G(2,1) =$ union of the horizontal sides of D_2, $R_{(i,j)}(i,x) = (j,x)$, for all $(i,j) \in E$.

Example 6 (Flow on the 2-torus $T^2(\alpha)$).
We have $\alpha > 0$, $n = 2$, $k = 2$, $E = \{(1,2),(2,1)\}$, $D_i = \{i\}\times K$, where $K = [0,1]\times[0,1]$ is the unit square, $X_1 = X_2 = (1,\alpha)^T$ are constant vector fields, $G(i,i) = \{i\}\times S_{\text{upper}}$, $G(i,j) = \{i\}\times S_{\text{right}}$, $R_{(i,i)}(i,x,1) = (i,x,0)$ and $R_{(i,j)}(i,1,y) = (j,0,y)$, where $i,j = 1,2$, $i \neq j$, $S_{\text{upper}} = [0,1]\times\{1\}$ and $S_{\text{right}} = \{1\}\times[0,1)$ denote the (closed) upper and (half-closed) right side of K. Note that $R_{(i,i)}(\{i\}\times S_{\text{upper}}) = \{i\}\times S_{\text{lower}}$ and $R_{(i,j)}(\{i\}\times S_{\text{right}}) = \{j\}\times S_{\text{left}}$, with the obvious meaning of S_{lower} and S_{left}.
If we proceed as is usually done in geometry and identify $\{i\}\times S_{\text{upper}}$ with $\{i\}\times S_{\text{lower}}$ via $R_{(i,i)}$ and $\{i\}\times S_{\text{right}}$ with $\{j\}\times S_{\text{left}}$ via $R_{(i,j)}$ (where $i,j = 1,2$, $i \neq j$), we obtain the standard 2-torus with a *smooth* flow with slope α on it. This is a baby-version of a construction we will later apply to more general hybrid systems.

Keeping in mind the examples above, we formally define the notion of an execution of a hybrid system.

Definition 2. *A (forward) hybrid time trajectory is a sequence (finite or infinite) $\tau = \{I_j\}_{j=0}^N$ of intervals such that $I_j = [\tau_j,\tau'_j]$ for all $j \geq 0$ if the sequence*

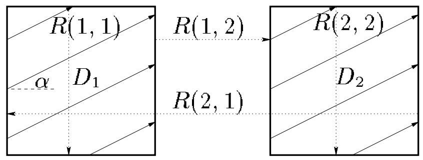

Fig. 3. $T^2(\alpha)$.

is infinite; if N *is finite, then* $I_j = [\tau_j, \tau'_j]$ *for all* $0 \leq j \leq N-1$ *and* I_N *is either of the form* $[\tau_N, \tau'_N]$ *or* $[\tau_N, \tau'_N)$*. The sequences* τ_j *and* τ'_j *satisfy:* $\tau_j \leq \tau'_j = \tau_{j+1}$*, for all* j*.*

One thinks of τ_j's as time instants when discrete transitions (or switches) from one domain to another take place. If τ is a hybrid time trajectory, we will call N its *size* and denote it by $N(\tau)$. Also, we use $\langle\tau\rangle$ to denote the set $\{0, \ldots, N(\tau)\}$ if $N(\tau)$ is finite, and $\{0, 1, 2, \ldots\}$ if $N(\tau)$ is infinite.

We will say that τ is a *prefix* of an execution $\tau' = \{I'_j\}_{j=0}^{N'}$ if $N \leq N'$ (where the inequality is taken in the extened real number system), and for $0 \leq j < N$, we have $I_j = I'_j$; furthermore, if τ has finite size, then we must also have $I_N \subset I'_N$.

Definition 3. *An execution (or forward execution) of a hybrid system* **H** *is a triple* $\chi = (\tau, q, x)$*, where* τ *is a hybrid time trajectory,* $q : \langle\tau\rangle \to Q$ *is a map, and* $x = \{x_j : j \in \langle\tau\rangle\}$ *is a collection of* C^1 *maps such that* $x_j : I_j \to D_{q(j)}$ *and for all* $t \in I_j$*,* $\dot{x}_j(t) = X_{q(j)}(x_j(t))$*. Furthermore, for all* $j \in \langle\tau\rangle$*, we have* $(q(j), q(j+1)) \in E$*,* $x_j(\tau'_j) \in G(q(j), q(j+1))$*, and* $(x_j(\tau'_j), x_{j+1}(\tau_{j+1})) \in R_{(q(j),q(j+1))}$*.*

For an execution $\chi = (\tau, q, x)$, denote by $\tau_\infty(\chi)$ its *execution time*: $\tau_\infty(\chi) = \sum_{j=0}^{N(\tau)}(\tau'_j - \tau_j) = \lim_{j \to N(\tau)} \tau'_j - \tau_0$.

Definition 4. *An execution* χ *is called:*

- *infinite, if* $N(\tau) = \infty$ *or* $\tau_\infty(\chi) = \infty$*;*
- *a Zeno execution if* $N(\tau) = \infty$ *and* $\tau_\infty(\chi) < \infty$*;*
- *maximal if it is not a strict prefix of any other execution of* **H**.

The last statement means that there exists no other execution $\chi' = (\tau', q', x')$ such that τ is a strict prefix of τ' and $x = x'$ on τ (in the sense that $x_j = x'_j$ on I_j for all $j \in \langle\tau\rangle$).

