Proceedings of the 37th Chinese Control Conference
July 25-27, 2018, Wuhan, China

Long-term behavior of cross-dimensional linear dynamical systems

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Abstract: Let $\mathcal{M}$ and $\mathcal{V}$ denote the sets of finite-dimensional matrices and finite-dimensional column vectors, respectively. Based on the semitensor product and the vector addition, $\mathcal{M}$ and $\mathcal{V}$ both form a monoid, where $\mathcal{V}$ is commutative. In addition, based on an equivalence relation $\leftrightarrow$ on $\mathcal{V}$, the induced quotient space $\mathcal{V}/_{\leftrightarrow}$ forms a vector space. In this paper, we give a basis for the vector space $\mathcal{V}/_{\leftrightarrow}$, showing that $\mathcal{V}/_{\leftrightarrow}$ is of countably infinite dimension. In addition, we give an explicit characterization for how the dimension of a vector in $\mathcal{V}$ changes caused by the repetitive actions of a matrix in $\mathcal{M}$ on the vector, and characterize the generalized inverse behavior of the repetitive actions.

Key Words: Long-term behavior, cross-dimensional vector space, cross-dimensional linear dynamical system, dimension-boundedness, basis, Drazin inverse

1 Introduction and preliminaries

The phenomenon of dimension variation can be found almost everywhere in the nature, e.g., the entrance or departure of a bird in a group of birds, the birth or death of a cell in an organ. This phenomenon can also be found in manufacturing processes, e.g., entering of parts or leaving of an entire product in a production line. Due to the semitensor product for all finite-dimensional matrices [5] and the vector addition for all finite-dimensional vectors [3], such a phenomenon can be formulated as so-called cross-dimensional dynamical systems. In this paper, motivated by the new construction in [3], we characterize the basis for a so-called cross-dimensional vector space and the long-term behavior of a cross-dimensional dynamical system in the framework of the semitensor product and the vector addition. Necessary notations are shown as below. Note that throughout this paper, all results hold when extending $\mathbb{R}$ to an arbitrary field.

- $\mathbb{R}^n$: the n-dimensional real column vector space
- $\mathcal{V}$: $\cup_{n=1}^{\infty}\mathbb{R}^n$
- $\mathbb{R}^{m\times n}$: the space of $m\times n$ real matrices
- $\mathcal{M}$: $\cup_{m,n=1}^{\infty}\mathbb{R}^{m\times n}$
- $\mathbb{N}$: the set of natural numbers
- $\mathbb{Z}_+$: the set of positive integers
- $\emptyset$: the empty set
- $\mathbf{1}_k$: the k-length column vector with all entries 1
- $\mathbf{0}_k$: the k-length column vector with all entries 0
- $\mathbf{0}_{m\times n}$: the $m\times n$ matrix with all entries be 0 (or briefly as $\mathbf{0}$ when dimension is known.)
- I_n: the $n\times n$ identity matrix
- $\operatorname{rank}(A)$: the rank of matrix A
- $\ker(A)$: the kernel of matrix A
- $\operatorname{im}(A)$: the image of matrix A
- $\dim(V)$: the dimension of a vector space V
- A^D: the Drazin inverse of a square matrix A
- $\operatorname{lcm}(p,q)$: the least common multiple of positive integers p and q
- $\gcd(p,q)$: the greatest common divisor of positive integers p and q
- $p\mid q$: integer p divides integer q
- $p\nmid q$: integer p does not divide integer q

In order to obtain the main results, we will use the well known associative law and the homogeneity of the least common multiple:

Proposition 1.1 *Let a,b,c be positive integers. Then*

1) $\operatorname{lcm}(a,\operatorname{lcm}(b,c)) = \operatorname{lcm}(\operatorname{lcm}(a,b),c)$ (associative law);

2) $a\operatorname{lcm}(b,c)=\operatorname{lcm}(ab,ac)$ (homogeneity).

Let us recall the semitensor product of matrices, which was originally proposed by Daizhan Cheng about twenty years ago [2].

Definition 1.2 *[[5]] Let $A\in\mathbb{R}^{m\times n}$ and $B\in\mathbb{R}^{p\times q}$. The semitensor product of A and B, denoted by $A\ltimes B$, is defined as $A\ltimes B=(A\otimes I_{l/n})(B\otimes I_{l/p})$, where $\otimes$ means the Kronecker product, $l=\operatorname{lcm}(n,p)$.*

It is known that $\ltimes$ preserves the associative law, and is an extension of the conventional matrix product [5]. Hence we can use some notations of the conventional matrix product without any confusion, e.g., for a matrix $A\in\mathcal{M}$, we can use A^n to denote $\ltimes_{i=1}^{n}A$.

The index [1] of a matrix $A\in\mathbb{R}^{n\times n}$ is the least natural number i such that $\operatorname{rank}(A^i)=\operatorname{rank}(A^{i+1})$, i.e., $\min\{i\in\mathbb{N}|\operatorname{rank}(A^i)=\operatorname{rank}(A^{i+1})\}=:\operatorname{ind}(A)$.

For a matrix $A\in\mathbb{R}^{n\times n}$, the matrix $X\in\mathbb{R}^{n\times n}$ is called the Drazin inverse [1] of A, denoted by $X=:A^D$, if $A^{\operatorname{ind}(A)+1}X=A^{\operatorname{ind}(A)}$, $AX=XA$, and $XAX=X$. For each matrix $A\in\mathbb{R}^{n\times n}$, A has a unique Drazin

This work was supported by Knut and Alice Wallenberg Foundation, Swedish Foundation for Strategic Research, and Swedish Research Council.

inverse, and satisfies that $\operatorname{im}(A^0) \supsetneq \operatorname{im}(A^1) \supsetneq \cdots \supsetneq \operatorname{im}(A^{\operatorname{ind}(A)}) = \operatorname{im}(A^i)$ for all integers $i > \operatorname{ind}(A)$ [1].

The vector addition of vectors in $\mathbb{R}^p$ can be extended to the following "vector addition" of vectors in $\mathcal{V}$.

