

Towards a Geometric Theory of Hybrid Systems

Slobodan N. Simić¹ Karl Henrik Johansson²

John Lygeros³ and Shankar Sastry⁴

¹Department of Mathematics
San José State University, San José, CA 95192, USA

²Department of Signals, Sensors & Systems, Royal Institute of Technology, 100 44
Stockholm, Sweden

³Department of Electrical and Computer Engineering, University of Patras, Rio, Patras,
GR 26500, Greece

⁴Department of Electrical Engineering and Computer Sciences
University of California, Berkeley, CA 94720, USA

Abstract. We propose a framework for a geometric theory of hybrid systems. Given a deterministic, non-blocking hybrid system, we introduce the notion of its hybridfold with the associated hybrid flow on it. This enables us to study hybrid systems from a global geometric perspective as (generally non-smooth) dynamical systems. This point of view is adopted in studying the Zeno phenomenon. We show that it is due to nonsmoothness of the hybrid flow. We introduce the notion of topological equivalence of hybrid systems and locally classify isolated Zeno states in dimension two.

Keywords. Hybrid system; dynamical system; hybridfold; Zeno; topological equivalence.

AMS (MOS) subject classification: 37C10, 37C99, 34C99, 37N35

1 Introduction

Any system which changes in time and involves interactions between continuous processes and discrete automata can be viewed as a hybrid system. The name comes from the hybrid nature of its evolution: intervals of continuous-time change are interleaved with instantaneous, discrete jumps. Hybrid systems are a topic of great interest, primarily in the control and computer science communities. Emerging applications in areas such as air traffic management [37], automotive control [5, 1], real-time software verification [2], transportation systems [39, 25], manufacturing [32], mobile robotics [6], and process industry [13], motivate work on hybrid systems modeling [11, 28, 4, 38, 10, 20], analysis [3, 40, 9, 22, 17, 23], and controller design [29, 8, 10, 27, 7]. Although extensive research efforts have been made in the area of hybrid systems, the understanding of the fundamental dynamical properties of systems

with interacting continuous-time and discrete-event dynamics [11, 20, 16, 21, 42] is not satisfactory.

In this paper we present a geometric approach for treatment of a class of hybrid systems which resemble deterministic continuous-time piecewise smooth dynamical systems. Early results in this direction were first reported in [33]. We introduce the notions of the *hybrifold* and *hybrid flow*, which enable us to study the hybrid system “in one piece”, that is, as a single, piecewise smooth dynamical system. It is well known that even simple *smooth* dynamical systems can exhibit very complicated behavior which makes their global study using analytical methods very difficult. This is why developing qualitative (i.e., geometric and topological) techniques has been at the center of modern dynamics, ever since Poincaré’s seminal work at the end of the 19th and beginning of the 20th century. For a beautiful introduction to this field, the reader is referred to [31]. In this article, we make a first step towards developing such techniques for hybrid systems.

Having established a suitable framework, we focus particularly on the *Zeno phenomenon*, which does not occur in smooth dynamical systems. We study its underlying geometric causes and classify it in dimension two. This classification is with respect to the notion of *topological equivalence* borrowed from dynamical systems: two systems are equivalent if they are qualitatively the same.

The advantages of the hybrifold method are the following.

- It enables one to use the standard techniques of the theory of dynamical systems to study hybrid systems (which have a hybrifold).
- It provides a convenient framework for global analysis of hybrid systems, giving insight into global behavior of hybrid executions, which is not possible to obtain by local, domain-by-domain, analysis.
- It introduces a convenient setting for qualitative analysis of hybrid systems; for instance, it simplifies the definitions of topological equivalence and structural stability of hybrid systems.

Furthermore, we distill the key properties of a class of examples into a list of assumptions, and study all hybrid systems which share these properties.

The organization of the paper is the following. In Section 2 we define the model, present some basic examples, and formulate the standing assumptions. In Section 3 we introduce and discuss the basic properties of the hybrifold and hybrid flow. Section 4 is concerned with the geometry of the Zeno phenomenon and Section 5 introduces topological equivalence of hybrid systems and classifies isolated Zeno states in dimension two. Finally, we make a few concluding remarks in Section 6.

2 Preliminaries

Definitions and examples

Let us first define our model and give some basic examples which will be useful later in the paper. The following definitions are motivated by [20, 26].

Definition 2.1. *An n -dimensional hybrid system is a 6-tuple*

$$\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R}),$$

where:

- $Q = \{1, \dots, k\}$ is the finite set of (discrete) states of \mathbf{H} , where $k \geq 1$ is an integer;
- $E \subset Q \times Q$ is the collection of edges;
- $\mathcal{D} = \{D_i : i \in Q\}$ is the collection of domains of \mathbf{H} , where $D_i \subset \{i\} \times \mathbb{R}^n$, for all $i \in Q$;
- $\mathcal{X} = \{X_i : i \in Q\}$ is the collection of vector fields such that X_i is Lipschitz on D_i for all $i \in Q$; we denote the local flow of X_i by $\{\phi_i^t\}$.
- $\mathcal{G} = \{G(e) : e \in E\}$ is the collection of guards, where for each $e = (i, j) \in E$, $G(e) \subset D_i$;
- $\mathcal{R} = \{R_e : e \in E\}$ is the collection of resets, where for each $e = (i, j) \in E$, R_e is a relation between elements of $G(e)$ and elements of D_j , i.e., $R_e \subset G(e) \times D_j$.

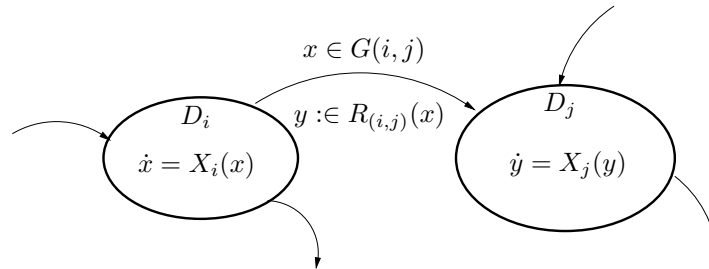


Figure 1: A hybrid system.

Remark 1. (a) Our terminology differs slightly from the usual terminology for hybrid systems, where the *domains* are called the *invariants*. Since there is nothing dynamically invariant about these sets, we prefer to reserve the term invariant set for later, more appropriate use.

- (b) We do not consider the set of *initial states* as a separate item in the definition of a hybrid system. This is because we restrict ourselves to studying only so called non-blocking systems in which all points can be initial conditions. The only assumption we will make subsequently is that the initial state is in one of the domains.
- (c) The above definition clearly allows a hybrid system to be a wild object. An important question is: *what properties should the domains, guards, resets and vector fields of \mathbf{H} satisfy to get a large enough class of hybrid systems about which something useful can be said?* Soon we will deal with this question in detail and focus our attention on one such class of hybrid systems (which we will call *regular*).
- (d) If a reset relation R_e is a map $G(e) \rightarrow D_j$, with $e = (i, j) \in E$, we write $y = R_e(x)$ instead of $(x, y) \in R_e$.
- (e) We also set

$$D = \bigcup_{i \in Q} D_i,$$

and call this set the *total domain* of \mathbf{H} . If the resets are maps, we write

$$G = \bigcup_{e \in E} G(e), \quad R = \bigcup_{e \in E} R_e(G(e)),$$

$$\overline{G} = \{\overline{G(e)} : e \in E\}, \quad \overline{R} = \{\overline{R_e(G(e))} : e \in E\}.$$

Therefore, G is the *union* of all guards, \overline{G} is the *collection* of closures of all guards, etc.

- (f) Observe that domains D_i lie in distinct copies of \mathbb{R}^n , namely $\{i\} \times \mathbb{R}^n$, $i \in Q$. However, to simplify things, we will sometimes abuse the notation and drop the i . More precisely, we identify $\{i\} \times \mathbb{R}^n$ with \mathbb{R}^n via the diffeomorphism $F_i : (i, x) \mapsto x$, and write $D_i, G(e), R_e, X_i, \dots$ for $F_i(D_i), F_i(G(e)), F_i \circ R_e \circ F_j^{-1}, TF_i \circ X_i \circ F_i^{-1}, \dots$, where $e = (i, j) \in E$ and TF_i is the derivative or tangent map of F_i .

Given \mathbf{H} , the basic idea is that starting from a point in some domain D_i , we flow according to X_i until (and if) we reach some guard $G(i, j)$. At that point it is possible for the state to switch via the reset $R_{(i, j)}$ and continue flowing in D_j according to X_j , and so on.

Example 2.2 (Water Tank *WT*). Consider the water tank system discussed in [4, 18, 20] (Fig. 2). For $i \in Q = \{1, 2\}$, x_i denotes the volume of water in tank i , v_i is the constant rate of flow of water out of tank i , and l_i is the desired volume of water in tank i . The constant rate of water flow into the system, dedicated exclusively to one tank at a time, is denoted by w . The control task is to keep the water volume above l_1 and l_2 (assuming the

initial volumes are above l_1 and l_2 respectively) by a strategy that switches the inflow to the first tank whenever $x_1 = l_1$ and to the second tank whenever $x_2 = l_2$.

Here $n = 2$, $k = 2$, $E = \{(1, 2), (2, 1)\}$,

$$D_1 = \{1\} \times C, \quad D_2 = \{2\} \times C,$$

where $C = [l_1, \infty) \times [l_2, \infty)$,

$$X_1 = (w - v_1, -v_2)^T, \quad X_2 = (-v_1, w - v_2)^T,$$

$$G(1, 2) = \{(1, x_1, x_2) \in D_1 : x_2 = l_2\}, \quad G(2, 1) = \{(2, x_1, x_2) \in D_2 : x_1 = l_1\},$$

and

$$R_{(1,2)}(1, x_1, l_2) = (2, x_1, l_2), \quad R_{(2,1)}(2, l_1, x_2) = (1, l_1, x_2).$$

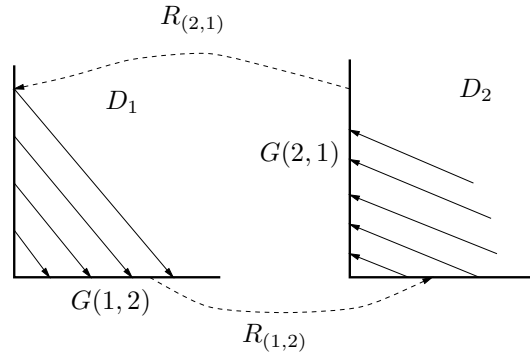


Figure 2: The water tank example.

We will assume that $\max(v_1, v_2) < w < v_1 + v_2$. This means that more water is flowing out of than into the tanks.