Note that in Examples 1 (WT), 2 (BB) and 3 ($BB(m)$) every execution is Zeno. The same can be shown for Examples 4 ($BBS(N)$) if $0 < c < 1$ and 5 ($S2(\lambda)$) if $0 < \lambda < 1$. On the other hand, every execution in Example 6 ($T^2(\alpha)$) is infinite with infinite execution time.

We say that an execution $\chi = (\tau, q, x)$ starts at a point $p \in D$ if $p = x_0(\tau_0)$ and $\tau_0 = 0$. It *passes through* p if $p = x_j(t)$ for some $j \in \langle\tau\rangle$, $t \in I_j$, $t > \tau_0$.

Given $p \in D$, it is not difficult to see that there are many ways in which a hybrid system can *accept* several executions starting from or passing through p. For instance, this happens if at least one of the resets is a relation which is not a function.

Definition 5. *A hybrid system is called deterministic if for every* $p \in D$ *there exists at most one maximal execution starting from* p*. It is called* non-blocking *if for every* $p \in D$ *there is at least one infinite execution starting from* p*.*

Necessary and sufficient conditions for a hybrid system to be deterministic and non-blocking can be found in [LJSE]. Roughly speaking, resets have to be functions, guards have to be mutually disjoint and whenever a continuous trajectory of one of the vector fields in $\mathcal{X}$ is about to exit the domain in which it lies, it has to hit a guard.

2.2 Standing assumptions

From now on we will assume that every hybrid system $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$ in this paper satisfies the following assumptions.

(A1) *$\mathbf{H}$ is deterministic and non-blocking.*[2]

This means that every point in D is the starting point of a unique infinite (and therefore maximal) execution of $\mathbf{H}$.

(A2) *Each domain D_i is a contractible n-dimensional smooth submanifold of $\mathbb{R}^n$, with piecewise smooth boundary. No two smooth components of the boundary meet at a zero angle.*

The non-zero angle requirement eliminates, for instance, cusps in dimension two, but does not eliminate "corners". Thus for domains of a hybrid system we allow disks, half-spaces, rectangles, etc.

(A3) *Each guard is a piecewise smooth $(n-1)$-dimensional submanifold of the boundary of the corresponding domain. The boundary of each guard is piecewise smooth (or possibly empty).*

(A4) *Each reset is a piecewise smooth homeomorphism onto its image. The image of every reset lies in the boundary of the corresponding domain.*

(A5) *Any sets in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ (i.e. closures of guards and images of resets) can intersect only along their boundaries. Furthemore, if $p \in \overline{G} \cup \overline{R}$, then p can be of only one of the following four types (cf. Fig. 4):*

Type I *: $p \in \operatorname{int} G \cup \operatorname{int} R$;*
Type II *: $p \in \partial G \cup \partial R$ and there exists exactly one set $S \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ which contains p;*
Type III *: $p \in \partial G \cup \partial R$ and there exist sets $S_1, \ldots, S_l \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ ($l \geq 2$) such that $p \in \partial S_1 \cap \ldots \cap \partial S_l$ and some neighborhood of p in $S_1 \cup \ldots \cup S_l$ is homeomorphic to $\mathbb{R}^{n-1}$;*
Type IV *: $p \in \partial G \cup \partial R$ and there exist sets $S_1, \ldots, S_l \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ ($l \geq 2$) such that $p \in \partial S_1 \cap \ldots \cap \partial S_l$ and some neighborhood of p in $S_1 \cup \ldots \cup S_l$ is homeomorphic to $\mathbb{R}^{n-1}_+$.*

Assumption (A5) ensures that intersections of guards and images of resets (that is, their closures) are sufficiantly nice. This in particular means that the configuration around p_5 in Fig. 4 is not allowed.

(A6) *For all $e = (i, j) \in E$, X_i points outside D_i along $G(e)$, and X_j is points inside D_j along* $\operatorname{im} R_e$.

This means that if $p \in G(i,j)$, $q = R_{(i,j)}(p)$, then there exists $\epsilon > 0$ such that $\phi^i_{-t}(p) \in \operatorname{int} D_i$ and $\phi^j_t(q) \in \operatorname{int} D_j$, for all $0 < t < \epsilon$, where int denotes the interior of a set. In particular, we have that X_i is transverse to the smooth part

[2] These assumptions can be relaxed. However, to simplify the exposition and avoid some nonessential technical difficulties in the subsequent construction, we keep them in the present form.

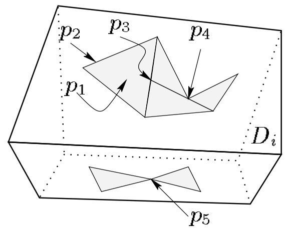

Fig. 4. p_i is of Type (Roman) i $(1 \leq i \leq 4)$.

of $G(e)$ and X_j is transverse to the smooth part of $\operatorname{im} R_e$, the image of the map R_e.

(A7) *Each reset map* R_e *extends to a map* $\tilde{R}_e$ *defined on a neighborhood of* $\overline{G(e)}$ *(the closure of* $G(e)$*) in* D_i *such that* $\tilde{R}_e$ *is a piecewise smooth homeomorphism onto its image, which, in turn, is a neighborhood of* $\operatorname{im} R_e$ *in* D_j*. Each vector field* X_i *can be smoothly extended to a neighborhood of* D_i *in* $\{i\} \times \mathbb{R}^n$*.*

The last one is a fairly technical assumption the need for which will become apparent later. Note that all the examples provided above satisfy this (as well as all other) assumptions. For instance, in Example 2 (BB), we can take $\tilde{R}_{(1,1)}(x_1, x_2) = (x_1, -cx_2)$.