Definition 1.3 **([3])** *Let $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, and $r = \operatorname{lcm}(p,q)$. The vector addition of x and y, denoted by $x\vec{\pm}y$, is defined as*

$$x\vec{\pm}y = x \otimes \mathbf{1}_{r/p} + y \otimes \mathbf{1}_{r/q}. \tag{1}$$

Similarly, the vector subtraction of x and y, denoted by $x\vec{\vdash}y$, is defined as

$$x\vec{\vdash}y = x \otimes \mathbf{1}_{r/p} - y \otimes \mathbf{1}_{r/q}. \tag{2}$$

It is not difficult to see that $\vec{\pm}$ preservers the commutative law and the associative law.

Proposition 1.4 *Let $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, and $z \in \mathbb{R}^r$. Then*

1) $x\vec{\pm}y = y\vec{\pm}x$ (the communicative law);
2) $(x\vec{\pm}y)\vec{\pm}z = x\vec{\pm}(y\vec{\pm}z)$ (the associative law).

Proof The communicative law holds naturally, we only verify the associative law. Let $\operatorname{lcm}(p,q) = u$, $\operatorname{lcm}(u,r) = v$, $\operatorname{lcm}(q,r) = w$, and $\operatorname{lcm}(p,w) = s$. Then

$$\begin{aligned}
&(x\vec{\pm}y)\vec{\pm}z \\
=&(x \otimes \mathbf{1}_{u/p} + y \otimes \mathbf{1}_{u/q})\vec{\pm}z \\
=&(x \otimes \mathbf{1}_{u/p} \otimes \mathbf{1}_{v/u} + y \otimes \mathbf{1}_{u/q} \otimes \mathbf{1}_{v/u}) + z \otimes \mathbf{1}_{v/r} \\
=&x \otimes \mathbf{1}_{v/p} + y \otimes \mathbf{1}_{v/q} + z \otimes \mathbf{1}_{v/r}, \\
&x\vec{\pm}(y\vec{\pm}z) \\
=&x\vec{\pm}(y \otimes \mathbf{1}_{w/q} + z \otimes \mathbf{1}_{w/r}) \\
=&x \otimes \mathbf{1}_{s/p} + (y \otimes \mathbf{1}_{w/q} \otimes \mathbf{1}_{s/w} + z \otimes \mathbf{1}_{w/r} \otimes \mathbf{1}_{s/w}) \\
=&x \otimes \mathbf{1}_{s/p} + y \otimes \mathbf{1}_{s/q} + z \otimes \mathbf{1}_{s/r}.
\end{aligned}$$

By Proposition 1.1 we have $v = \operatorname{lcm}(u,r) = \operatorname{lcm}(\operatorname{lcm}(p,q),r) = \operatorname{lcm}(p,\operatorname{lcm}(q,r)) = \operatorname{lcm}(p,w) = s$. Hence the associative law holds. □

It it is natural to ask whether $(\mathcal{V},\vec{\pm},\cdot)$ forms a vector space, where $\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ is the conventional scalar multiplication of a real number and a real vector. To this end, we should first find a zero element. Note that in $\mathcal{V}$, only the real number 0 satisfies that $0\vec{\pm}x = x\vec{\pm}0 = x$ for any $x \in \mathcal{V}$. Hence only 0 can be the potential zero element. However, it is easy to see that $(\mathcal{V},\vec{\pm})$ is not an Abelian group when 0 is regarded as the zero element, since only real numbers have inverse elements. As a result, $(\mathcal{V},\vec{\pm},\cdot)$ is not a vector space. Despite of this, $(\mathcal{V},\vec{\pm})$ forms a commutative monoid with 0 the identity element.

2 Long-term behavior of the action of $\mathcal{M}$ on $\mathcal{V}$

In this section, we characterize the long-term behavior of the repetitive actions of a matrix M in $\mathcal{M}$ on a vector x in $\mathcal{V}$. One main result is that in such a trajectory, the dimensions of vectors will be either eventually constant or eventually strictly increasing, where for the former case, the matrix is called *dimension-bounded* [3]. Actually, compared to these results, coarser results have been given in [3, 4]. In this paper, we will use different methods to give more refined characterization. In addition, for a dimension-bounded matrix in $\mathcal{M}$, we characterize the limit set of the system generated by its repetitive actions on a vector in $\mathcal{V}$, and also the generalized inverse system of the system.

Next we show our results, where necessary known results are also introduced. A vector product of a matrix A in $\mathcal{M}$ and a vector x in $\mathcal{V}$ is defined as follows.

Definition 2.1 *[[3]] Let $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^t$. The vector product of A and x, denoted by $A\vec{\ltimes}x$, is defined as*

$$A\vec{\ltimes}x = (A \otimes I_{l/n})(x \otimes \mathbf{1}_{l/t}), \tag{3}$$

where $l = \operatorname{lcm}(n,t)$.

Note that based on the vector product $\vec{\ltimes}$, a matrix A can be regarded as an operator on $\mathcal{V}$.

Next we characterize the composition of two matrices as operators on $\mathcal{V}$. By the following Proposition 2.2, one sees that the composition of two operators A and B on $\mathcal{V}$ is exactly their semitensor product. That is, the semitensor product of matrices and the action of $\mathcal{M}$ on $\mathcal{V}$ are consistent.

Proposition 2.2 *[[3]] Let $A, B \in \mathcal{M}$ and $x \in \mathcal{V}$. Then*

$$A\vec{\ltimes}(B\vec{\ltimes}x) = (A \ltimes B)\vec{\ltimes}x. \tag{4}$$

Here we use the associative law and homogeneity of the least common multiple to give a more concise proof than the one in [3].