Example 2.3 (Bouncing Ball BB). This is a simplified model of an elastic ball that is bouncing and losing a fraction of its energy with each bounce, discussed in [20]. We denote by x_1 the altitude of the ball and by x_2 its vertical speed. Here $n = 2$, $k = 1$, $E = \{(1, 1)\}$,

$$D_1 = \{(1, x_1, x_2) : x_1 \geq 0\}, \quad X_1(1, x_1, x_2) = (x_2, -g)^T,$$

$$G(1, 1) = \{(1, 0, x_2) : x_2 \leq 0\}, \quad R_{(1,1)}(1, 0, x_2) = (1, 0, -cx_2),$$

where g is the acceleration due to gravity and $0 < c < 1$ (Fig. 3).

Example 2.4 (Bouncing m -Ball $BB(m)$). The only difference between this and the previous example is that we have m different domains in which the ball can bounce. After each bounce, the ball switches to the next domain

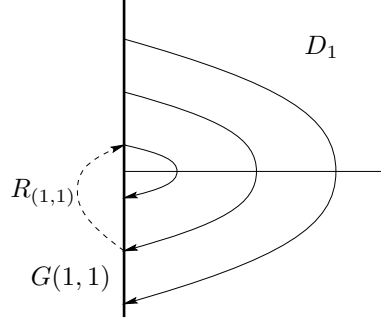


Figure 3: Bouncing ball.

in a cyclical order. That is, $n = 2$, $k = m > 1$, $E = \{(1, 2), (2, 3), \dots, (m - 1, m), (m, 1)\}$, and for all $i \in Q$,

$$D_i = \{i\} \times \{(x_1, x_2) : x_1 \geq 0\}, G(i, i + 1) = \{i\} \times \{(0, x_2) : x_2 \leq 0\},$$

$$R_{(i, i+1)}(i, 0, x_2) = (i + 1, 0, -cx_2),$$

where we conveniently identify $m + 1 := 1$. Note that here the domains are just different copies of the closed right half-plane in \mathbb{R}^2 .

Example 2.5 (Ball Bouncing on an N -step Staircase $BBS(N)$). Here we have a staircase with steps $i = 1, \dots, N$ of width $w_i > 0$ and height $h_i > 0$ on which a ball is bouncing. Define $\hat{w}_m = \sum_{i=1}^m w_i$ and $\hat{h}_m = \sum_{i=m}^N h_i$. Assume also that with each bounce the ball loses a proportional amount of its vertical velocity (x_2) and that it has constant horizontal speed (x_3). Denote by x_1 its vertical position. Then we have: $Q = \{1, \dots, N + 1\}$, $E = \{(i, i) : 1 \leq i \leq N + 1\} \cup \{(1, 2), \dots, (N, N + 1)\}$, and for $1 \leq i \leq N$,

$$D_i = \{i\} \times [\hat{h}_i, \infty) \times (-\infty, \infty) \times (\hat{w}_{i-1}, \hat{w}_i],$$

$$G(i, i) = \{(i, x_1, x_2, x_3) \in D_i : x_1 = \hat{h}_i, x_2 \leq 0, x_3 < \hat{w}_i\},$$

$$G(i, i + 1) = \{(i, x_1, x_2, x_3) \in D_i : x_3 = \hat{w}_i\}, \quad R_{(i, i+1)}(i, \mathbf{x}) = (i + 1, \mathbf{x}).$$

For $1 \leq i \leq N + 1$,

$$X_i(i, x_1, x_2, x_3) = (x_2, -g, v)^T, \quad R_{(i, i)}(i, x_1, x_2, x_3) = (i, x_1, -cx_2, x_3).$$

Finally,

$$G(N + 1, N + 1) = \{(N + 1, x_1, x_2, x_3) \in D_{N+1} : x_1 = 0, x_2 \leq 0\},$$

$$D_{N+1} = \{N + 1\} \times [0, \infty) \times (-\infty, \infty) \times (\hat{w}_N, \infty).$$

Example 2.6 (Two Saddles $S2(\lambda)$). Here (see Fig. 4) $n = 2$, $k = 2$, $0 < \lambda < 1$, $E = \{(1, 2), (2, 1)\}$, the domains are two copies of the square $S = [-1, 1] \times [-1, 1]$, i.e. for $i \in Q$,

$$D_i = \{i\} \times S, \quad X_1(1, x_1, x_2) = (\lambda x_1, -x_2)^T, \quad X_2(x_1, x_2) = (2, -x_1, \lambda x_2)^T,$$

$$G(1, 2) = \text{union of the vertical sides of } D_1,$$

$$G(2, 1) = \text{union of the horizontal sides of } D_2,$$

$$R_{(i,j)}(i, x) = (j, x), \text{ for all } (i, j) \in E.$$

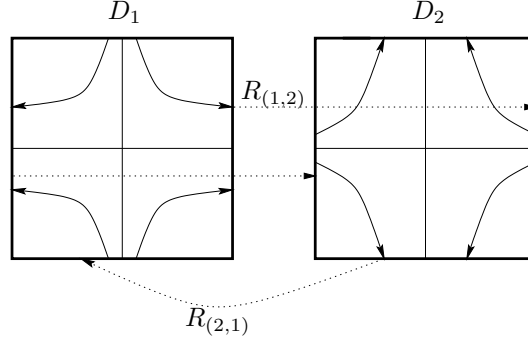


Figure 4: $S2(\lambda)$

Example 2.7 (Flow on the 2-torus $T^2(\alpha)$). We have $\alpha > 0$, $n = 2$, $k = 2$, $E = \{(1, 2), (2, 1)\}$, $D_i = \{i\} \times K$, where $K = [0, 1] \times [0, 1]$ is the unit square, $X_1 = X_2 = (1, \alpha)^T$ are constant vector fields,

$$G(i, i) = \{i\} \times S_{\text{upper}}, \quad G(i, j) = \{i\} \times S_{\text{right}},$$

$$R_{(i,i)}(i, x, 1) = (i, x, 0) \quad \text{and} \quad R_{(i,j)}(i, 1, y) = (j, 0, y),$$

where $i, j = 1, 2$, $i \neq j$, $S_{\text{upper}} = [0, 1] \times \{1\}$ and $S_{\text{right}} = \{1\} \times [0, 1]$ denote the (closed) upper and (half-closed) right side of K . Note that $R_{(i,i)}(\{i\} \times S_{\text{upper}}) = \{i\} \times S_{\text{lower}}$ and $R_{(i,j)}(\{i\} \times S_{\text{right}}) = \{j\} \times S_{\text{left}}$, with the obvious meaning of S_{lower} and S_{left} (see Fig. 5).

If we proceed in the usual way and identify $\{i\} \times S_{\text{upper}}$ with $\{i\} \times S_{\text{lower}}$ via $R_{(i,i)}$ and $\{i\} \times S_{\text{right}}$ with $\{j\} \times S_{\text{left}}$ via $R_{(i,j)}$ (where $i, j = 1, 2$, $i \neq j$), we obtain the standard 2-torus with a *smooth* straight-line flow with slope α on it. This is a “baby-version” of a construction we will later apply to more general hybrid systems.

Keeping in mind the examples above, we now define the notions of a hybrid time trajectory and an execution of a hybrid system.

Definition 2.8. A (forward) hybrid time trajectory is a sequence (finite or infinite) $\tau = \{I_j\}_{j=0}^N$ of intervals such that $I_j = [\tau_j, \tau'_j]$ for all $j \geq 0$ if the sequence is infinite; if N is finite, then $I_j = [\tau_j, \tau'_j]$ for all $0 \leq j \leq N-1$ and I_N is either of the form $[\tau_N, \tau'_N]$ or $[\tau_N, \tau'_N)$. Furthermore, $\tau_j \leq \tau'_j = \tau_{j+1}$, for all j .

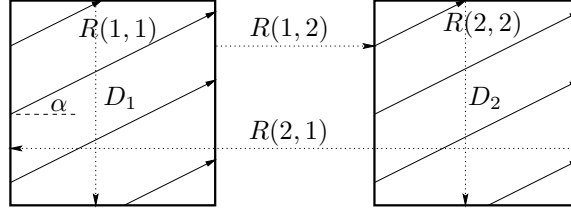


Figure 5: $T^2(\alpha)$.

One thinks of τ_j 's as time instants when discrete transitions (or switches) from one domain to another take place. If $\tau = \{I_j\}_{j=0}^N$ is a hybrid time trajectory, we will call N its *size* and denote it by $N(\tau)$. Also, we use $\langle \tau \rangle$ to denote the set $\{0, \dots, N(\tau)\}$ if $N(\tau)$ is finite, and $\{0, 1, 2, \dots\}$ if $N(\tau)$ is infinite.

We will say that τ is a *prefix* of a hybrid time trajectory τ' if $N(\tau) \leq N(\tau')$ (where the inequality is taken in the extended real number system), and for $0 \leq j < N(\tau)$, we have $I_j = I'_j$; furthermore, if $N(\tau) < \infty$, then we must also have $I_{N(\tau)} \subset I'_{N(\tau)}$.

Definition 2.9. An execution (or forward execution) of a hybrid system \mathbf{H} is a triple $\chi = (\tau, q, x)$, where τ is a hybrid time trajectory, $q : \langle \tau \rangle \rightarrow Q$ is a map, and $x = \{x_j : j \in \langle \tau \rangle\}$ is a collection of C^1 maps such that $x_j : I_j \rightarrow D_{q(j)}$ and for all $t \in I_j$,

$$\dot{x}_j(t) = X_{q(j)}(x_j(t)). \quad (1)$$

Furthermore, for all $j \in \langle \tau \rangle$ such that $j < N(\tau)$, we have

$$(q(j), q(j+1)) \in E,$$

and

$$(x_j(\tau'_j), x_{j+1}(\tau_{j+1})) \in R_{(q(j), q(j+1))}.$$

Observe the abuse of notation in (1): the curve $x_j(t)$ is in $D_{q(j)} \subset \{q(j)\} \times \mathbb{R}^n$, but we identify its tangent with a vector in \mathbb{R}^n , dropping the discrete part $q(j)$. See also Remark (f) above.

We say that an execution $\chi = (\tau, q, x)$ *starts* at a point $p \in D$ if $p = x_0(\tau_0)$ and $\tau_0 = 0$. It *passes through* p if $p = x_j(t)$ for some $j \in \langle \tau \rangle$, $t \in I_j$. A hybrid system is usually said to *accept* an execution. Given $p \in D$, it is not difficult

to see that there are many ways in which a hybrid system can accept several executions starting from or passing through p . For instance, this happens if at least one of the resets is a relation which is not a function.