Definition 6. *A hybrid system which satisfies assumptions* (A1) - (A7) *will be called* regular*.*

Given $\mathbf{H}$, define a map $\Phi^{\mathbf{H}} : \Omega_0 \to D$, (where $\Omega_0 \subset \mathbb{R} \times D$ will be specified later) as follows. Let $p \in D$ be arbitrary. Because of (A1), there exists a unique infinite execution $\chi(p) = (\tau, q, x)$ starting at p. For any $0 \leq t < \tau_\infty(\chi(p))$ there exist a unique $j \in Q$ such that $t \in [\tau_j, \tau_j')$. Then define $\Phi^{\mathbf{H}}(t,p) = x_j(t)$. To define $\Phi^{\mathbf{H}}(t,p)$ for negative t, set $\Phi^{\mathbf{H}}(t,p) = \Phi^{\mathbf{H}'}(-t,p)$, where $\mathbf{H}'$ is the *reverse* hybrid system $(Q', E', \mathcal{D}', \mathcal{X}', \mathcal{G}', \mathcal{R}')$ defined by: $Q' = Q$, $\mathcal{D}' = \mathcal{D}$, $X_i' = -X_i$, for all $i \in Q$; $(i,j) \in E'$ if and only if $(j,i) \in E$; and for every $e = (i,j) \in E'$, we have $G'(e) = R_{(j,i)}(G(j,i))$ and $R_e' = R_e^{-1}$.

It can easily be checked that $\mathbf{H}'$ satisfies (A1) - (A7) if $\mathbf{H}$ does. Now let Ω_0 be the largest subset of $\mathbb{R} \times D$ on which $\Phi^{\mathbf{H}}$ is defined.

For instance, in Example 2, for any $p \neq \mathbf{0}$, $\Phi^{BB}(t,p) \to \mathbf{0}$, as $t \to \tau_\infty(\chi(p))$, where $\chi(p)$ is the unique infinite execution starting at p. Note, however, that $\chi(\mathbf{0})$ makes no time progress, i.e. $\tau_j = 0$ for all $j \geq 0$, but it involves infinitely many switches at the same, i.e. initial point, which happens to be fixed by the reset map.

Theorem 1. **(a)** Ω_0 *contains a neighborhood of* $\{0\} \times \operatorname{int} D$ *in* $\mathbb{R} \times D$*.*
(b) *For all* $p \in D$, $\Phi^{\mathbf{H}}(0,p) = p$*. Furthermore,* $\Phi^{\mathbf{H}}(t, \Phi^{\mathbf{H}}(s,p)) = \Phi^{\mathbf{H}}(t+s,p)$*, whenever both sides are defined.*

3 The hybrid manifold and hybrid flow

The basic idea in construction of the hybrid manifold from a hybrid system is simple: "glue" the closure of each guard to the image of the corresponding extended reset via the extended reset map. Some relatively similar ideas appear in [GJ].

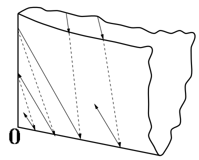

Fig. 5. Hybrifold and an orbit of the hybrid flow for WT.

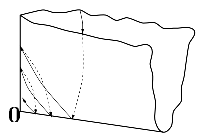

Fig. 6. Hybrifold and an orbit of the hybrid flow for BB.

3.1 The hybrifold

Let $\mathbf{H}$ be a *regular* hybrid system. On D let $\sim$ be the equivalence relation generated by $p \sim \tilde{R}_e(p)$, for all $e \in E$ and $p \in \overline{G(e)}$. Collapse each equivalence class to a point to obtain the quotient space $M_{\mathbf{H}} = D/\sim$.

Definition 7. *We call $M_{\mathbf{H}}$ the hybrid manifold or hybrifold of* $\mathbf{H}$.[3]

Denote by π the natural projection $D \to M_{\mathbf{H}}$ which assigns to each p its equivalence class $p/\sim$. Put the *quotient topology* on $M_{\mathbf{H}}$. Recall that this is the smallest topology that makes π continuous, i.e. a set $V \subset M_{\mathbf{H}}$ is open if and only if $\pi^{-1}(V)$ is open in D.

Define the *hybrid flow* of $\mathbf{H}$, $\Psi^{\mathbf{H}} : \Omega \to M_{\mathbf{H}}$, by $\Psi^{\mathbf{H}}(t, \pi(p)) = \pi\Phi^{\mathbf{H}}(t,p)$. Here $\Omega = \{(t, \pi(p)) : (t,p) \in \Omega_0\}$. In other words, orbits of $\Psi^{\mathbf{H}}$ are obtained by projecting orbits of $\Phi^{\mathbf{H}}$ by π. By the $\Phi^{\mathbf{H}}$-orbit of p we mean the collection of points $\Phi^{\mathbf{H}}(t,p)$ for all possible t (i.e. all t such that $(t,p) \in \Omega_0$).

Let us run this construction on some of the examples listed above.

Example 7 (WT continued).

Without loss we assume that $l_1 = l_2 = 0$. To obtain M_{WT} we have to identify the x_1-axis from D_1 with the same axis from D_2 via $R_{(1,2)}$ and similarly with the x_2-axis.

It is not difficult to see that M_{WT} is homeomorphic to $\mathbb{R}^2$ (see Fig. 5). However, M_{WT} has a singularity (or "corner") at $\mathbf{0} = \pi(1,0,0)$, i.e. π does not define a smooth structure on M_{WT}. Note that every execution starting at $x \neq \mathbf{0}$ converges to $\mathbf{0}$.