Proof [of Propostion 2.2] Assume $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, and $x \in \mathbb{R}^t$. Then we have

$$\begin{aligned}
&(A \ltimes B)\vec{\ltimes}x \\
=&((A \otimes I_{r/n})(B \otimes I_{r/p}))\vec{\ltimes}x \\
=&(((A \otimes I_{r/n})(B \otimes I_{r/p})) \otimes I_{sp/qr})(x \otimes \mathbf{1}_{s/t}), \\
=&(A \otimes I_{r/n} \otimes I_{sp/qr})(B \otimes I_{r/p} \otimes I_{sp/qr})(x \otimes \mathbf{1}_{s/t}) \\
=&(A \otimes I_{sp/qn})(B \otimes I_{s/q})(x \otimes \mathbf{1}_{s/t}),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&A\vec{\ltimes}(B\vec{\ltimes}x) \\
=&A\vec{\ltimes}((B \otimes I_{u/q})(x \otimes \mathbf{1}_{u/t})), \\
=&(A \otimes I_{v/n})(((B \otimes I_{u/q})(x \otimes \mathbf{1}_{u/t})) \otimes \mathbf{1}_{vq/pu}) \\
=&(A \otimes I_{v/n})(B \otimes I_{u/q} \otimes I_{vq/pu})(x \otimes \mathbf{1}_{u/t} \otimes \mathbf{1}_{vq/pu}) \\
=&(A \otimes I_{v/n})(B \otimes I_{v/p})(x \otimes \mathbf{1}_{vq/pt}),
\end{aligned} \tag{6}$$

where $r = \operatorname{lcm}(n,p)$, $s = \operatorname{lcm}(qr/p,t)$, $u = \operatorname{lcm}(q,t)$, $v = \operatorname{lcm}(n,pu/q)$.

By Proposition 1.1, we have

$$\begin{aligned}
sp &= \operatorname{lcm}(qr/p,t)p = \operatorname{lcm}(qr,tp) \\
&= \operatorname{lcm}(q\operatorname{lcm}(n,p),tp) = \operatorname{lcm}(\operatorname{lcm}(nq,pq),tp), \\
vq &= \operatorname{lcm}(n,pu/q)q = \operatorname{lcm}(nq,pu) \\
&= \operatorname{lcm}(nq,p\operatorname{lcm}(q,t)) = \operatorname{lcm}(nq,\operatorname{lcm}(pq,tp)) \\
&= \operatorname{lcm}(\operatorname{lcm}(nq,pq),tp) = sp.
\end{aligned}$$

Then we have $sp/qn = v/n$, $s/q = v/p$, and $s/t = vq/tp$. By (5) and (6), (4) holds. □

By Proposition 2.2, we obtain a dynamical system

$$x(\tau+1) = A \vec{\ltimes} x(\tau), \tag{7}$$

where $A \in \mathbb{R}^{m\times n}$, $\tau = 0,1,\dots$, $x(\tau) \in \mathcal{V}$. Note that here we cannot call (7) a linear dynamical system, as $\mathcal{V}$ is not a vector space.

Now we can consider the long-term action of a matrix on $\mathcal{V}$, e.g., system (7). Note that the action of a matrix on a vector may change the dimension of the vector, next we characterize when the action of a matrix does not change the dimension. Here we use dimension to represent the following result. Actually this result has been given in [3], we give a different proof.

Theorem 2.3 *[[3]] Let A be in $\mathbb{R}^{m\times n}$ and t in $\mathbb{Z}_+$. Then*

$$A\vec{\ltimes}\mathbb{R}^t := \{A\vec{\ltimes}x | x \in \mathbb{R}^t\} \subset \mathbb{R}^t$$

if and only if

$$m \mid n,\ m \mid t,\ \text{and}\ \gcd(n/m, t/m) = 1.$$

Proof Denote $\mathrm{lcm}(n,t) = r$. Then for each $x \in \mathbb{R}^t$, $A\vec{\ltimes}x \in \mathbb{R}^{mr/n}$.

"if":

By assumption we can denote $n = mk_1$ and $t = mk_2$, where $k_1, k_2 \in \mathbb{Z}_+$. Then $\gcd(n/m, t/m) = \gcd(k_1,k_2) = 1$, $r = \mathrm{lcm}(n,t) = \mathrm{lcm}(mk_1, mk_2) = m\,\mathrm{lcm}(k_1,k_2) = mk_1k_2$, $mr/n = mmk_1k_2/n = mmk_1k_2/mk_1 = mk_2 = t$.

"only if":

By assumption we have $mr/n = t$. Denote $r = nl_1 = tl_2$, where $l_1,l_2 \in \mathbb{Z}_+$. Then $nt = mr = mnl_1 = mtl_2$, $t = ml_1$, $n = ml_2$, $m \mid t$, $m \mid n$, $r = \mathrm{lcm}(t,n) = lcm(ml_1, ml_2) = m\,\mathrm{lcm}(l_1,l_2) = nl_1 = ml_2l_1$, $\mathrm{lcm}(l_1,l_2) = l_1l_2$, hence $\gcd(l_1,l_2) = \gcd(t/m, n/m) = 1$. □

The following result directly follows from Theorem 2.3.

Corollary 2.4 *[[3]] Let A be in $\mathbb{R}^{m\times n}$ and t in $\mathbb{Z}_+$. If $A\vec{\ltimes}\mathbb{R}^t \subset \mathbb{R}^t$ then A has the representation $A_{\mathcal{L}} = (A\otimes I_{r/n})(I_t \otimes \mathbf{1}_{r/t})$, where $r = \mathrm{lcm}(n,t)$. That is, $A\vec{\ltimes}x = A_{\mathcal{L}}x$ for each $x \in \mathbb{R}^t$. (Note that $A_{\mathcal{L}} \in \mathbb{R}^{t\times t}$.)*

More generally, we next characterize when the action of a matrix eventually does not change the dimension of vectors.

Definition 2.5 *[[3]] Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{Z}_+$. A is called dimension-bounded with respect to t if there exist $i_0, t' \in \mathbb{Z}_+$ both depending on t such that for each $x_0 \in \mathbb{R}^t$, $A^i\vec{\ltimes}x_0 \in \mathbb{R}^{t'}$ for all integers $i \geq i_0$.*

Although the next result has been given in [3], here we give a different proof which yields a more refined result, i.e., Theorem 2.7, as our first main result.