For an execution $\chi = (\tau, q, x)$, denote by $\tau_\infty(\chi)$ its (*forward*) *execution time*:

$$\tau_\infty(\chi) = \sum_{j=0}^{N(\tau)} (\tau'_j - \tau_j) = \lim_{j \rightarrow N(\tau)} \tau'_j - \tau_0.$$

Definition 2.10. *An execution χ is called:*

- *infinite, if $N(\tau) = \infty$ or $\tau_\infty(\chi) = \infty$;*
- *a Zeno execution if $N(\tau) = \infty$ and $\tau_\infty(\chi) < \infty$;*
- *maximal if it is not a strict prefix of any other execution of \mathbf{H} .*

In other words, an execution is Zeno if it takes an infinite number of discrete transitions in a finite amount of time. An execution is maximal if there exists no other execution $\chi' = (\tau', q', x')$ such that τ is a strict prefix of τ' and $x_j = x'_j$ on I_j for all $j \in \langle \tau \rangle$.

We will call a Zeno execution *static* if there exists $l \geq 0$ such that $\tau'_j = \tau_j$, for all $j \geq l$. Otherwise, it will be called *dynamic*.

In Examples 2.2 (*WT*), 2.3 (*BB*), and 2.4 (*BB(m)*) for every $p \neq (0, 0)$, the maximal execution starting from p is dynamic Zeno. The same can be shown for Examples 2.5 (*BBS(N)*) if $0 < c < 1$ and 2.6 (*S2(λ)*) if $0 < \lambda < 1$ and p belongs to neither axis. On the other hand, every maximal execution in Example 2.7 (*T²(α)*) is infinite with infinite execution time. For more examples and a connection with Filippov solutions in the context of differential equations with discontinuous right-hand sides, see [14].

Definition 2.11. *A hybrid system is called deterministic if for every $p \in D$ there exists at most one maximal execution starting from p . It is called non-blocking if for every $p \in D$ there is at least one infinite execution starting from p .*

Necessary and sufficient conditions for a hybrid system to be deterministic and non-blocking can be found in [26, 24, 21] and for special classes of systems in [38, 19, 15]. Roughly speaking, resets have to be functions, guards have to be mutually disjoint and whenever a continuous trajectory of one of the vector fields in \mathcal{X} is about to exit the domain in which it lies, it has to hit a guard. It is easy to show (using the conditions in [26, 21] for example) that all the examples introduced above are non-blocking and deterministic.

Standing assumptions

To obtain a class of hybrid systems sufficiently tame for our study yet sufficiently broad to include a fair number of important examples, we impose the

following standing assumptions on the hybrid system $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$ under consideration. They will hold in the rest of the paper. Observe that our assumptions on the continuous dynamics are very moderate when compared to, for instance, timed hybrid automata; in each domain, we allow general nonlinear vector fields with a reasonable degree of regularity.

The first assumption deals with uniqueness of executions.

(A1) \mathbf{H} is deterministic and non-blocking.

This means that every point in D is the starting point of a unique infinite (and therefore maximal) execution of \mathbf{H} .

The second assumption deals with the geometry of domains.

(A2) There exists a d such that each domain D_i is a connected n -dimensional smooth submanifold of \mathbb{R}^d , with piecewise smooth boundary. The angle between any two intersecting smooth components of the boundary is nonzero.

Recall that a manifold is called *piecewise smooth* if, intuitively speaking, it is the union of finitely many smooth pieces. By smooth we will mean of class C^∞ , unless specified otherwise. Since each domain is embedded into \mathbb{R}^d , it inherits from it the standard Riemann structure so the notion of angle is defined. The non-zero angle requirement eliminates, for instance, “cusps” in dimension two, but does not eliminate “corners”. Thus for domains of a hybrid system we allow disks, half-spaces, rectangles, cubes, etc. The non-zero angle assumption can be easily relaxed for most of the results, in fact it is only necessary in Section 4.

The next assumption deals with the geometry of guards.

(A3) Each guard is a smooth $(n-1)$ -dimensional submanifold of the boundary of the corresponding domain. The boundary of each guard is piecewise smooth (or possibly empty).

There is an intuitive reasoning behind this assumption which explains why we don’t allow guards to be n -dimensional and why we require them to be part of the boundary of domains. If $G(i, j)$ is a guard of dimension n , e.g., a closed set with nonempty interior in D_i , and $x(t)$ is a trajectory in the domain D_i which meets $G(i, j)$, the corresponding execution χ jumps to D_j as soon as $x(t)$ hits the boundary of $G(i, j)$. Therefore, the interior of $G(i, j)$ will be irrelevant for the evolution of χ . If we ignore the hybrid dynamics starting in the interior of $G(i, j)$, then we can in effect remove that interior from the system as superfluous. This way we create $(n-1)$ -dimensional guards which lie on the boundary of the new domains.

The following is an assumption on the resets.

(A4) Each reset is a diffeomorphism from its domain $G(e)$ onto its image.

The image of every reset lies on the boundary of the corresponding domain. Moreover, if $e = (i, j) \in E$ and $R_e(p) = q$, then $X_i(p) = 0$ if and only if $X_j(q) = 0$.

Thus resets take equilibria to equilibria.

The next assumption deals with the relation between guards and images of resets.

(A5) Elements of $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ (i.e., sets which are closures of guards and images of resets) can intersect only along their boundaries. Furthermore, if $p \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$, then p can be of only one of the following four types (Fig. 6):

Type I $p \in \text{int } G \cup \text{int } R$;

Type II $p \in \partial G \cup \partial R$ and there exists exactly **one** set $S \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ which contains p ;

Type III $p \in \partial G \cup \partial R$ and there exist exactly **two** sets $S_1, S_2 \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$, such that $p \in \partial S_1 \cap \partial S_2$ and some neighborhood of p in $S_1 \cup S_2$ is homeomorphic to \mathbb{R}^{n-1} ;

Type IV $p \in \partial G \cup \partial R$ and there exist exactly **two** sets $S_1, S_2 \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$, such that $p \in \partial S_1 \cap \partial S_2$ and some neighborhood of p in $S_1 \cup S_2$ is homeomorphic to \mathbb{R}_+^{n-1} .

Furthermore, resets preserve the type.

The last sentence means that if p is a point of type $T \in \{I, II, III, IV\}$ and f is a reset map, then $f(p)$ is also of type T , provided that $f(p)$ is defined. Further, int and ∂ of G and R are the interior and boundary of G and R , taken in the relative topology of these sets as subsets of the boundary of D . Assumption (A5) ensures that intersections of guards and images of resets (in fact, their closures) are sufficiently nice. The shaded regions in Figure 6 represent sets in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$. They are 2-dimensional manifolds with boundary which lie on the boundary of the 3-dimensional domain D_i . Note the following: a point p of Type III lies on the boundary of two sets in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ and is surrounded by them from all sides; that is, there exists a neighborhood of p in the union of these sets which is homeomorphic to the Euclidean space of the same dimension $(n - 1)$. A point p is of Type IV if it lies on the boundary of two sets in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$, but is not surrounded by them; instead, there is a neighborhood of p in the union of these sets which is homeomorphic to the Euclidean half-space of the same dimension $(n - 1)$. Note that three or more objects in $\overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ are not allowed to meet. We also refer the reader to the proof of Theorem 3.7 where utility of (A5) will become apparent in the construction of the hybrid fold.

The following assumption deals with the vector fields along guards and images of resets.

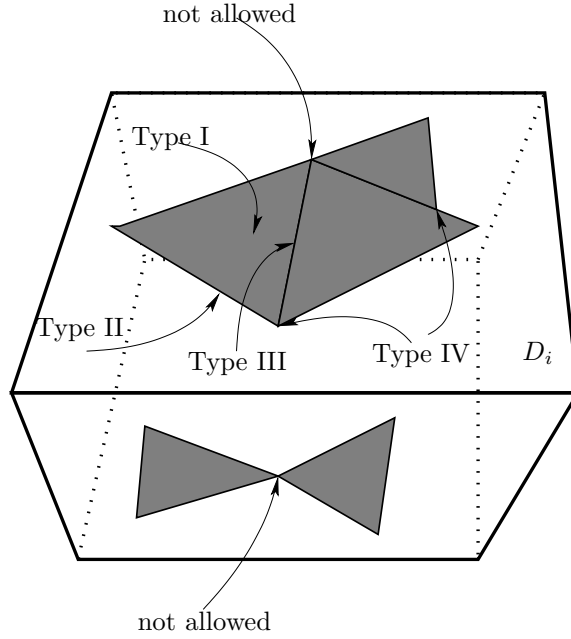


Figure 6: Types of points in (A5).

(A6) For all $e = (i, j) \in E$, the following holds: on $\text{int } G(e)$, X_i points outside $\text{int } D_i$; on $\text{int}(\text{im } R_e)$, X_j points inside \bar{D}_j .

Here the bar denotes closure and im the image (or range) of a map. As above, $\text{int } G(e)$ and $\text{int}(\text{im } R_e)$ denote the interior of $G(e)$ and $\text{im } R_e$ in ∂D_i and ∂D_j , respectively. That a vector field X with flow ϕ_t points inside a set A at $p \in \partial A$ means that there exists $\epsilon > 0$ such that $\phi_t(p) \in A$, for all $0 < t < \epsilon$. It points outside the same set if $-X$ points inside.

The next assumption states that resets and vector fields are smoothly extendable objects.

(A7) Each vector field X_i is the restriction to D_i of some smooth vector field, which we also denote by X_i , defined on a neighborhood of D_i in $\{i\} \times \mathbb{R}^n$. Each reset map R_e extends to a map \tilde{R}_e defined on a neighborhood of $\overline{G(e)}$ in D_i such that \tilde{R}_e is a diffeomorphism onto its image, which is a neighborhood of $\text{im } R_e$ in D_j .

Furthermore, suppose $f_1, \dots, f_m, f_{m+1} := f_1$ is a collection of maps such that for each j there exists $e_j \in E$ with the property that either $f_j = \tilde{R}_{e_j}$ or $f_j = \tilde{R}_{e_j}^{-1}$, and the image of f_j meets the domain of f_{j+1} . Then we have the consistency condition

$$f_m \circ \dots \circ f_2 \circ f_1 = \text{identity}, \quad (2)$$

wherever the left-hand side is defined.

The importance of this assumption is that it enables us to “glue” domains via the extended reset maps and obtain a reasonable geometric object. The condition $f_m \circ \cdots \circ f_2 \circ f_1 = \text{identity}$ simply means that a point of Type III or IV, which lies on the common boundary $C = \partial S_1 \cap \partial S_2$ of some sets $S_1, S_2 \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ will not be identified with any other point in C in the gluing process. In the language of Section 3, for every $p \in C$, $C \cap p / \sim = \{p\}$.

The last assumption makes sure that all the above ones remain satisfied when time is reversed.

(A8) *If $p \in D_i$ is on the boundary of D_i and $X_i(p)$ points inside D_i then p is in the image of some reset.*

Observe that all our examples satisfy all the assumptions above.