Example 8 (BB continued).

Here we have to identify the negative part with the positive part of the x_2-axis. The resulting space M_{BB} is again homeomorphic to $\mathbb{R}^2$ (see Fig. 6), but π again does not define a smooth structure on it. As in the previous example, $\Psi^{BB}(t,x) \to \mathbf{0}$, as $t \to \tau_\infty(\chi(x))$, for all $x \neq \mathbf{0}$.

Example 9 ($BB(m)$ continued).

For simplicity assume $m = 2$. It is not difficult to see that $M_{BB(2)}$ is smooth (in the sense explained above) and *diffeomorphic* to $\mathbb{R}^2$. However, the hybrid flow is not smooth.

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[3] The authors thank Renaud Dreyer for suggesting the term hybrifold. The term "manifold" will be justified by Theorem 2.

Example 10 ($S2(\lambda)$ continued).

$M_{S2(\lambda)}$ is homeomorphic to the 2-sphere; it is not equipped with a smooth structure by π.

Example 11 ($T^2(\alpha)$ continued).

We already observed that $M_{T^2(\alpha)}$ is the standard 2-torus and $\Psi^{T^2(\alpha)}$ is a smooth linear flow on it. If α is rational, then every orbit is closed; if α is irrational, then every orbit is dense in T^2.

Theorem 2. **(a)** *$M_{\mathbf{H}}$ defined above is a topological n-manifold with boundary.*
(b) *Both $M_{\mathbf{H}}$ and its boundary are piecewise smooth.*
(c) *The restriction $\pi|_{\text{int } D} : \text{int } D \to \pi(\text{int } D)$ is a diffeomorphism.*

Recall that M is called a *topological n-manifold with boundary* if it is Hausdorff and every point in M has a neighborhood homeomorphic to either $\mathbb{R}^n$ or the closed upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n \geq 0\}$. Points having the latter property are said to be on the boundary ∂M, which is a topological $(n-1)$-manifold.

3.2 The hybrid flow

Let $\Psi := \Psi^{\mathbf{H}}$ be the hybrid flow of $\mathbf{H}$, as defined above. For each $t \in \mathbb{R}$ and $x \in M_{\mathbf{H}}$, let $M(t) = \{y \in M_{\mathbf{H}} : \Psi(t, y) \text{ is defined}\}$, and $J(x) = \{s \in \mathbb{R} : \Psi(s, x) \text{ is defined}\}$. Observe that if $x = \pi(p)$, then $J(x) \cap [0, \infty) = [0, \tau_\infty(\chi(p)))$. Also, for $t > 0$, $M(t)$ contains all points $x = \pi(p)$ such that $\tau_\infty(\chi(p))) > t$. As usual, $\chi(p)$ denotes the unique execution of $\mathbf{H}$ starting at p.

If $M(t)$ is not empty, denote by $\Psi_t : M(t) \to M_{\mathbf{H}}$ the *time t map of Ψ*, defined by $\Psi_t(x) = \Psi(t, x)$. Recall that a function (in particular, vector field) is said to be smooth on a *closed* set F if it is the restriction of a smooth function defined on a neighborhood of F. Then we have the following theorem.

Theorem 3. *Suppose each vector field X in $\mathcal{X}$ is smooth (in addition to being globally Lipschitz). Then:*

(a) *For each $x \in M_{\mathbf{H}}$ the map $t \mapsto \Psi_t(x)$ is continuous and, if $J(x)$ is not a single point, piecewise smooth on $J(x)$. More precisely, it is smooth except at (at most) countably many points in $J(x)$. Furthermore, each map Ψ_t is injective.*
(b) *Whenever both sides are defined: $\Psi^{\mathbf{H}}_t \Psi^{\mathbf{H}}_s(x) = \Psi^{\mathbf{H}}_{t+s}(x)$.*
(c) *There is an open and dense subset of Ω on which Ψ is smooth.*

4 ω-limit sets and the Zeno phenomenon

It has to be pointed out that Zeno executions do not arise in physical systems and are a consequence of modeling over-abstraction. Therefore, one wishes to avoid them. However, from a mathematical viewpoint, the Zeno phenomenon poses numerous interesting questions. In this section we show that, in short, the topological cause of Zenoness is a lack of smoothness in the hybrid flow and that the Zeno phenomenon can be removed by smoothing out the hybrifold and the hybrid flow on it.

Definition 8. *A point $y \in M_{\mathbf{H}}$ is called an ω-limit point of $x \in M_{\mathbf{H}}$ if $y = \lim_{m\to\infty} \Psi^{\mathbf{H}}_{t_m}(x)$, for some increasing sequence (t_m) in $J(x)$ such that $t_m \to \tau_\infty(x)$, as $m \to \infty$. The set of all ω-limit points of x is called the ω-limit set of x and is denoted by $\omega(x)$.*

By $\tau_\infty(x)$ we denote the execution time of the unique execution of $\mathbf{H}$ starting from p, where $x = \pi(p)$; that is, $\tau_\infty(x) = \tau_\infty(\chi(p))$. It is easy to check that this is a well defined element of the extended real number system. In other words, ω-limit points for x are accumulation points of the orbit of x.

Suppose $x \in M_{\mathbf{H}}$ and denote by $E_\infty(x)$ the set of discrete transitions which occur infinitely many times in the execution starting from x. If $E_\infty(x)$ is empty, then the orbit of x eventually ends up in a single domain D_i (that is, its image under π in the hybrifold) in which case $\omega(x) \subset \pi(\overline{D_i})$. This means that every point $y \in \omega(x)$ is an accumulation point of the orbit of a single vector field, namely X_i. We will call such a point y, a *pure ω-limit point*.