Theorem 2.6 *[[3]] Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{Z}_+$. Matrix A is dimension-bounded with respect to t if and only if $m \mid n$.*

Proof Arbitrarily chosen $x_0 \in \mathbb{R}^t$, we have

$$A\vec{\ltimes}x_0 \in \mathbb{R}^{m\,\mathrm{lcm}(n,t)/n},$$

where $m\,\mathrm{lcm}(n,t)/n =: f_1$;

$$A^2\vec{\ltimes}x_0 = A\vec{\ltimes}(A\vec{\ltimes}x_0) \in \mathbb{R}^{m\,\mathrm{lcm}(n,f_1)/n},$$

where

$$\begin{aligned} & m\,\mathrm{lcm}(n,f_1)/n \\ =& m\,\mathrm{lcm}(n, m\,\mathrm{lcm}(n,t)/n)/n \\ =& \mathrm{lcm}(mn^2, m^2\,\mathrm{lcm}(n,t))/n^2 \\ =& \mathrm{lcm}(mn^2, \mathrm{lcm}(m^2n, m^2t))/n^2 =: f_2; \end{aligned}$$

by induction we can obtain that $A^i\vec{\ltimes}x_0 \in \mathbb{R}^{f_i}$ for each $i \in \mathbb{Z}_+$, where

$$f_i = \mathrm{lcm}(\mathrm{lcm}_{k=1}^i m^k n^{i+1-k}, m^i t)/n^i. \tag{8}$$

"if":

By $m \mid n$ we next prove that

$$\mathrm{lcm}(mn^{r+1}, m^r nt) = \mathrm{lcm}(mn^{r+1}, m^{r+1}t) \tag{9}$$

for all sufficiently large integers r.

Denote $n = mk$, where $k \in \mathbb{Z}_+$. We have

$$\begin{aligned} \mathrm{lcm}(mn^{r+1}, m^r nt) &= m^{r+1}\,\mathrm{lcm}(mk^{r+1}, kt), \\ \mathrm{lcm}(mn^{r+1}, m^{r+1}t) &= m^{r+1}\,\mathrm{lcm}(mk^{r+1}, t). \end{aligned}$$

If $k = 1$ or all prime factors of t are also factors of k, then (9) obviously holds for all sufficiently large r. Next we assume that $k > 1$ and t has a prime factor that is not a factor of k. Based on this assumption, we have

$$\begin{aligned} k =& k_1^{\alpha_1}\cdots k_p^{\alpha_p}, \\ t =& k_1^{\gamma_1}\cdots k_p^{\gamma_p} t_1^{\delta_1}\cdots t_q^{\delta_q}, \\ mk^{r+1} =& k_1^{\alpha_1(r+1)+\epsilon_1}\cdots k_p^{\alpha_p(r+1)+\epsilon_p} t_1^{\mu_1}\cdots t_q^{\mu_q} m_1^{\nu_1}\cdots m_s^{\nu_s}, \end{aligned}$$

where $k_1,\dots,k_p,t_1,\dots,t_q,m_1,\dots,m_s$ are pairwise different prime numbers; $\alpha_1,\dots,\alpha_p \in \mathbb{Z}_+$; $\gamma_1,\dots,\gamma_p \in \mathbb{N}$; $\delta_1,\dots,\delta_q \in \mathbb{Z}_+$; $\epsilon_1,..,\epsilon_p \in \mathbb{N}$; $\mu_1,\dots,\mu_q \in \mathbb{N}$; $\nu_1,\dots,\nu_s \in \mathbb{N}$.

When r is sufficiently large, we have

$$\begin{aligned} & \mathrm{lcm}(mk^{r+1}, kt) \\ =& k_1^{\alpha_1(r+1)+\epsilon_1}\cdots k_p^{\alpha_p(r+1)+\epsilon_p} t_1^{\max\{\delta_1,\mu_1\}}\cdots \\ & t_q^{\max\{\delta_q,\mu_q\}} m_1^{\nu_1}\cdots m_s^{\nu_s} \\ =& \mathrm{lcm}(mk^{r+1}, t). \end{aligned}$$

Hence (9) holds for all sufficiently large r.

By $m \mid n$ we have

$$f_i = \mathrm{lcm}(mn^i, m^i t)/n^i \tag{10}$$

for each $i \in \mathbb{Z}_+$. Then by the above analysis, for sufficiently large r, we have

$$\begin{aligned} f_r &= \mathrm{lcm}(mn^r, m^r t)/n^r = \mathrm{lcm}(mn^{r+1}, m^r nt)/n^{r+1} \\ &= \mathrm{lcm}(mn^{r+1}, m^{r+1}t)/n^{r+1} = f_{r+1}, \end{aligned}$$

which completes the "if" part.

Actually, from the above analysis, we also have if $m \mid n$, then for each $s \in \mathbb{Z}_+$, the corresponding f_i satisfies $f_r = f_{r+1}$ for all sufficiently large integers r. We also have that for all sufficiently large $r \in \mathbb{Z}_+$, $f_{r+1}/m = t_1^{\max\{\mu_1,\delta_1\}-\mu_1} \cdots t_q^{\max\{\mu_q,\delta_q\}-\mu_q}$, hence $m \mid f_{r+1}$ and $\gcd(n/m, f_{r+1}/m) = 1$, which is consistent with Theorem 2.3.

"only if":

By assumption we have $f_r = f_{r+1}$ for all sufficiently large integer r. Denote

$$A_r := \operatorname{lcm}(\operatorname{lcm}_{k=1}^{r} m^k n^{r+1-k}, m^r t),$$

then $f_{r+1} = m \operatorname{lcm}(n^{r+1}, A_r)/n^{r+1}$. By $f_r = f_{r+1}$, we have $nA_r = m \operatorname{lcm}(n^{r+1}, A_r)$, hence $m \mid n$, which completes the proof.

□

From the above analysis, we see for each $t \in \mathbb{Z}_+$, $f_r = f_{r+1}$ for some r implies $m \mid n$. In addition, we can prove one more result as below, i.e., for each $t \in \mathbb{Z}_+$, $f_r = f_{r+1}$ for some r implies $f_r = f_s$ for all $s \geq r$. To this end, we only need to prove $f_r = f_{r+1}$ implies $f_{r+1} = f_{r+2}$ for any r.