Definition 2.12. *A hybrid system which satisfies assumptions (A1)–(A8) will be called regular.*

Given \mathbf{H} , define a map $\Phi^{\mathbf{H}}$ on a subset of $\mathbb{R} \times D$ as follows. Let $p \in D$ be arbitrary. By (A1), there exists a unique infinite execution (τ, q, x) starting at p . We will denote it by $\chi(p)$. Set $\Phi^{\mathbf{H}}(0, p) = p$. Assume that $\tau_\infty(\chi(p)) > 0$. For any $0 < t < \tau_\infty(\chi(p))$ there exist a unique $j \in \langle \tau \rangle$ such that $t \in [\tau_j, \tau'_j)$ (even though there may be multiple $j \in \langle \tau \rangle$ for which $t \in [\tau_j, \tau'_j)$). Then define

$$\Phi^{\mathbf{H}}(t, p) = x_j(t).$$

To define $\Phi^{\mathbf{H}}(t, p)$ for negative t , set

$$\Phi^{\mathbf{H}}(t, p) = \Phi^{\mathbf{H}'}(-t, p),$$

where \mathbf{H}' is the *reverse* hybrid system $(Q', E', \mathcal{D}', \mathcal{X}', \mathcal{G}', \mathcal{R}')$ defined by:

- $Q' = Q$, $\mathcal{D}' = \mathcal{D}$, $X'_i = -X_i$ for all $i \in Q$;
- $(i, j) \in E'$ if and only if $(j, i) \in E$;
- for all $e = (i, j) \in E'$, $G'(e) = R_{(j,i)}(G(j, i))$ and $R'_{(i,j)} = R_{(j,i)}^{-1}$.

It can be checked without difficulty that \mathbf{H}' satisfies (A1)–(A8) if \mathbf{H} does. Denote by Ω_0 the largest subset of $\mathbb{R} \times D$ on which $\Phi^{\mathbf{H}}$ is defined.

In Example 2.3, for instance, for any $p \neq \mathbf{0}$, $\Phi^{BB}(t, p) \rightarrow \mathbf{0}$, as $t \rightarrow \tau_\infty(\chi(p))$. Note, however, that $\chi(\mathbf{0})$ makes no time progress, i.e., $\tau_j = 0$ for all $j \geq 0$, but it involves infinitely many switches at the same (i.e., initial) point, which happens to be a fixed point of the reset map.

Theorem 2.13. *(a) Ω_0 contains a neighborhood of $\{0\} \times \text{int } D$ in $\mathbb{R} \times D$, where $\text{int } D = D - \partial D$.*

(b) For all $p \in D$, $\Phi^{\mathbf{H}}(0, p) = p$. Furthermore,

$$\Phi^{\mathbf{H}}(t, \Phi^{\mathbf{H}}(s, p)) = \Phi^{\mathbf{H}}(t + s, p),$$

whenever both sides are defined.

Proof. (a) If $p \in \text{int } D_i$, then since X_i is Lipschitz on D_i , $t \mapsto \phi_t^i(p)$ is defined on a neighborhood of 0. Furthermore, there exists a neighborhood U of p in $\text{int } D_i$ and $\epsilon > 0$ such that for each $p' \in U$, $t \mapsto \phi_t^i(p')$ is defined on $(-\epsilon, \epsilon)$. Thus $(-\epsilon, \epsilon) \times U$ is a neighborhood of $(0, p)$ in $\mathbb{R} \times \text{int } D_i$. This proves (a).

(b) The first statement is true by definition. The second statement follows from uniqueness of executions, guaranteed by (A1). \square

3 The hybrid and hybrid flow

The basic idea behind the construction of the hybrid of a hybrid system is simple: “glue” each guard to the image of the corresponding reset map. We make this more precise below and then show basic properties of the newly constructed object.

The hybrid

Let \mathbf{H} be a regular hybrid system. On D , let \sim be the equivalence relation generated by

$$p \sim \tilde{R}_e(p),$$

for all $e \in E$ and $p \in \overline{G(e)}$. Collapse each equivalence class to a point to obtain the quotient space

$$M_{\mathbf{H}} = D / \sim.$$

Definition 3.1. We call $M_{\mathbf{H}}$ the hybrid¹ of \mathbf{H} .

Note that $M_{\mathbf{H}}$ does not depend on the choice of the extensions for the reset maps. This is because by continuity any two extensions of a fixed reset R_e necessarily agree on $\overline{G(e)}$. Denote by π the natural projection $D \rightarrow M_{\mathbf{H}}$ which assigns to each p its equivalence class p / \sim . Note also that π can easily be extended to \overline{D} . Put the *quotient topology* on $M_{\mathbf{H}}$. Recall that this is the smallest topology that makes π continuous, i.e., a set $V \subset M_{\mathbf{H}}$ is open if and only if $\pi^{-1}(V)$ is open in D .

Let us apply this construction to some of the examples above.

Example 3.2 (WT continued). Without loss we assume that $l_1 = l_2 = 0$. To obtain M_{WT} we have to identify the x_1 -axis from D_1 with the same axis from D_2 via $R_{(1,2)}$ and similarly with the x_2 -axis (see Fig. 2). It is not

¹The authors thank Renaud Dreyer for suggesting the term hybridfold.

difficult to see that M_{WT} is homeomorphic to \mathbb{R}^2 (see Fig. 7). However, M_{WT} has a singularity (or “corner”) at $\mathbf{0} = \pi(1, 0, 0)$ in the following sense. The space D is equipped with a differentiable structure – the one it inherits from individual domains D_i . The space M_{WT} inherits its own differentiable structure from the ambient space \mathbb{R}^3 . With respect to these two differentiable structures, π is not a smooth map. When this is the case, we will say that π does not define a smooth structure on M_{WT} . As an alternative possibility, consider the standard construction of the 2-torus by identifying the opposite sides of a square (cf., Example 3.6); there, π is a smooth map in the above sense.

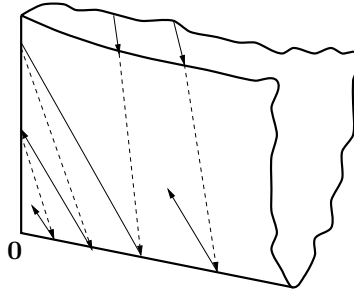


Figure 7: Hybridfold and two orbits of the hybrid flow for WT .

Example 3.3 (BB continued). Here we have to identify the negative part with the positive part of the x_2 -axis (see Fig. 3). The resulting space M_{BB} is again homeomorphic to \mathbb{R}^2 (see Fig. 8), but π again does not define a smooth structure on it in the same sense as above.

Example 3.4 ($BB(m)$ continued). For simplicity assume $m = 2$. Then we see from Fig. 9 that $M_{BB(2)}$ is smooth (in the sense that π is a smooth map, as above) and *diffeomorphic* to \mathbb{R}^2 .

Example 3.5 ($S^2(\lambda)$ continued). $M_{S^2(\lambda)}$ is homeomorphic to the 2-sphere, but is not equipped with a smooth structure by π in the sense explained above.

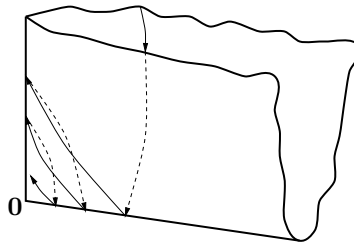


Figure 8: Hybridfold and an orbit of the hybrid flow for BB .

Example 3.6 ($T^2(\alpha)$ continued). We already observed that $M_{T^2(\alpha)}$ is the standard 2-torus.

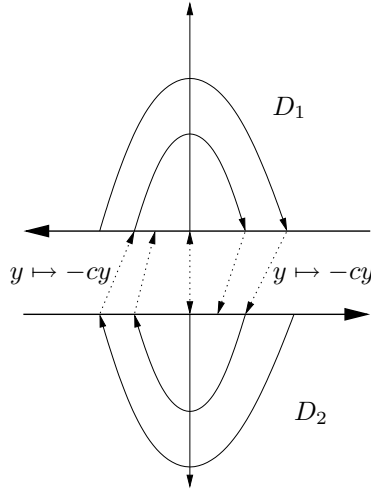


Figure 9: Hybridflow and several orbits of the hybrid flow for $BB(2)$.

The following theorem establishes some basic properties of the hybridflow. Recall that M is called a *topological n -manifold with boundary* if it is Hausdorff and every point in M has a neighborhood homeomorphic to either \mathbb{R}^n or the closed upper half-space $\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_n \geq 0\}$. Points having the latter property are said to be on the boundary ∂M , which in itself is a topological $(n - 1)$ -manifold.

Theorem 3.7. (a) $M_{\mathbf{H}}$ is a topological n -manifold with boundary.

(b) Both $M_{\mathbf{H}}$ and its boundary are piecewise smooth.

(c) The restriction $\pi|_{\text{int } D} : \text{int } D \rightarrow \pi(\text{int } D)$ is a diffeomorphism.

Proof. (a) Recall [30] that the quotient of a manifold M by an equivalence relation ρ is Hausdorff if and only if the graph of ρ , given by $\{(p, q) \in M \times M : (p, q) \in \rho\}$, is closed in $M \times M$. In our case, the relation $\rho = \sim$ is the symmetric and transitive closure of the relation ρ_0 given by

$$\Delta_D \cup \bigcup_{e \in E} \{(p, \tilde{R}_e(p)) : p \in \overline{G(e)}\},$$

where $\Delta_D = \{(p, p) : p \in D\}$ is the diagonal of $D \times D$. Observe that the graph of ρ_0 is a closed set. Taking the symmetric and transitive closure of ρ_0 amounts to adding to ρ_0 the graphs of all maps $f_k \circ \dots \circ f_1$, where f_1, \dots, f_k is a collection of maps such that for some $e_j \in E$, either $f_j = \tilde{R}_{e_j}$ or $f_j = \tilde{R}_{e_j}^{-1}$

($1 \leq j \leq k$), and the image of f_j meets the domain of f_{j+1} ($1 \leq j \leq k-1$). Since the sets $\overline{G(e)}$ (the domain of \tilde{R}_e) and $\overline{\text{im } \tilde{R}_e}$ (the image of \tilde{R}_e) are closed for all $e \in E$, and E is finite, the consistency condition from (A7) guarantees that we are adding a *finite* number of closed sets to ρ_0 . Therefore, the graph of \sim is closed and $M_{\mathbf{H}} = D/\sim$ is a Hausdorff space.

Assume $x \in M_{\mathbf{H}}$. We need to show that there exists a neighborhood of x in $M_{\mathbf{H}}$ which is homeomorphic to \mathbb{R}^n or \mathbb{R}_+^n , where $n = \dim \mathbf{H}$. These neighborhoods will be the charts of the topological manifold $M_{\mathbf{H}}$.