If $E_\infty(x)$ is nonempty, then every ω-limit point for x is a result of both the continuous *and* discrete (i.e. hybrid) dynamics of $\mathbf{H}$ and will accordingly be called a *hybrid ω-limit point of x*.

Theorem 4. *For every $x \in M_{\mathbf{H}}$, $\omega(x)$ is invariant with respect to the hybrid flow. That is, if $y \in \omega(x)$, then $\Psi^{\mathbf{H}}_t(y) \in \omega(x)$, for all $t \in J(y)$.*

4.1 Properties of Zeno executions

Definition 9. *A point $z \in M_{\mathbf{H}}$ is called a Zeno state for x if $z \in \omega(x)$ and $\tau_\infty(x) < \infty$.*

We will also refer to points in $\pi^{-1}(z)$ as *Zeno states* in $\mathbf{H}$. For example, the "origin" of M_{WT} (as well as M_{BB} and $M_{BB(2)}$) is a Zeno state for every point. Moreover, for each x, $\omega(x)$ contains only one Zeno state. We now show this is always the case.

Theorem 5. *If the execution starting from $x \in M_{\mathbf{H}}$ is Zeno, then $\omega(x)$ consists of exactly one Zeno state for x and $\omega(x) \subset \bigcap_{e \in E_\infty(x)} \pi(\overline{G(e)})$.*

Note than in all the Zeno examples above none of the flows involved in creating the Zeno state has an equilibrium at the Zeno state. The following lemma shows that this is not a coincidence.

Lemma 1. *A Zeno state is not a standard equilibrium (cf. Def. 12). More specifically, if $z \in M_{\mathbf{H}}$ is a Zeno state, then for every $p \in \pi^{-1}(z)$, if $p \in D_i$, then $X_i(p) \neq 0$.*

Example 12 (equilibrium + cusp = Zeno).

Consider the following one-domain hybrid system: $D = \{(x, y) \in \mathbb{R}^2 : y \geq 0,\ \ -f(y) \leq x \leq f(y)\}$, $G = \{(-f(y), y) : y \geq 0\}$, $\ R(-f(y), y) = (f(cy), cy)$, $X(x, y) = (-x - y, x - y)^T$. Here $0 < c < 1$, $f : [0, \infty) \to [0, \infty)$ is a smooth function such that $f(0) = 0$ and for all $y \geq 0$, $f(y) \leq y^2$. In particular, $f'(0) = 0$, which means that D has a cusp at $\mathbf{0}$. It is not difficult to check that $\mathbf{0}$ is a Zeno state despite the fact that it is an equlibrium for X. This shows the importance of geometry of domains and assumption (A2).

Theorem 6. *Suppose $\mathbf{H}$ is a hybrid system such that its hybrid flow $\Psi^{\mathbf{H}}$ is smooth. (This in particular means that its hybrifold $M_{\mathbf{H}}$ is smooth.) Then $\mathbf{H}$ admits no Zeno executions or equivalently, there are no Zeno states in $M_{\mathbf{H}}$.*

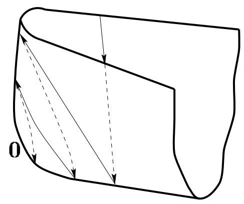

Fig. 7. Smoothed water tank M_{WT}^{smooth}.

In general it may not be easy to check whether, given $\mathbf{H}$, the hybrifold $M_{\mathbf{H}}$ is smooth. Even if it were, non-smoothness of the hybrid flow may cause Zeno (cf. $BB(2)$). However, the following result provides an easily verifiable criterion for smoothness of $\Psi^{\mathbf{H}}$.

Theorem 7. *Suppose that $M_{\mathbf{H}}$ is smooth and for every $e = (i,j) \in E$, X_i and X_j are $\tilde{R}_e$-related on $\overline{G(e)}$. That is, for every $p \in \overline{G(e)}$: $T\tilde{R}_e(X_i(p)) = X_j(\tilde{R}_e(p))$. Then the hybrid flow is smooth.*

Example 13. Consider $BB(2)$. Here we have: $X_1(x_1, x_2) = (x_2, -g)^T = X_2$, $\tilde{R}_{(i,j)}(i, x_1, x_2) = (j, x_1, -cx_2)$, where $(i,j) = (1,2)$ or $(2,1)$. Therefore, $T\tilde{R}_{(1,2)}(X_1) = (x_2, cg)^T \neq X_2$, so the hybrid flow for $BB(2)$ is not smooth, as we already knew.

Example 14. It is not difficult to check that in case of $T^2(\alpha)$, the condition from Theorem 7 is satisfied for every $\alpha > 0$. Thus $T^2(\alpha)$ does not admit Zeno, as was already shown above.

Corollary 1. *If* $\mathbf{H}$ *is a hybrid system satifying condition from Theorem 7, then* $\mathbf{H}$ *accepts no Zeno executions.*

4.2 Removal of Zeno

Suppose that $\mathbf{H}$ is a regular hybrid system and that $z \in M_{\mathbf{H}}$ is a Zeno state. We have seen that $M_{\mathbf{H}}$ in a certain sense has a singularity at z. Consider the following ways of removing such singularities.