Next we fix t and r. By $f_r = f_{r+1}$ we have $m \mid n$. Then $f_l = \operatorname{lcm}(mn^l, m^l t)/n^l$ for any $l \in \mathbb{Z}_+$. Using $mk = n$, we have $f_l = \operatorname{lcm}(mk^l, t)/k^l$ for any l. Then $f_r = f_{r+1}$ implies $\operatorname{lcm}(mk^{r+1}, kt) = \operatorname{lcm}(mk^{r+1}, t)$. We then have $\operatorname{lcm}(mk^{r+2}, \operatorname{lcm}(mk^{r+1}, kt)) = \operatorname{lcm}(mk^{r+2}, \operatorname{lcm}(mk^{r+1}, t))$, i.e., $\operatorname{lcm}(mk^{r+2}, kt) = \operatorname{lcm}(mk^{r+2}, t)$, then $f_{r+1} = \operatorname{lcm}(mk^{r+2}, kt)/k^{r+2} = \operatorname{lcm}(mk^{r+2}, t)/k^{r+2} = f_{r+2}$.

Besides, by $m \mid n$ we have $f_l = \operatorname{lcm}(mk^l, t)/k^l$ for any $l \in \mathbb{Z}_+$, then

$$\begin{aligned}
&\operatorname{lcm}(f_l, f_{l+1}) \\
=&\operatorname{lcm}(\operatorname{lcm}(mk^l, t)/k^l, \operatorname{lcm}(mk^{l+1}, t)/k^{l+1}) \\
=&\operatorname{lcm}(\operatorname{lcm}(mk^{l+1}, kt)/k^{l+1}, \operatorname{lcm}(mk^{l+1}, t)/k^{l+1}) \\
=&\operatorname{lcm}(\operatorname{lcm}(mk^{l+1}, kt), \operatorname{lcm}(mk^{l+1}, t))/k^{l+1} \\
=&\operatorname{lcm}(mk^{l+1}, kt)/k^{l+1} \\
=&\operatorname{lcm}(mk^l, t)/k^l \\
=&f_l.
\end{aligned}$$

Hence $f_{l+1} \mid f_l$ for each $l \in \mathbb{Z}_+$.

Based the above analysis and Theorem 2.6, we obtain our first main result.

Theorem 2.7 *Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{Z}_+$. Let f_i be as in (8).*

1) *If matrix A is dimension-bounded with respect to some $u \in \mathbb{Z}_+$, then it is dimension-bounded with respect to any $v \in \mathbb{Z}_+$.*
2) *If $m \mid n$ then function f_i is strictly decreasing on $\{1, \dots, i_0\}$ for some $i_0 \in \mathbb{Z}_+$ depending on t, constant on $\{i_0, i_0+1, \dots\}$, and satisfies $f_{l+1} \mid f_l$ for any $l \in \mathbb{Z}_+$.*

By Theorem 2.7, Definition 2.5 can be equivalently rewritten as follows.

Definition 2.8 *Let $A \in \mathbb{R}^{m\times n}$. A is called dimension-bounded if for each $t \in \mathbb{Z}_+$, there exist $i_0, t' \in \mathbb{Z}_+$ both depending on t such that for each $x_0 \in \mathbb{R}^t$, $A^i \vec{\ltimes} x_0 \in \mathbb{R}^{t'}$ for all integers $i \geq i_0$. Here the minimal such i_0 is called the index of m, n, t, and denoted by $\operatorname{ind}(m, n, t)$.*

Then similar to Theorem 2.6, we have the following result.

Theorem 2.9 *[[3]] Let $A \in \mathbb{R}^{m\times n}$. Matrix A is dimension-bounded if and only if $m \mid n$.*

Remark 2.1 *One sees that whether a matrix is dimension-bounded only depends on its dimension, but does not depend its entries.*

Next we characterize the matrices that are not dimension-bounded.

Corollary 2.10 *Let $A \in \mathbb{R}^{m\times n}$ be such that $m \nmid n$.*

1) *For each $t \in \mathbb{Z}_+$, the corresponding function f_i as in (8) satisfies that $f_r \neq f_{r+1}$ for all $r \in \mathbb{Z}_+$.*
2) *If $n \mid m$ and $m \neq n$ then for each $t \in \mathbb{Z}_+$, the corresponding f_i is strictly increasing and satisfies $mf_l = nf_{l+1}$ for any $l \in \mathbb{Z}_+$.*

Proof 1) This conclusion directly follows from Theorems 2.7 and 2.9.

2) By $n \mid m$ we have $f_i = k^i \operatorname{lcm}(n, t)$, where $k = m/n$. The conclusion follows. □

Furthermore, we give a complete characterization for the matrices that are not dimension-bounded, i.e., Theorem 2.11, as our second main result. Specifically, the next result shows that for each matrix $A \in \mathcal{M}$ that is not dimension-bounded and each positive integer t, the corresponding function f_i as in (8) is injective and eventually strictly increasing. In [4], it was shown that $\lim_{i\to\infty} f_i = \infty$. Hence our result is more refined.

Theorem 2.11 *Let $A \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{Z}_+$. Let f_i be as in (8). Assume that M is not dimension-bounded, i.e., $m \nmid n$.*

1) *Function f_i is injective.*
2) *Function f_i is strictly increasing on $\{i_0, i_0+1, \dots\}$ for some $i_0 \in \mathbb{Z}_+$ depending on m, n, t; and $f_{r+1}/f_r = m/\gcd(m, n)$ for all integers $r \geq i_0$. (Here we also call the minimal such i_0 the index of m, n, t.)*