Let $x = \pi(p)$, for some $p \in D$. If p is not identified with any other points, i.e., if it is not in $\overline{G} \cup \overline{R}$, then either $p \in \text{int } D$ or $p \in \partial D - (\overline{G} \cup \overline{R})$. In the former case, p has a neighborhood V contained in the interior of a single domain, homeomorphic to \mathbb{R}^n ; since π is one-to-one on V , $\pi(V)$ is homeomorphic to \mathbb{R}^n . In the latter case, p has a neighborhood W contained in a single domain, disjoint from $\overline{G} \cup \overline{R}$ and homeomorphic to \mathbb{R}_+^n . Since π is one-to-one on W (nothing in W gets glued to anything else), $\pi(W)$ is homeomorphic to \mathbb{R}_+^n . This completes the case when p is identified with no other points.

Before we go ahead with the proof, we state the following lemma (see Fig. 10).

Lemma 3.8 (Gluing homeomorphisms). *Suppose $h_+ : A_+ \rightarrow \mathbb{R}_+^n$ and $h_- : A_- \rightarrow \mathbb{R}_-^n$ are homeomorphisms, where A_+ and A_- are disjoint topological spaces. Let $H_s = h_s^{-1}(\mathbb{R}^{n-1} \times \{0\})$, for $s \in \{-, +\}$, and assume that there exists a homeomorphism $g : H_+ \rightarrow H_-$ such that $h_+|_{H_+} = h_- \circ g$. Let $A = (A_+ \cup A_-)/\sim$ be the quotient space, where \sim is the smallest equivalence relation generated by $x \sim g(x)$, for all $x \in H_+$. Denote by x/\sim the equivalence class of $x \in A_+ \cup A_-$. Then the map $h : A \rightarrow \mathbb{R}^n$ defined by*

$$h(x/\sim) = \begin{cases} h_+(x) & \text{if } x \in A_+ \\ h_-(x) & \text{if } x \in A_- \end{cases}$$

is a homeomorphism.

Proof. Surjectivity, injectivity, and continuity of h can be checked without difficulty. To show that h^{-1} is continuous, let $U \subset A$ be an open set. Denote by $p : A_+ \cup A_- \rightarrow A$ the quotient map. We will show that $V = h(U)$ is open in \mathbb{R}^n . To see that, observe first that $p^{-1}(U) = U_+ \cup U_-$ (disjoint union), where U_s is open in A_s , $s \in \{-, +\}$ (Fig. 10). Therefore, $V_s = h_s(U_s)$ is open in \mathbb{R}_s^n . Consider an arbitrary $x \in V_+ \cap \mathbb{R}^{n-1} \times \{0\}$. Then $h_+^{-1}(x) \in H_+$. Therefore, $x = h_- \circ g \circ h_+^{-1}(x)$. Since $g \circ h_+^{-1}(x) \in H_-$, we have that $x \in V_- \cap \mathbb{R}^{n-1} \times \{0\}$. Conversely, take an arbitrary $x \in V_- \cap \mathbb{R}^{n-1} \times \{0\}$. Then there exists $y \in H_-$ such that $x = h_-(y)$. Hence, there exists $w \in H_+$ such that $y = g(w)$. Then $x = h_- \circ g(w) = h_+(w) \in V_+ \cap \mathbb{R}^{n-1} \times \{0\}$. In summary, $V_+ \cap \mathbb{R}^{n-1} \times \{0\} = V_- \cap \mathbb{R}^{n-1} \times \{0\}$ and hence $V = V_- \cup V_+$ is open. \square

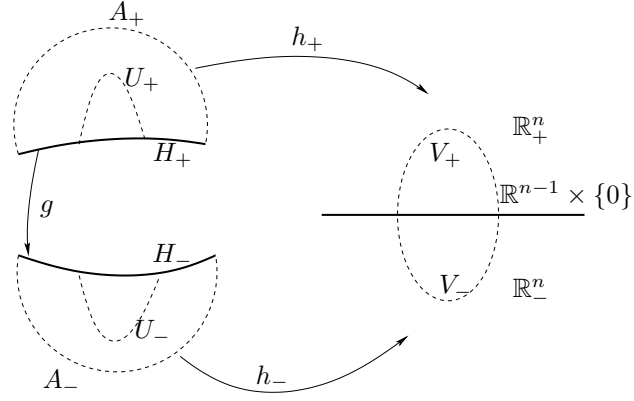


Figure 10: Gluing homeomorphisms in Lemma 3.8.

We now continue with the proof of the theorem.

If $p \in \overline{G} \cup \overline{R}$, then according to (A5) we must consider the following four cases. See Figure 11 for an illustration of Cases 1 and 2 and Figure 12 for Cases 3 and 4.

Case 1: p is of Type I Then there exists $e = (i, j) \in E$ such that either $p \in \text{int } G(e)$ or $p \in \text{int}(\text{im } R_e)$. Without loss we can assume the former. Since interiors of sets in $\mathcal{G} \cup \mathcal{R}$ are not allowed to meet, this e is unique. The point p is identified with $p' = R_e(p)$; note that $p / \sim = \{p, p'\}$. There exist neighborhoods V (in D_i) and V' (in D_j) of p and p' respectively, homeomorphic to \mathbb{R}_+^n and \mathbb{R}_-^n (the closed upper and lower half plane) via some homeomorphisms h and h' , respectively. Since R_e is a diffeomorphism, without loss we can assume that $h|_{V \cap G(e)} = h' \circ R_e|_{V \cap G(e)}$.² Then, by Lemma 3.8, $U = \pi(V \cup V')$ is homeomorphic to \mathbb{R}^n via a homeomorphism h_U . Note that h_U is, in fact, piecewise smooth. Thus $x \in \text{int } M_{\mathbf{H}}$ and (U, h_U) is a chart of $M_{\mathbf{H}}$ at x .

Case 2: p is of Type II Then p is on the boundary of exactly one set $S \in \overline{G} \cup \overline{R}$. Without loss we may assume $S = \overline{G(e)}$, for a unique $e = (i, j) \in E$. Hence p gets identified with (a unique point) $p' = \tilde{R}_e(p) \in \partial(\text{im } R_e)$; observe that $p / \sim = \{p, p'\}$. There exist neighborhoods V (in D_i) and V' (in D_j) of p and p' , homeomorphic to \mathbb{R}_+^n and \mathbb{R}_-^n via some homeomorphisms h and h' , respectively. Note that only a proper subset of the boundary of V (namely, $V \cap \overline{G(e)}$) is identified with a proper subset of the boundary of V' (namely, $V' \cap \text{im } R_e$). Since \tilde{R}_e is a diffeomorphism, without loss we can assume that $h|_{V \cap \overline{G(e)}} =$

²If not, instead of h consider the homeomorphism $\hat{h} : V \rightarrow \mathbb{R}_+^n$ defined by $\hat{h} = F \circ h$, where $F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is any homeomorphism which extends $h' \circ R_e \circ (h|_{V \cap G(e)})^{-1} : \mathbb{R}^{n-1} \times \{0\} \rightarrow \mathbb{R}^{n-1} \times \{0\}$. Then it is easy to check that $\hat{h}|_{V \cap G(e)} = h' \circ R_e|_{V \cap G(e)}$.

$h' \circ \tilde{R}_e|_{V \cap \overline{G(e)}}$. By a construction analogous to the one in Lemma 3.8, it follows that $U = \pi(V \cup V')$ is homeomorphic to \mathbb{R}_+^n via some h_U . Thus $x \in \partial M_{\mathbf{H}}$ and (U, h_U) is a chart of $M_{\mathbf{H}}$ at x . Observe that the restriction of h_U is a piecewise smooth homeomorphism between $U \cap \partial M_{\mathbf{H}}$ and \mathbb{R}^{n-1} . Therefore, $(U \cap \partial M_{\mathbf{H}}, h_U|_{U \cap \partial M_{\mathbf{H}}})$ is a chart for the boundary of $M_{\mathbf{H}}$.

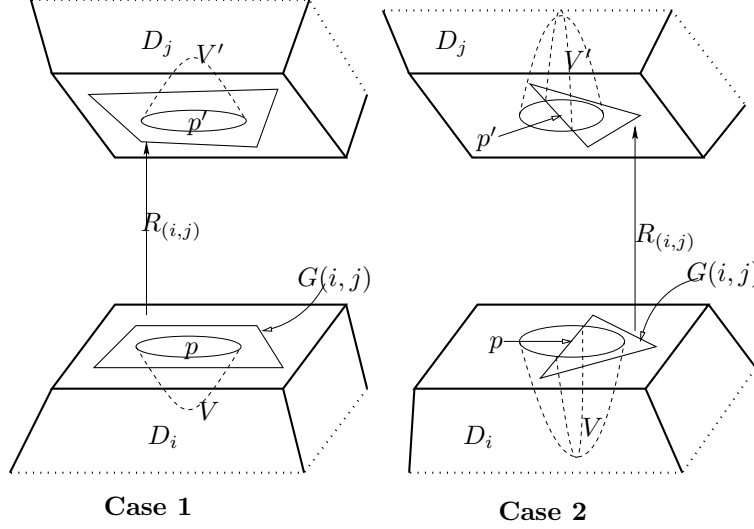


Figure 11: Proof of Theorem 3.7(a), Cases 1 and 2.

Case 3: p is of Type III Assume $p/\sim = \{p_1, \dots, p_m\}$, where $p_1 = p$ and $p_j \in D_{i_j}$. For each j , there exist sets $S_1^j, S_2^j \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$ such that $p_j \in \partial S_1^j \cap \partial S_2^j$. Further, there exist maps f_1, \dots, f_m such that for each j , f_j is either \tilde{R}_{e_j} or $\tilde{R}_{e_j}^{-1}$, for some $e_j \in E$, and f_j maps S_2^j onto one of the sets $S_1^1, S_2^1, \dots, S_1^m, S_2^m$ other than S_2^j . Without loss of generality, we can assume that

$$f_j : S_2^j \rightarrow S_1^{j+1},$$

where $S_1^{m+1} = S_1^1, S_2^{m+1} = S_2^1$. See Fig. 12 (where $n = 3, m = 3$). Since p_j is of Type III, there exists a neighborhood V_j of p_j in $S_1^j \cup S_2^j$, such that V_j is homeomorphic to \mathbb{R}^{n-1} . We require that $V_j \cap \partial S_i^j$ ($i = 1, 2, 1 \leq j \leq m$) be a neighborhood of p_j in ∂S_i^j (with respect to the relative topology ∂S_i^j inherits from S_i^j). By the consistency condition (2) of (A7), it is possible to choose V_1, \dots, V_m so that for all j ,

$$f_j(S_2^j \cap V_j) = S_1^{j+1} \cap V_{j+1}.$$

This means that $\pi(V_1) = \dots = \pi(V_m)$.