Smoothing. Suppose that $M_{\mathbf{H}}$ can be equipped with a smooth structure which induces the same topology as the original one and denote the smoothed hybrifold by $M_{\mathbf{H}}^{smooth}$ (cf. Fig. 7). Note that $M_{\mathbf{H}}$ and $M_{\mathbf{H}}^{smooth}$ are homeomorphic. It is not guaranteed that the hybrid flow $\Psi^{\mathbf{H}}$ will be smooth on $M_{\mathbf{H}}^{smooth}$. If, however, $\Psi^{\mathbf{H}}$ *is* smooth with respect to the differentiable structure on $M_{\mathbf{H}}^{smooth}$, then Theorem 6 implies that there are no Zeno states in $M_{\mathbf{H}}^{smooth}$. We say that we have removed Zeno by *smoothing.*

Hybrid suspension. [4] The basic idea is to "interpolate" executions between guards and images of corresponding resets, i.e. to make "instantaneous" discrete transitions given by reset maps "last" some time ϵ. The constructions goes as follows. Let $\epsilon > 0$ be arbitrary and assume $e = (i,j) \in E$. Instead of gluing $\overline{G(e)}$ to $\overline{\operatorname{im} R_e}$ via $\tilde{R}_e$, first enlarge the domain D_i by $D_i^\epsilon = D_i \cup (\overline{G(e)} \times [0, \epsilon])$, and then identify $(p, \epsilon) \sim \tilde{R}_e(p)$, for every $p \in \overline{G(e)}$. Denote the space obtained by this identification for all $e \in E$ by $S^\epsilon M_{\mathbf{H}}$ and by π^ϵ the quotient (i.e. identification) map. On each $\overline{G(e)} \times [0, \epsilon]$, consider the

[4] We thank Morris W. Hirsch for suggesting this idea in a recent conversation.

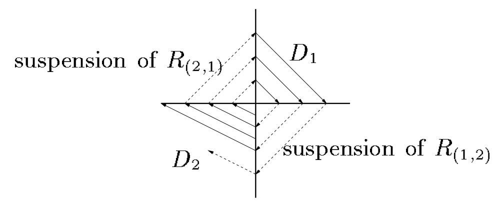

Fig. 8. ϵ-suspended water tank $S^\epsilon M_{WT}$.

trivial "vertical" flow: $(p,s,t) \mapsto (p, s+t)$ $(p \in \overline{G(e)},\ 0 \le s \le \epsilon,\ t \in \mathbb{R})$. Denote by $S^\epsilon \Psi^{\mathbf{H}}$ the flow on $S^\epsilon M_{\mathbf{H}}$ obtained by projecting via π^ϵ this flow (for each $e \in E$) as well as $\Phi^{\mathbf{H}}$. We will call $S^\epsilon M_{\mathbf{H}}$ the *ϵ-suspended hybrid manifold* and $S^\epsilon \Psi^{\mathbf{H}}$ the associated *ϵ-suspended hybrid flow* (see Fig. 8). (This construction resembles the standard suspension of a map; cf. e.g. [PdM].) It is immediate by construction that for ever $\epsilon > 0$, $S^\epsilon \Psi^{\mathbf{H}}$ accepts no Zeno-type executions.

5 Conjugacy of hybrid systems and classification of Zeno states in dimension two

In this section we discuss the following question: when are two hybrid systems qualitatively the same? For that purpose we borrow the notion of conjugacy from the theory of dynamical systems. Roughly speaking, two dynamical systems are conjugate if their phase portraits look qualitatively (or topologically) the same. Similarly, two hybrid systems are conjugate if their hybrid flows are conjugate. We now make this more precise.

Definition 10. *Two hybrid systems $\mathbf{H_1}$ and $\mathbf{H_2}$ are said to be topologically conjugate (denoted by $\mathbf{H_1} \approx \mathbf{H_2}$) if there exists a homeomorphism $h : M_{\mathbf{H_1}} \to M_{\mathbf{H_2}}$ which sends orbits of $\Psi^{\mathbf{H_1}}$ to orbits of $\Psi^{\mathbf{H_2}}$. If $M_{\mathbf{H_1}}$ and $M_{\mathbf{H_2}}$ happen to be smooth manifolds of class C^r $(r \ge 1)$ and h is a C^r diffeomorphism, then $\mathbf{H_1}$ and $\mathbf{H_2}$ are said to be C^r-conjugate.*

As usual, by the *orbit* of a point x under a (local) flow $\{\phi_t\}$ we mean the set of points $\phi_t(x)$ for all t for which $\phi_t(x)$ is defined. We usually think of h as a change of coordinates so that two hybrid systems are topologically conjugate if their hybrid flows are the same up to a continuous coordinate change. Note that conjugacy does not necessarily preserve the time parameter t. If it does, it is called *equivalence*.

Example 15. WT is topologically conjugate to BB. This can be seen by suitably projecting M_{WT} and M_{BB} onto $\mathbb{R}^2$ so that both Ψ^{WT} and Ψ^{BB} look like a spiral sink at the origin. For more details, see [SJSL]. We will see later that in dimension two this picture is typical.

Example 16. $T^2(1)$ is not conjugate to $T^2(\sqrt{2})$. Even though the hybrifold for both hybrid systems is the same (the 2-torus), every orbit of $T^2(1)$ is closed, while every orbit of $T^2(\sqrt{2})$ is dense in T^2.

Even though it is not possible to classify all hybrid systems up to conjugacy (this attempt fails even for smooth dynamical systems), the next theorem shows that near a Zeno state, every 2-dimensional hybrid flow looks like Ψ^{WT} near $\mathbf{0}$.