Proof If $n \mid m$ then 2) of Corollary 2.10 implies 1) and 2) of this theorem. Next we assume that $n \nmid m$. We have

$$\begin{aligned}
m =&s_1^{\alpha_1} \cdots s_p^{\alpha_p} m_1^{\beta_1} \cdots m_q^{\beta_q}, \\
n =&s_1^{\alpha_1} \cdots s_p^{\alpha_p} n_1^{\gamma_1} \cdots n_u^{\gamma_u}, \\
t =&s_1^{\delta_1} \cdots s_p^{\delta_p} m_1^{\epsilon_1} \cdots m_q^{\epsilon_q} n_1^{\mu_1} \cdots n_u^{\mu_u} t_1^{\nu_1} \cdots t_v^{\nu_v},
\end{aligned}$$

where $s_1, \dots, s_p, m_1, \dots, m_q, n_1, \dots, n_u, t_1, \dots, t_v$ are pairwise different prime numbers; $\alpha_1, \dots, \alpha_p \in \mathbb{N}$; $\beta_1, \dots, \beta_q \in \mathbb{Z}_+$; $\gamma_1, \dots, \gamma_u \in \mathbb{Z}_+$; $\delta_1, \dots, \delta_p \in \mathbb{N}$; $\epsilon_1, \dots, \epsilon_q \in \mathbb{N}$; $\mu_1, \dots, \mu_u \in \mathbb{N}$; $\nu_1, \dots, \nu_v \in \mathbb{N}$; $s_1^{\alpha_1} \cdots s_p^{\alpha_p} = \operatorname{lcm}(m, n)$.

By a direct computation, we have

$$\begin{aligned}f_i =& s_1^{\max\{\alpha_1,\delta_1\}}\cdots s_p^{\max\{\alpha_p,\delta_p\}} m_1^{i\beta_1+\epsilon_1}\cdots m_q^{i\beta_q+\epsilon_q}\\ & n_1^{\max\{i\gamma_1,\mu_1\}-i\gamma_1}\cdots n_u^{\max\{i\gamma_u,\mu_u\}-i\gamma_u}\\ & t_1^{\nu_1}\dots t_v^{\nu_v}.\end{aligned}$$

Then for all positive integers j,k, $f_j=f_{j+k}$ implies that $j\beta_1+\epsilon_1=(j+k)\beta_1+\epsilon_1$, $\dots$, $j\beta_q+\epsilon_q=(j+k)\beta_q+\epsilon_q$, hence $\beta_1=\dots=\beta_q=0$, i.e., $m\mid n$, which is a contradiction. That is, 1) holds.

On the other hand, for each sufficiently large $r\in\mathbb{Z}_+$, we have

$$\begin{aligned}f_{r+1}/f_r =& m_1^{\beta_1}\cdots m_q^{\beta_q}\\ & n_1^{\max\{(r+1)\gamma_1,\mu_1\}-\max\{r\gamma_1,\mu_1\}-\gamma_1}\cdots\\ & n_u^{\max\{(r+1)\gamma_u,\mu_u\}-\max\{r\gamma_u,\mu_u\}-\gamma_u}\\ =& m_1^{\beta_1}\cdots m_q^{\beta_q}=m/\gcd(m,n),\end{aligned}$$

i.e., 2) holds, which completes the proof. □

Remark 2.2 *It is easy to obtain that for $m=2$, $n=3$, $t=9$, the corresponding f_i satisfies that $f_1=6$, $f_i=2^i$, where $1<i\in\mathbb{Z}_+$. That is, when $m\nmid n$, f_i is not alway strictly increasing.*

We next characterize the long-term behavior of system (7) as our third main result.

Definition 2.12 *A system (7) is called dimension-bounded if $m\mid n$. Consider a dimension-bounded system (7) and a positive integer t, denote the index of m,n,t by $i_0=\operatorname{ind}(m,n,t)$ and the representation of A by $A_{\mathcal{L}}=(A\otimes I_{r/n})(I_{f_{i_0}}\otimes \mathbf{1}_{r/f_{i_0}})\in\mathbb{R}^{f_{i_0}\times f_{i_0}}$, where f_{i_0} is as in (10), $r=\operatorname{lcm}(n,f_{i_0})$. The limit set of a dimension-bounded system (7) with respect to t is defined as $\Omega_A:=\cap_{s=i_0}^{\infty}A^s\vec{\ltimes}\mathbb{R}^t$. The generalized inverse system of a dimension-bounded system with respect to t is defined as the system*

$$x(\tau+1)=(A_{\mathcal{L}})^D\vec{\ltimes}x(\tau), \tag{11}$$

where $\tau=0,1,\dots$, $x(\tau)\in\mathcal{V}$.

For a matrix $A\in\mathbb{R}^{m\times n}$ satisfying $m\mid n$, i.e., A is dimension-bounded, and a positive integer t, denote the index of m,n,t by i_0, we have $A^{i_0}\in\mathbb{R}^{m\times(n^{i_0}/m^{i_0-1})}$. Hence

$$\begin{aligned}A^{i_0}\vec{\ltimes}\mathbb{R}^t&=(A^{i_0}\otimes I_{f_{i_0}/m})(I_t\otimes\mathbf{1}_{(n^{i_0}f_{i_0})/(m^{i_0}t)})\mathbb{R}^t,\\ &=:A_{\mathcal{L}_0}\mathbb{R}^t,\end{aligned} \tag{12}$$

which is a subspace of $\mathbb{R}^{f_{i_0}}$, where f_{i_0} is as in (10), $A_{\mathcal{L}_0}\in\mathbb{R}^{f_{i_0}\times t}$. Hence $\Omega_A=\cap_{i=0}^{\infty}(A_{\mathcal{L}})^iA_{\mathcal{L}_0}\mathbb{R}^t$, where $A_{\mathcal{L}}\in\mathbb{R}^{f_{i_0}\times f_{i_0}}$ is as in Definition 2.12. Based on these analysis, the long-term behavior of the dimension-bounded matrix A on $\mathbb{R}^t$ is as shown in (13).

By Theorem 2.11, for a matrix $A\in\mathbb{R}^{m\times n}$ satisfying $m\nmid n$, i.e, A is not dimension-bounded, and a positive integer t, denote the index of m,n,t by i_0, we have that function f_i as in (8) is injective and satisfies $f_{i_0}<f_{i_0+1}<\cdots$. The long-term behavior of the non-dimension-bounded matrix A on $\mathbb{R}^t$ is as shown in (14).