Furthermore, for each j , there exists a set U_j open in D_{i_j} , such that U_j is homeomorphic to \mathbb{R}_+^n and $V_j = U_j \cap \partial D_{i_j}$. Let $U = \pi(U_1 \cup \dots \cup U_m)$; this set is obtained by gluing U_1, \dots, U_m along V_1, \dots, V_m via the maps f_1, \dots, f_m . It follows by construction that U is a neighborhood of $x = \pi(p_1)$ in $M_{\mathbf{H}}$ and that U is homeomorphic to \mathbb{R}^n . This in particular means that p is in the interior of $M_{\mathbf{H}}$ and U is a chart of $M_{\mathbf{H}}$ as a topological manifold. This completes Case 3.

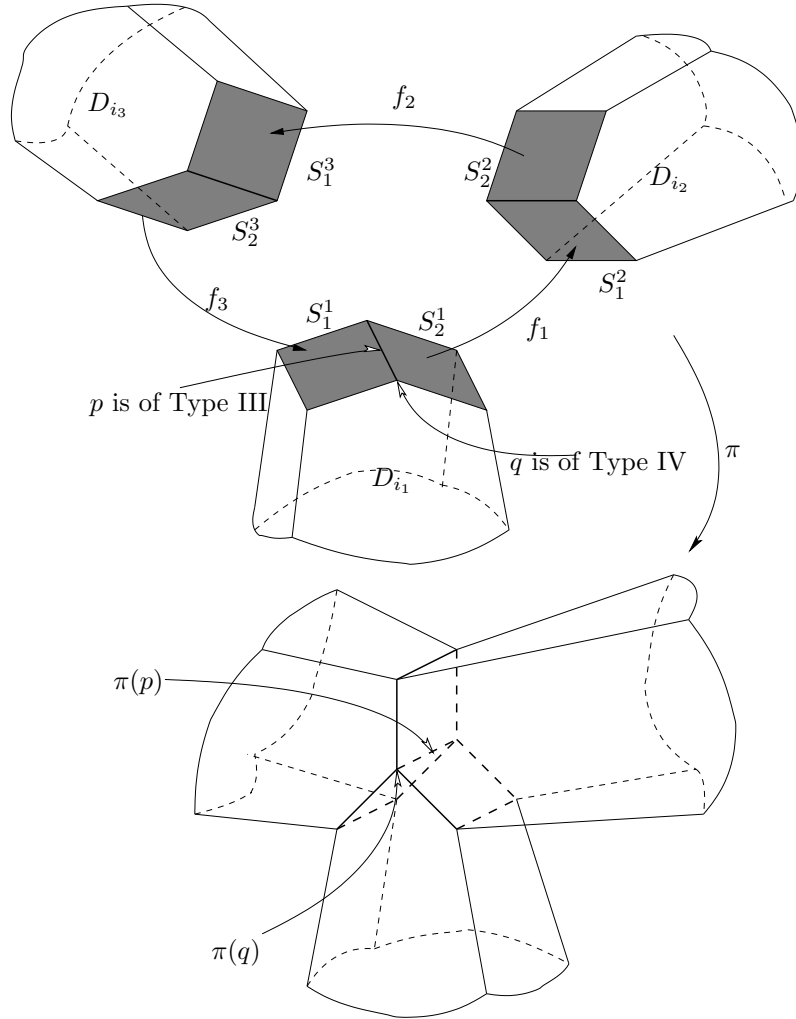


Figure 12: Proof of Theorem 3.7(a), Cases 3 and 4.

Case 4: p is of Type IV This proof is similar to the previous one. Consider the point q of Type IV in Fig. 12. Analogously as in the proof

of Case 3, we obtain sets $S_1^1, S_2^1, \dots, S_1^m, S_2^m \in \overline{\mathcal{G}} \cup \overline{\mathcal{R}}$, U_1, \dots, U_m , and V_1, \dots, V_m such that: (i) U_j is open in D_{i_j} and homeomorphic to \mathbb{R}_+^n ; (ii) V_j is a neighborhood of p_j in $S_1^j \cup S_2^j$ homeomorphic to \mathbb{R}_+^{n-1} ; (iii) $V_j = U_j \cap \partial D_{i_j}$; and (iv) $\pi(V_1) = \dots = \pi(V_m)$. Then $U = \pi(U_1 \cup \dots \cup U_m)$ is homeomorphic to \mathbb{R}_+^n via a piecewise smooth homeomorphism. Furthermore, $y = \pi(q)$ lies on the boundary of $M_{\mathbf{H}}$ and U is a chart of $M_{\mathbf{H}}$ at y ; also, $U \cap \partial M_{\mathbf{H}}$ is a neighborhood of y in $\partial M_{\mathbf{H}}$, homeomorphic (via a piecewise smooth homeomorphism) to \mathbb{R}_+^{n-1} , hence a chart of $\partial M_{\mathbf{H}}$ at y .

(b) Piecewise smoothness of $M_{\mathbf{H}}$ and its boundary follow from piecewise smoothness of their charts introduced in (a).

(c) A direct consequence of the above construction.

□

Remark 2. It is clear that the hybridfold can be a very complicated object, especially in dimensions greater than two (consider, for instance, $M_{BBS(N)}$). The complexity of the hybridfold is simply the reflection of the complexity of the original hybrid system; it is a measure of how complicated the interplay is between the discrete-time and continuous-time dynamics in the original hybrid system.

Remark on metrizability

We point out that every hybridfold $M_{\mathbf{H}}$ is *metrizable*, that is, it admits a metric $d_{\mathbf{H}} : M_{\mathbf{H}} \times M_{\mathbf{H}} \rightarrow \mathbb{R}_+$ which generates the already existing (quotient) topology on $M_{\mathbf{H}}$. This follows from Urysohn's metrization theorem (every regular space with a countable basis is metrizable) and from the just proven fact that $M_{\mathbf{H}}$ is a manifold. See for example, [30]. Observe, however, that if D is given a metric space structure, in general it may not be possible to define $d_{\mathbf{H}}$ so that the projection π is an isometry – a quotient of a metric space may not even be metrizable. Also note that the usual definition of distance between points as on a Riemannian manifold may not work either. Since metrizability is not used in the rest of the paper, we will not discuss this point any further.

The hybrid flow

Define the *hybrid flow* $\Psi^{\mathbf{H}}$ by

$$\Psi^{\mathbf{H}}(t, \pi(p)) = \pi\Phi^{\mathbf{H}}(t, p).$$

This possibly set-valued map is defined on

$$\Omega = \{(t, \pi(p)) : (t, p) \in \Omega_0\}.$$

Orbits of $\Psi^{\mathbf{H}}$ are obtained by projecting orbits of $\Phi^{\mathbf{H}}$ by π . By the $\Phi^{\mathbf{H}}$ -orbit of p we mean the collection of points $\Phi^{\mathbf{H}}(t, p)$ for all t such that $(t, p) \in \Omega_0$.

Example 3.9. Consider the examples from the previous subsection. Observe that if $\mathbf{H} = WT, BB$ or $BB(m)$, then for all $x \in M_{\mathbf{H}}$, we have $\Psi^{\mathbf{H}}(t, x) \rightarrow \pi(\mathbf{0})$, as $t \rightarrow \tau_{\infty}(\chi(x))$. In other words, all executions “spiral” towards zero. If $\mathbf{H} = S2(\lambda)$, then the hybrid flow has two equilibria corresponding to the saddles in the original system and four “Zeno states”, that is, points at which trajectories accumulate in a finite amount of time. The hybrid flow of $\Psi^{T^2(\alpha)}$ is just the linear flow on the 2-torus. If α is rational, every orbit is closed; if it is irrational, every orbit is dense in the torus.

Fix \mathbf{H} and set $\Psi := \Psi^{\mathbf{H}}$. For each $t \in \mathbb{R}$ and $x \in M_{\mathbf{H}}$, let

$$M(t) = \{y \in M_{\mathbf{H}} : \Psi(t, y) \text{ is defined}\},$$

and

$$J(x) = \{s \in \mathbb{R} : \Psi(s, x) \text{ is defined}\}.$$

Observe that if $x = \pi(p)$, then $J(x) \cap [0, \infty) = [0, \tau_{\infty}(\chi(p))]$. Also, for $t > 0$, $M(t)$ contains all points $x = \pi(p)$ such that $\tau_{\infty}(\chi(p)) > t$. As usual, $\chi(p)$ denotes the unique execution of \mathbf{H} starting at p .

If $M(t)$ is not empty, denote by Ψ_t the *time t map of Ψ* , defined by

$$\Psi_t(x) = \Psi(t, x).$$

Note again that the time t map of the hybrid flow of a regular hybrid system may be multi-valued. This occurs when, due to gluing, the projections of two disjoint executions of \mathbf{H} overlap in the hybrid manifold. This can happen at $x = \pi(p) \in M_{\mathbf{H}}$, where p is a point of type III or IV (cf. (A5)); since p is glued to two or more other points, the orbit through x could branch. We will deal with this situation shortly.

Definition 3.10. *Let X be a smooth vector field on a smooth manifold M , with flow ϕ_t . We say that $q \in M$ is X -reachable from a point $p \in M$, if $q = \phi_t(p)$, for some $t > 0$. A set S is X -reachable from a point p if there exists a point $q \in S$ such that q is X -reachable from p .*

To avoid the “branching scenario” described above we now introduce the following property of hybrid systems.

Definition 3.11. *A regular hybrid system \mathbf{H} is said to be without branching if for every point $p \in \partial D$ of type III or IV with $p/\sim = \{p_1, \dots, p_m\}$, where $p_j \in D_{i_j}$, there exists at most one k and at most one ℓ , $1 \leq k \neq \ell \leq m$, such that p_k is X_{i_k} -reachable from D_{i_k} and D_{i_ℓ} is X_{i_ℓ} -reachable from p_ℓ .*

From this point on, **all hybrid systems in this paper will be assumed to be regular and without branching**. Note that all our examples have these properties. Observe also that \mathbf{H} is without branching if and only if the reverse hybrid system \mathbf{H}' is. We can now state the following theorem.

Theorem 3.12. *Suppose that \mathbf{H} is regular and without branching. Then:*

- (a) *For each $(t, x) \in \Omega$, $\Psi_t(x)$ is a single point.*
- (b) *For each $x \in M_{\mathbf{H}}$, the map $t \mapsto \Psi_t(x)$ is continuous. Moreover, if $J(x)$ is not a single point, the map is smooth except at (at most) countably many points in $J(x)$.*
- (c) *Each map Ψ_t is one-to-one.*
- (d) *Whenever both sides are defined,*

$$\Psi_t \Psi_s(x) = \Psi_{t+s}(x).$$

- (e) *For each $t \in \mathbb{R}$, there exists an open and dense subset of $M(t)$ on which Ψ_t is continuous.*

Proof. (a) Nonuniqueness of executions in $M_{\mathbf{H}}$ can occur only at points $x = \pi(p) \in M_{\mathbf{H}}$ such that p is of Type III or IV, and more than one trajectory of Ψ enters or leaves x . This situation is eliminated by the no branching condition.