Theorem 8. *Let* $\mathbf{H}$ *be a 2-dimensional hybrid system and suppose that* $z \in M_{\mathbf{H}}$ *is a Zeno state. Then there is a neighborhood* U *of* z *in* $M_{\mathbf{H}}$ *and a neighborhood* V *of* $\mathbf{0}$ *in* M_{WT} *such that* $\Psi^{\mathbf{H}}|_U$ *is topologically conjugate to* $\Psi^{WT}|_V$*.*

6 Stability of Hybrid Equilibria

Recall that if ϕ_t is a local flow generated by a smooth vector field X on some set U (in $\mathbb{R}^n$ or any manifold), then $p \in U$ is an *equilibrium* for X (equivalently: for ϕ_t) if $X(p) = 0$ (equivalently: if $\phi_t(p) = p$ for all $t \in \mathbb{R}$). In case of a hybrid system there is usually more than one vector field at play, and even in the case when there is only one, resets are involved in generating the hybrid dynamics. Taking this into account we define a hybrid equilibrium as follows.

Definition 11. *Let* $\mathbf{H}$ *be a hybrid system. A point* $x \in M_{\mathbf{H}}$ *is called an (hybrid) equilibrium for the hybrid flow* $\Psi^{\mathbf{H}}$ *if* $\Psi^{\mathbf{H}}(t, x) = x$ *for all* $t \in J(x)$*.*

Equivalently, $x \in M_{\mathbf{H}}$ is a hybrid equilibrium if the hybrid dynamics of $\mathbf{H}$, consisting of reset maps and local flows of $\mathbf{H}$, map $\pi^{-1}(x)$ to itself. For example, any Zeno state is a hybrid equilibrium despite Lemma 1; however, hybrid dynamics make no time progress at this kind of equilibrium. The following definition distinguishes those hybrid equilibria which are created from equilibria of vector fields in $\mathbf{H}$ in the standard sense.

Definition 12. *A point* $x \in M_{\mathbf{H}}$ *is called a* standard equilibrium *for* $\Psi^{\mathbf{H}}$ *if it is a hybrid equilibrium and for each* $p \in \pi^{-1}(x)$*, if* $p \in D_i$*, then* p *is an equilibrium for* X_i *(i.e.* $X_i(p) = 0$*). It is called a* pure equilibrium *if it is standard and belongs to* $\pi(\mathit{int}\, D)$*.*

Note that the only dynamics involved in creating a pure equilibrium are those of a single vector field. We now define the notions of (Lyapunov) stability and asymptotic stability of hybrid equilibria in analogy with those from dynamical systems.

Definition 13. *An equilibrium* x_* *of* $\Psi^{\mathbf{H}}$ *is called (Lyapunov) stable if for every neighborhood* U *of* x_* *in* $M_{\mathbf{H}}$ *there exists a neighborhood* V *of* x_* *in* U *such that for every* $x \in V$*,* $\Psi^{\mathbf{H}}_t(x) \in U$ *for all* $t \in [0, \tau_\infty(x))$*. If* V *can be chosen so that in addition to the properties described above,* $\lim_{t \to \tau_\infty(x)} \Psi^{\mathbf{H}}_t(x) = x_*$*, then* x_* *is* asymptotically stable*.*

Example 17. There are well known 2-dimensional hybrid systems (and they are also not difficult to construct from scratch; cf. [SJSL]) with a standard hybrid equilibrium which can described as follows: stable + stable = unstable, or unstable + stable = stable, or unstable + unstable = stable. This means that (in the case of the first example) the unstable hybrid equilibrium in question is created by stable equilibria for the vector fields at play in the hybrid system. These examples show us that extra caution is needed in analyzing stability of hybrid equilibria.

In the subsequent text, we use the following notation: if X is a vector field on a manifold M with local flow ϕ_t and $f : M \to \mathbb{R}$ a function, Xf will denote the derivative of f in the direction of X: $(Xf)(x) = Tf(X(x))$. For a map $h : (A, d_A) \to (B, d_B)$ between metric spaces, let $\mathrm{Lip}_p(f) = \sup_{q \in A - \{p\}} \frac{d_B(f(q), f(p))}{d_A(q,p)}$. This is the *Lipschitz constant of* f *at* p.

The following theorem is an analog of the linearization theorem for stability of equilibria of a single dynamical system. In the hybrid case, the linearized data include, besides the derivatives of the vector fields at the equilibrium, the tangent spaces at the equilibrium of guards and images of resets involved in the hybrid dynamics near the equilibrium. Here, for a manifold A with boundary and $p \in \partial A$, we denote by $T_p^+ A$ the set of all vectors $v \in T_p A$ which point inside A (i.e. there exists $\epsilon > 0$ and a smooth curve $c : [0, \epsilon] \to A$ such that $c(0) = p$, $\dot{c}(0) = v$ and $c(t) \in A - \partial A$ for $0 < t \leq \epsilon$).