Since for each $i\in\mathbb{N}$, $(A_{\mathcal{L}})^iA_{\mathcal{L}_0}\mathbb{R}^t$ is a subspace of $\mathbb{R}^{f_{i_0}}$, $\cap_{k=0}^{i}(A_{\mathcal{L}})^k(A_{\mathcal{L}_0}\mathbb{R}^t)=:\mathcal{A}_i$ is also a subspace of $\mathbb{R}^{f_{i_0}}$, and $\mathcal{A}_{i+1}\subset\mathcal{A}_i$, we have $\Omega_A=\cap_{k=0}^{\infty}\mathcal{A}_k=\mathcal{A}_l=\mathcal{A}_{l+l'}$ for some $l\in\mathbb{Z}_+$ and all $l'\in\mathbb{Z}_+$. On the other hand, we have $\Omega_A=\cap_{i=0}^{\infty}(A_{\mathcal{L}})^iA_{\mathcal{L}_0}\mathbb{R}^t=\cap_{i=0}^{\infty}(A_{\mathcal{L}})^i\operatorname{im}(A_{\mathcal{L}_0})\subset\cap_{i=0}^{\infty}(A_{\mathcal{L}})^i\mathbb{R}^{f_{i_0}}=\operatorname{im}((A_{\mathcal{L}})^{\operatorname{ind}(A_{\mathcal{L}})})$. That is, the following theorem holds.

Theorem 2.13 *For a dimension-bounded system (7) with respect to $t\in\mathbb{Z}_+$, its limit set Ω_A is a subspace of $\mathbb{R}^{f_{i_0}}$, satisfies $\Omega_A\subset\operatorname{im}((A_{\mathcal{L}})^{\operatorname{ind}(A_{\mathcal{L}})})$, where $i_0=\operatorname{ind}(m,n,t)$, f_{i_0} is as in (10), $A_{\mathcal{L}}$ is as in Definition 2.12.*

Remark 2.3 *For a dimension-bounded system (7) with $m=n$ with respect to m (i.e., a standard discrete-time linear dynamical system), it is obvious that its limit set Ω_A equals $\operatorname{im}(A^{\operatorname{ind}(A)})$. Particularly if A is invertible, then the generalized inverse system is*

$$x(\tau+1)=A^{-1}x(\tau), \tag{15}$$

where $\tau=0,1,\dots$.

Next we give an algorithm to compute its generalized inverse system. The following proposition which can be seen as an extension of [7, Theorem 4.1] over the real field $\mathbb{R}$, is the basis for the designed algorithm. Note that the proof for Proposition 2.14 does not hold for a right Ore domain studied in [7].

Proposition 2.14 *Consider a matrix $A\in\mathbb{R}^{n\times n}$. Then*

$$A^D=A^{\operatorname{ind}(A)}X^{\operatorname{ind}(A)+1}, \tag{16}$$

where $X\in\mathbb{R}^{n\times n}$ satisfies that $A^{\operatorname{ind}(A)+1}X=A^{\operatorname{ind}(A)}$ (Note that such X always exists).

Proof By induction on the dimension, it can be proved that for a matrix $A\in\mathbb{R}^{n\times n}$, there exist invertible matrices $P\in\mathbb{R}^{n\times n}$ and $C\in\mathbb{R}^{r\times r}$ and nilpotent matrix $N\in\mathbb{R}^{(n-r)\times(n-r)}$ such that

$$A=P\left[C\oplus N\right]P^{-1}. \tag{17}$$

Then we have $N^{\operatorname{ind}(A)}=\mathbf{0}$. If we choose $X=P\left[\begin{smallmatrix}X & Y\\ Z & W\end{smallmatrix}\right]P^{-1}\in\mathbb{R}^{n\times n}$ satisfying $A^{\operatorname{ind}(A)+1}X=A^{\operatorname{ind}(A)}$, where $X\in\mathbb{R}^{r\times r}$, then $X=C^{-1}$, $Y=\mathbf{0}$, and $A^{\operatorname{ind}(A)}X^{\operatorname{ind}(A)+1}=P\left[C^{-1}\oplus\mathbf{0}\right]P^{-1}=A^D$. □

Algorithm 2.15 *1) Input a matrix $A\in\mathbb{R}^{n\times n}$, find $\operatorname{ind}(A)$ (e.g., by definition).*

2) Find a solution to linear equation $A^{\operatorname{ind}(A)+1}X=A^{\operatorname{ind}(A)}$ (e.g., by using the Gaussian elimination).

3) Compute the Drazin inverse of A: $A^D=A^{\operatorname{ind}(A)}X^{\operatorname{ind}(A)+1}$.

3 Action of $\mathcal{M}/_\sim$ on $\mathcal{V}/_\leftrightarrow$

Previously we showed that $(\mathcal{V},\vec{\vdash},\cdot)$ does not form a vector space. However, the quotient space of $\mathcal{V}$ under an equivalence relation $\leftrightarrow$ forms a vector space [3].

$$
\begin{array}{ccccccccc}
\mathbb{R}^t & \xrightarrow{A^{i_0}} & A_{\mathcal{L}_0}\mathbb{R}^t & \xrightarrow{A} & A_{\mathcal{L}}A_{\mathcal{L}_0}\mathbb{R}^t & \xrightarrow{A} & (A_{\mathcal{L}})^2A_{\mathcal{L}_0}\mathbb{R}^t & \xrightarrow{A} & \cdots \\
 & & \cap & & \cap & & \cap & & \\
 & & \mathbb{R}^{f_{i_0}} & & \mathbb{R}^{f_{i_0}} & & \mathbb{R}^{f_{i_0}} & &
\end{array} \tag{13}
$$

$$
\begin{array}{ccccccccccc}
\mathbb{R}^t & \xrightarrow{A} & A\vec{\ltimes}\mathbb{R}^t & \xrightarrow{A} & \cdots & \xrightarrow{A} & A^{i_0}\vec{\ltimes}\mathbb{R}^t & \xrightarrow{A} & A^{i_0+1}\vec{\ltimes}\mathbb{R}^t & \xrightarrow{A} & \cdots \\
 & & \cap & & & & \cap & & \cap & & \\
 & & \mathbb{R}^{f_1} & & & & \mathbb{R}^{f_{i_0}} & & \mathbb{R}^{f_{i_0+1}} & &
\end{array} \tag{14}
$$

Definition 3.1 *[[3]] For all $x, y \in \mathcal{V}$,*

$$x \leftrightarrow y \text{ if and only if } x \otimes \mathbf{1}_s = y \otimes \mathbf{1}_t \tag{18}$$

for some $s, t \in \mathbb{Z}_+$.