(b) Let $x = \pi(p)$ for some $p \in D$. Let $\chi(p) = (\tau, q, x)$ be the unique execution of \mathbf{H} starting at p (i.e., $\tau_0 = 0$ and $x_0(0) = p$). Recall that for positive t , $\Psi_t(x) = \pi \Phi^{\mathbf{H}}(t, p) = \pi x_j(t)$, if $t \in [\tau_j, \tau'_j)$. Thus it is enough to check continuity of $t \mapsto \Psi_t(x)$ at $\tau'_j = \tau_{j+1}$, for $j \geq 0$. But recall that $t \mapsto \Phi^{\mathbf{H}}(t, p)$ is continuous from the right, with discontinuities of the first kind only at τ_j , $j \geq 0$. Since $x_j(\tau_j) \in G$ and

$$\pi(x_j(\tau'_j)) = \pi(x_{j+1}(\tau_{j+1})),$$

for all $j \geq 0$, it follows that

$$\lim_{t \rightarrow \tau'_j-} \Psi_t(x) = \lim_{t \rightarrow \tau'_j-} \pi x_j(t) = \pi x_{j+1}(\tau_{j+1}) = \Psi_{\tau'_j}(x),$$

which shows that $t \mapsto \Psi_t(x)$ is continuous on $J(x) \cap [0, \infty)$.

Continuity of $t \mapsto \Psi_t(x)$ for negative t follows by observing that $\Psi_{-t}^{\mathbf{H}}(x) = \Psi_t^{\mathbf{H}'}(x)$ ($t > 0$), where \mathbf{H}' is the reverse hybrid system to \mathbf{H} .

The extreme case when $J(x) = \{0\}$ happens when $x = \pi(p)$ for some $p \in D$ such that:

$$p \in \overline{G(e_0)}, p_1 = \tilde{R}_{e_0}(p) \in \overline{G(e_1)}, p_2 = \tilde{R}_{e_1}(p_1) \in \overline{G(e_2)}, \text{ etc.},$$

for a sequence e_0, e_1, e_2, \dots in E . Then $t \mapsto \Psi_t(x)$ is trivially continuous.

Assume now that $J(x)$ is not a single point. With the notation as above, we have that for $\tau_j < t < \tau'_j$,

$$\frac{d}{dt} \Psi_t(x) = T\pi(X_{q(j)}(x_j(t))),$$

which proves that $t \mapsto \Psi_t(x)$ is piecewise smooth. Here $T\pi$ denotes the tangent (or derivative) map of π . It is defined at $x_j(t)$ because $x_j(t) \in \text{int } D$, for $\tau_j < t < \tau'_j$, and π is smooth on $\text{int } D$.

(c) Injectivity of Ψ_t follows directly from uniqueness of executions through any point.

(d) Follows from (a) and the analogous property of $\Phi^{\mathbf{H}}$.

(e) It is enough to show this for $t > 0$ with $M(t) \neq \emptyset$. For $x \in M(t)$, denote by $\mathcal{O}_{[0,t]}(x)$ the set of points $\Psi_s(x)$, for all $0 \leq s \leq t$. Let

$$M_d(t) = \{x \in M(t) : \mathcal{O}_{[0,t]}(x) \text{ intersects } \pi(\partial G)\},$$

and $M_c(t) = M(t) - M_d(t)$. So $M_d(t)$ is the set of all points in $M(t)$ such that on the time interval $[0, t]$ their Ψ -orbits pass through the projection of the boundary of some guard (and thus the boundary of the image of some reset). On the other hand, the $[0, t]$ -segment of the orbit of any point in $M_c(t)$ passes only through the projection of interiors of guards or is possibly trapped in some domain.

It is not difficult to see that $M_c(t)$ is open and dense in $M(t)$. Furthermore, we claim that Ψ_t is continuous on $M_c(t)$. To see this, let $x_0 \in M_c(t)$ be arbitrary and let $y_0 = \Psi_t(x_0)$. Let N be the number of guards which $\mathcal{O}_{[0,t]}(x_0)$ intersects. We prove that Ψ_t is continuous at x_0 by induction on N . If $N = 0$, we are dealing with a flow of a single vector field on an invariant set, so the result follows from standard theory of differential equations. If $N > 0$, since continuity is preserved under composition, it suffices to treat the case $N = 1$. So let $N = 1$ and let V be a neighborhood of y_0 in $M_{\mathbf{H}}$. Assume $x_0 \in \pi(D_i)$ and $y_0 \in \pi(D_j)$. Let w_0 be the intersection of $\mathcal{O}_{[0,t]}(x_0)$ with the only guard (more precisely, its projection by π) it intersects, namely, $\Gamma = \pi(G(i, j))$. Then $w_0 \in \text{int } \Gamma$, and $w_0 = \Psi_s(x_0)$, for some $0 \leq s \leq t$ (see Fig. 13). We assume that $0 < s < t$; other cases are treated analogously. Then, there exists a neighborhood $W \subset \Gamma$ of w_0 such that for every $w \in W$, there exist r with $0 < r < t$ such that $\Psi_r(w) \in V$. This follows from continuity of the flow of X_j . Since the flow of X_i is also continuous, there exists a neighborhood U of x_0 such that for all $x \in U$, $\mathcal{O}_{[0,t]}(x)$ passes through W . By taking a smaller U if necessary, we get that for every $x \in U$, $\Psi_t(x) \in V$. Therefore, Ψ_t is continuous at x_0 . Observe that Ψ may well be continuous also on $M_d(t)$ (e.g., Example 2.7). Compare also [36]. \square

Remark 3. For $x \in M_{\mathbf{H}}$ with $x = \pi(p)$, we set $\chi(x) = \pi(\chi(p))$ and refer to it as the execution (in $M_{\mathbf{H}}$) starting from x . Furthermore, properties of $\chi(p)$ transfer to properties of $\chi(x)$, e.g., we say that $\chi(x)$ is a Zeno execution if $\chi(p)$ is, and so on.

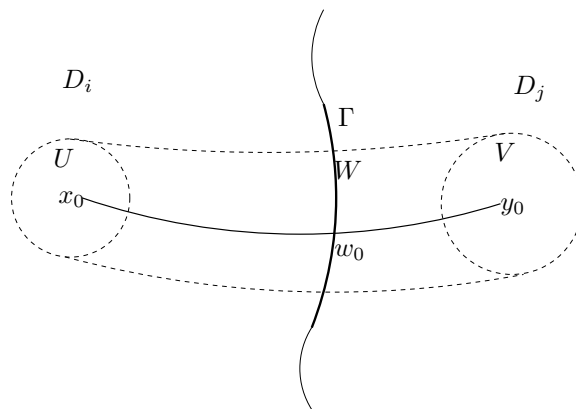


Figure 13: Proof of Theorem 3.12(e).

4 ω -limit sets and the Zeno phenomenon

Zeno executions do not arise in physical systems, but are a consequence of modeling over-abstraction. However, as we have seen, it is easy to construct seemingly realistic systems which admit Zeno behavior. Therefore, one would like to be able to identify and eliminate it. In this section we deal with the first problem. From a mathematical viewpoint, the Zeno phenomenon poses several interesting questions: for instance, what is its topological cause? Is there a checkable criterion which guarantees the non-occurrence of Zeno? How should the original system be modified to remove Zeno executions? In this section we show that, in short, the topological cause of Zenoness is lack of smoothness in the hybrid flow.

Since we would like to study the long term behavior of executions of hybrid systems, we define the following notion.

Definition 4.1. A point $y \in M_{\mathbf{H}}$ is called an ω -limit point of $x \in M_{\mathbf{H}}$ if

$$y = \lim_{m \rightarrow \infty} \Psi_{t_m}^{\mathbf{H}}(x),$$

for some increasing sequence (t_m) in $J(x)$ such that $t_m \rightarrow \tau_{\infty}(x)$, as $m \rightarrow \infty$. The set of all ω -limit points of x is called the ω -limit set of x and is denoted by $\omega(x)$.

In other words, ω -limit points for x are accumulation points of the orbit of x . By $\tau_{\infty}(x)$ we denote the execution time of the unique infinite execution of \mathbf{H} starting from some p , where $x = \pi(p)$; that is, $\tau_{\infty}(x) = \tau_{\infty}(\chi(p))$. It is easy to check that this is a well defined element of $[0, \infty]$.

Suppose $x \in M_{\mathbf{H}}$ and denote by $E_{\infty}(x) \subset E$ the set of discrete transitions which occur infinitely many times in the execution starting from x . If $E_{\infty}(x)$

is empty, then the orbit of x eventually ends up in a single domain D_i (that is, its image under π in the hybridfold) in which case

$$\omega(x) \subset \pi(\overline{D_i}).$$

This means that every point $y \in \omega(x)$ is an accumulation point of the orbit of a single vector field, namely X_i . If $E_\infty(x)$ is non-empty, but $\tau_i = \tau_\infty(x)$ for some $i \in \langle \tau \rangle$, then every point $y \in \omega(x)$ is an accumulation point of the orbit of a discrete-time system generated by the resets R_e , $e \in E_\infty(x)$. We will call points $y \in \omega(x)$ that involve only the continuous or only the discrete dynamics *pure ω -limit points*.

If $E_\infty(x)$ is nonempty and $\tau_i < \tau_\infty(x)$ for all $i \in \langle \tau \rangle$, then every ω -limit point for x is a result of both the continuous *and* discrete (i.e., hybrid) dynamics of \mathbf{H} and will accordingly be called a *hybrid ω -limit point of x* .

A special case of a hybrid ω -limit point is a Zeno state:

Definition 4.2. *A point $z \in M_{\mathbf{H}}$ is called a Zeno state for x if $z \in \omega(x)$ and $\chi(x)$ is a dynamic Zeno execution.*

In other words, a Zeno state is an accumulation point of a dynamic Zeno execution. (Recall that a Zeno execution is called dynamic if $\tau_j \neq \tau'_j$ for infinitely many j 's.) We will also refer to points in $\pi^{-1}(z)$ as *Zeno states in \mathbf{H}* . For example, the “origin” $\mathbf{0}$ of M_{WT} (as well as M_{BB} and $M_{BB(2)}$) is a Zeno state for every point different from $\mathbf{0}$. Moreover, for each $x \neq \mathbf{0}$, $\omega(x)$ contains only one Zeno state. We now show this is always the case.