Theorem 9 (Stability via Linearization).
Let $x_* \in M_{\mathbf{H}}$ be an isolated standard equilibrium for $\Psi^{\mathbf{H}}$ and $\pi^{-1}(x_*) = \{p_1, \dots, p_l\}$, where $p_j \in D_{i_j}$ and $1 \leq j \leq l$. Suppose that there exists a bounded neighborhood W of x_* and for each $1 \leq j \leq l$ a smooth function $f_j : U_j - \{p_j\} \to \mathbb{R}$, where U_j is a neighborhood of $D_{i_j} \cap \pi^{-1}(W)$ in $\{i_j\} \times \mathbb{R}^n$, such that:

(a) *$p_j \in A_j \cap B_j$, where $A_j = \overline{\mathrm{im} R_{(i_{j-1}, i_j)}} \cap U_j$, $B_j = \overline{G(i_j, i_{j+1})} \cap U_j$, for all $1 \leq j \leq l$. Assume further that A_j and B_j are differentiable at p_j.*
(b) *$a_j^- \leq f_j \leq a_j^+$ on A_j, and $B_j = \overline{f_j^{-1}(b_j)}$, for all j, for some numbers $a_j^- \leq a_j^+ < b_j$.*
(c) *$0 < m_j^- \leq X_{i_j} f_j \leq m_j^+$ on $U_j - \{p_j\}$ $(1 \leq j \leq l)$.*
(d) *For each j there exists $\tau_j > 0$ such that $e^{\tau_j L_j}(T_{p_j}^+ A_j) \subset T_{p_j}^+ B_j$, where $L_j = T_{p_j} X_{i_j}$.*

For $1 \leq j \leq l$, let S_j be an $n \times (n-1)$-matrix whose columns form an orthonormal basis for $T_{p_j} A_j$ and belong to $T_{p_j}^+ A_j$. Let

$$\mu_j = \sqrt{\lambda_{\max}[(e^{\tau_j L_j} S_j)^T e^{\tau_j L_j} S_j]},$$

and $\nu_j = \|T_{p_j} R_{(i_j, i_{j+1})}\|$. Define $\eta_{\mathbf{H}}(x_*) = \prod_{j=1}^{l} \mu_j \nu_j$. If $\eta_{\mathbf{H}}(x_*) < 1$, then x_* is an asymptotically stable hybrid equilibrium. If $\dim \mathbf{H} = 2$ and $\eta_{\mathbf{H}}(x_*) > 1$, then x_* is unstable.

Remarks.

(i) Condition (b) says that B_j is the closure of a level set of f_j while A_j is "almost" a level set of f_j. The function f_j measures the progress trajectories of X_{i_j} make towards B_j, starting from A_j.
(ii) Condition (c) says that the time-τ_j map of the linearization of the flow of X_{i_j} at p_j (i.e. $T\phi_t^{i_j}$) maps $T_{p_j}^+ A_j$ to $T_{p_j}^+ B_j$. This means that at least on the level of linearizations, B_j is reachable from A_j in a bounded amount of time.
(iii) Note that (unlike in [B] and [MH]) it is not necessary to integrate any vector fields and that all the input data of the theorem are computable (even though finding f_j's and τ_j's may be difficult).

Example 18. Define a 3-dimensional hybrid system $\mathbf{H}$ by: $D_1 = \{1\} \times S$, $D_2 = \{2\} \times \overline{\mathbb{R}^3 - S}$, where $S = \{(x, y, z) : x \geq 0,\ y \geq x^2,\ z \in \mathbb{R}\} \cup \{(x, y, x) : x \leq 0,\ y \geq -x(x - c),\ z \in \mathbb{R}\}$, and $G(1, 2) = \{(x, y, z \in D_1 : y = x^2\}$, $G(2, 1) = \{(x, y, z) \in D_2 : y = -x(x - c)\}$, for some constant c. Let $X_1(x, y, z) = (-x - y, x - y, -\lambda_1 z)$ and $X_2(x, y, z) = (x - y, x + y, \lambda_2 z)$, where $0 < \lambda_2 \leq 1 \leq \lambda_1$. Then it is not difficult to check that $\eta_{\mathbf{H}}(\mathbf{0}) = e^{-2\gamma}$, where $\gamma = \arctan c$, so if $c > 0$, then $\mathbf{0}$ is asymptotically stable.

Example 19. Let $\mathbf{H}$ be a 3-dimensional hybrid system with $D_1 = \{1\} \times K \times \mathbb{R}$ and $D_2 = \{2\} \times \overline{\mathbb{R}^2 - K} \times \mathbb{R}$, where $K = [0, \infty) \times [0, \infty)$. Let $G(1,2) = \{(x, y, z) \in D_1 : x = 0\}$, $G(2,1) = \{(x, y, z) \in D_2 : y = 0\}$, and $X_1(x, y, z) = (x - y, x + y, -\lambda_1 z)$, $X_2(x, y, z) = (-x - y, x - y, \lambda_2 z)$, where $\lambda_1, \lambda_2 > 0$. The resets are identity maps.

Then the full trajectories of X_1 are spirals around the z-axis which increase in radius and converge to the xy-plane. The full trajectories of X_2 are also spirals around the z-axis, but they decrease in radius and diverge from the xy-plane. It is not difficult to check that, with notation from Theorem 9, $\mu_1 = e^{\pi/2}$, $\mu_2 = e^{3\pi\lambda_2/2}$, so $\eta_{\mathbf{H}}(\mathbf{0}) > 1$ and the theorem is inconclusive.

However, the flows can be decoupled into their xy- and z-parts the analysis of which shows that if $\lambda_1 > 3\lambda_2$, then $\mathbf{0}$ is an asymptotically stable hybrid equilibrium of $\mathbf{H}$. The reason Theorem 9 does not provide the same answer, intuitively speaking, is because it is not able to measure the small amount of contraction around $\mathbf{0}$ in the flows of both X_1 and X_2, which turns out to be sufficient for asymptotic stability. Namely, on $G(2,1)$ the flow of X_1 contracts in only one direction (and expands in the other) and similarly for the flow of X_2 on $G(1,2)$.

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