Proposition 3.2 ([3]) *1) For all $x, y \in \mathcal{V}$, if $x \leftrightarrow y$ then $x = z \otimes \mathbf{1}_s$ and $y = z \otimes \mathbf{1}_t$ for some $z \in \mathcal{V}$ and $s, t \in \mathbb{Z}_+$.*

2) For all $x \in \mathcal{V}$, in the equivalence class $[x] := \{y \in \mathcal{V} | y \leftrightarrow x\}$, there exists a unique vector $x_0 \in \mathcal{V}$ (called the irreducible element) such that for any $y \leftrightarrow x$, $y = x_0 \otimes \mathbf{1}_k$ for some $k \in \mathbb{Z}_+$. Hence $[x] = \{x_0 \otimes \mathbf{1}_k | k \in \mathbb{Z}_+\}$.

3) For all $x, x', y, y' \in \mathcal{V}$, if $x \leftrightarrow x'$ and $y \leftrightarrow y'$ then $x \vec{\pm} y \leftrightarrow x' \vec{\pm} y'$ and $x \vec{\vdash} y \leftrightarrow x' \vec{\vdash} y'$.

By 3) of Proposition 3.2 the vector addition and vector subtraction of equivalence classes can be defined as follows.

Definition 3.3 *The vector addition and vector subtraction of equivalence classes induced by the equivalence relation $\leftrightarrow$ as in Definition 3.1 are defined as follow: For all $x, y \in \mathcal{V}$,*

$$[x] \vec{\pm} [y] := [x \vec{\pm} y], \quad [x] \vec{\vdash} [y] := [x \vec{\vdash} y]. \tag{19}$$

It is not difficult to verify that $(\mathcal{V}/_{\leftrightarrow}, \vec{\pm}, \cdot)$ ($\mathcal{V}/_{\leftrightarrow}$ for short) forms a vector space, where $\mathcal{V}/_{\leftrightarrow} := \{[x] | x \in \mathcal{V}\}$ is the quotient space induced by $\leftrightarrow$; scalar multiplication $\cdot : \mathbb{R} \times \mathcal{V}/_{\leftrightarrow} \to \mathcal{V}/_{\leftrightarrow}$ is as $\alpha[x] := [\alpha x]$ for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{V}$; $[0]$ is the zero element; for each $[x] \in \mathcal{V}/_{\leftrightarrow}$, its inverse element is $[-x]$.

Now we give a basis for space $\mathcal{V}/_{\leftrightarrow}$, which shows that $\mathcal{V}/_{\leftrightarrow}$ is of countably infinite dimension. Actually, this basis is similar to the one for a matrix quotient space based on the semitensor product and semitensor addition of matrices given in [6].

Theorem 3.4 *Consider vector space $\mathcal{V}/_{\leftrightarrow}$. The set*

$$\mathcal{B}_{\mathcal{V}} := \{[e_i^j] | i, j \in \mathbb{Z}_+, i \geq j, \gcd(i, j) = 1\} \tag{20}$$

is a basis of the space, where e_i^j is the j-th column of I_i.

Proof To prove this result, we only need to verify that 1) each $[e_i^j]$ is generated by $\mathcal{B}_{\mathcal{V}}$ and 2) every finite elements of $\mathcal{B}_{\mathcal{V}}$ is linearly independent, where $i, j \in \mathbb{Z}_+$, $i \geq j$.

We first verify 1). Given $[e_n^m]$, if $\gcd(m, n) = 1$ then $[e_n^m] \in \mathcal{B}_{\mathcal{V}}$. Next we assume $\gcd(m, n) = k > 1$. We have $e_{n/k}^{m/k} \otimes \mathbf{1}_k - e_n^m = \sum_{i=0}^{k-1} e_n^{m-i}$ and $[e_{n/k}^{m/k}] \in \mathcal{B}_{\mathcal{V}}$. For each $0 \leq i \leq k-1$, if $\gcd(m-i, n) = 1$ then $[e_n^{m-i}] \in \mathcal{B}_{\mathcal{V}}$; else, we do the same decomposition for e_n^{m-i} as for e_n^m. Repeat this step again and again, we obtain that $[e_n^m]$ is a linear combination of finitely many elements of $\mathcal{B}_{\mathcal{V}}$. Hence $\mathcal{V}/_{\leftrightarrow}$ is generated by $\mathcal{B}_{\mathcal{V}}$.

Second we verify 2). Actually, we only need to verify for each $k \in \mathbb{Z}_+$, the vectors $[e_i^j]$, $i, j \in \{1, \ldots, k\}, i \geq j, \gcd(i, j) = 1$ are linearly independent. Denoting $l := \operatorname{lcm}(1, \ldots, k)$, we obtain vectors $e_i^j \otimes \mathbf{1}_{l/i} \in \mathbb{R}^l$, $i, j \in \{1, \ldots, k\}, i \geq j, \gcd(i, j) = 1$, where for each e_i^j, the jl/i-th entry equals 1, and any t-th entry with $t > jl/i$ equals 0. Note that jl/i, where $i, j \in \{1, \ldots, k\}, i \geq j, \gcd(i, j) = 1$, are pairwise different, hence these vectors are linearly independent, and the vectors $[e_i^j]$, $i, j \in \{1, \ldots, k\}, i \geq j, \gcd(i, j) = 1$ are also linearly independent, which completes the proof. □

4 Conclusion

In this paper, we characterized a so-called cross-dimensional vector space and the long-term behavior of cross-dimensional dynamical systems. Specifically, we give a basis for the cross-dimensional vector space, showing that the space is of countably infinite dimension. In addition, we characterized the long-term behavior of repetitive actions of a matrix on a vector. Further results will be followed along this line.

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