Proposition 4.3. *If the execution starting from $x \in M_{\mathbf{H}}$ is Zeno and bounded, then $\omega(x)$ consists of exactly one Zeno state for x . Moreover,*

$$\omega(x) \subset \bigcap_{e \in E_\infty(x)} \pi(\overline{G(e)}) = \bigcap_{e \in E_\infty(x)} \pi(\partial G(e)). \quad (3)$$

Proof. Let $p \in \pi^{-1}(x)$ be arbitrary and, as before, let $\chi(p) = (\tau, q, x)$ be the unique execution starting from p . For $j \geq 0$, let $A_j = \pi(x_j(I_j))$. Then A_j is an arc in $\pi(D_{q(j)})$ of the $\Psi^{\mathbf{H}}$ -orbit of x . Since the set $A = \bigcup A_j$ is by assumption bounded, it must have an accumulation point, say z . Clearly, $z \in \omega(x)$ and z is a Zeno state for x . Furthermore, all vector fields X_i are bounded on $A = \bigcup A_j$. (This is true, for example, if the reset maps are non-expanding; see [41, 42].) Therefore,

$$\sum_{j=0}^{\infty} |A_j| < \infty,$$

where $|A_j|$ is the length of $x_j(I_j)$ as a curve in \mathbb{R}^d .

Suppose there exists another accumulation point of A or equivalently, another Zeno state z' for x . Then we have:

$$z = \lim_{m \rightarrow \infty} \Psi_{t_m}^{\mathbf{H}}(x), \quad z' = \lim_{m \rightarrow \infty} \Psi_{t'_m}^{\mathbf{H}}(x),$$

for some increasing sequences (t_m) and (t'_m) in $J(x)$ such that $t_m, t'_m \rightarrow \tau_\infty(x)$, as $m \rightarrow \infty$. If

$$x_m = \Psi_{t_m}^{\mathbf{H}}(x) \in A_{j_m} \quad \text{and} \quad x'_m = \Psi_{t'_m}^{\mathbf{H}}(x) \in A_{j'_m},$$

for some $j_m, j'_m \rightarrow \infty$, we obtain

$$|x_m x'_m| \leq \sum_{l=\min(j_m, j'_m)}^{\infty} |A_l| \rightarrow 0$$

as $m \rightarrow \infty$, where $|x_m x'_m|$ denotes the length of the segment of A between x_m and x'_m . Thus $z = z'$. This completes the first part of the proof.

To show (3), let $\omega(x) = \{z\}$, and let $e = (i, j) \in E_\infty(x)$. Then there exists a sequence (x_m) of points in D_i such that: x_m is on the forward orbit of x and $x_m \rightarrow z$ as $m \rightarrow \infty$. Thus: $z \in \pi(\overline{D_i})$. Similarly, $z \in \pi(\overline{D_j})$. But $\pi(\overline{D_i}) \cap \pi(\overline{D_j}) \subset \pi(\overline{G(e)})$, so $z \in \pi(\overline{G(e)})$. Since this holds for all $e \in E_\infty(x)$, the proof of (3) is complete. \square

Remark 4. Accumulation points of static Zeno executions are called *chattering states*. So a Zeno state is a hybrid ω -limit point whereas a chattering state is a pure discrete ω -limit point. Note that a chattering state for one execution may be a Zeno state for another. For instance, in M_{WT} and M_{BB} , $\mathbf{0}$ is a chattering state for $\chi(\mathbf{0})$, but a Zeno state for $\chi(x)$, for every $x \neq \mathbf{0}$.

Note that in all the Zeno examples above none of the flows involved in creating the Zeno state has an equilibrium at the Zeno state. (This is not necessarily the case for purely chattering states.) The following lemma shows that this is not a coincidence.

Proposition 4.4. *A Zeno state is not an equilibrium of any of the vector fields that participate in creating it. More precisely, if $z \in M_{\mathbf{H}}$ is a Zeno state for x , then for every $p \in \pi^{-1}(z)$, if $p \in \overline{D_i}$ and D_i is visited infinitely often by the lift by π of the orbit of x , then $X_i(p) \neq 0$.*

Proof. Let z be a Zeno state for x . Consider the lift of the orbit of x by π to D and let us concentrate on its “trace” in a particular domain which it visits infinitely often.

More precisely, there exist $j \in Q$, $e = (i, j)$, $e' = (j, i') \in E$ and $p_* \in D_j$ such that $\pi(p_*) = z$ and

$$p_* \in \overline{\text{im } R_e} \cap \overline{G(e')}.$$

Furthermore, there exists a sequence (p_m) in $\text{im } R_e$ converging to p_* , and a sequence (t_m) of positive numbers such that (see Fig. 14)

$$q_m = \phi_{t_m}^j(p_m) \in G(e') \quad \text{and} \quad T_\infty := \sum_{m=0}^{\infty} t_m < \infty.$$

Recall that $\{\phi_t^j\}$ is the (local) flow of the vector field $X := X_j$ on D_j .

Let A_m be the arc tangent to X and connecting p_m and q_m (i.e., A_m is the X -orbit of p_m), and denote by $|A_m|$ its length. Finally, let $K \subset D_j$ be a compact set containing p_* and A_m for all $m \geq 0$. It exists, because $p_m \rightarrow p_*$, as $m \rightarrow \infty$.

The intuition is as follows: we start from p_0 , flow for time t_0 to q_0 when we reach the guard $G(e')$ and are taken outside of D_j by a reset. We enter D_j again at p_1 , flow for time t_1 until we reach q_1 , etc.

Now assume that p_* is an equilibrium for X . Then $\phi_t^j(p_*) = p_*$ for all $t \in \mathbb{R}$. If $p \in A_m$, then $p = \phi_t^j(p_m)$ for some $0 \leq t \leq t_m$, so we have:

$$\begin{aligned} \|X(p)\| &= \|X(p) - X(p_*)\| \\ &\leq L\|p - p_*\| \\ &= L\|\phi_t^j(p_m) - \phi_t^j(p_*)\| \\ &\leq C\|p_m - p_*\|, \end{aligned}$$

where L is the Lipschitz constant of X and $C = L \max\{\|T_q \phi_t^j\| : 0 \leq t \leq T_\infty, q \in K\} < \infty$. From this inequality we get that

$$\|X\|_{A_m} := \max_{p \in A_m} \|X(p)\| \leq C\|p_m - p_*\|,$$

for all $m \geq 0$.

Next observe that

$$|A_m| \leq t_m \|X\|_{A_m},$$

so $|A_m|/\|X\|_{A_m} \rightarrow 0$, as $m \rightarrow \infty$. However, by the non-zero angle requirement in (A2), there exists a constant $a > 0$ such that $|A_m| \geq a\|p_m - p_*\|$, for all $m \geq 0$. Thus:

$$\frac{|A_m|}{\|X\|_{A_m}} \geq \frac{a}{C} > 0,$$

a contradiction. Therefore, $X(p_*) \neq 0$. □

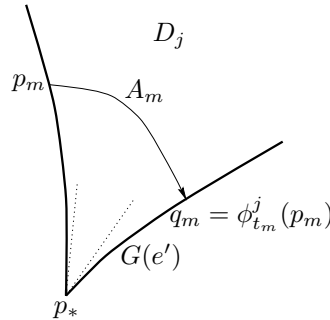


Figure 14: Proof of Proposition 4.4.

The following example indicates that the non-zero angle assumption in (A2) was necessary in Proposition 4.4.

Example 4.5 (equilibrium + cusp = Zeno). Consider the following one-domain hybrid system (see Fig. 15):

$$\begin{aligned} D &= \{(x, y) \in \mathbb{R}^2 : y \geq 0, -f(y) \leq x \leq f(y)\} \\ G &= \{(-f(y), y) : y \geq 0\}, \quad R(-f(y), y) = (f(cy), cy), \\ X(x, y) &= (-x - y, x - y)^T. \end{aligned}$$

Here $0 < c < 1$ and $f(y) = y^2$ (in fact, any smooth f such that $0 \leq f(y) \leq y^2$ would do). Since $f'(0) = 0$, D has a cusp at $\mathbf{0}$.

The vector field X has a spiral sink at the origin, and the time t map of its flow is the composition of the counterclockwise rotation by t (in radians) and contraction by e^{-t} . Let p_0 be an arbitrary nonzero point on the right part S of the boundary of D and let χ the execution starting from p_0 . Let (p_m) be the sequence of intersections of χ with S ; let $p_m = (f(y_m), y_m)$. Let t_m be the time it takes for the positive X -orbit of p_m to reach G . Then $\|p_{m+1}\| = c e^{-t_m} \|p_m\|$, so

$$\|p_m\| = c^m \exp\left(-\sum_{i=0}^{m-1} t_i\right) \|p_0\|, \quad \text{and} \quad y_m \leq c^m y_0.$$

Let θ_m be the angle between the line $\mathbf{0}p_m$ and the positive y -axis and η_m the angle between the positive y -axis and the line $\mathbf{0}p'_m$, where p'_m is the intersection of the positive X -orbit of p_m and G . Then

$$t_m = \theta_m + \eta_m \leq 2\theta_m = 2 \arctan \frac{f(y_m)}{y_m} \leq 2y_m \leq 2c^m y_0.$$

Therefore, $\sum t_m$ converges and $\mathbf{0}$ is a Zeno state despite the fact that it is an equilibrium for X . This shows the importance of geometry of domains and assumption (A2).

Before we proceed, we need to remind the reader of the following *flow box theorem* for smooth flows [31]. Namely, assume that X is a smooth vector field on an open set $U \subset \mathbb{R}^n$, $p \in U$ and $X(p) \neq 0$. Then there exists a neighborhood V of p in U (called a *flow box for X at p*) and a smooth coordinate system on V relative to which X equals the vector field $\partial/\partial x_1$, i.e., the flow of X looks like

$$\phi_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n).$$

This means that in a neighborhood of any of its nonsingular points, the flow of a smooth vector field has a particularly simple form.

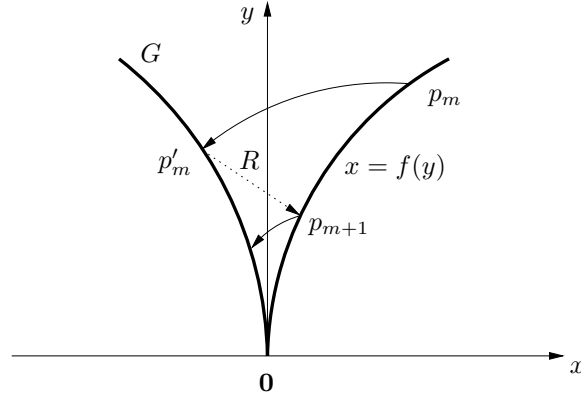


Figure 15: Zeno state at a cusp which is an equilibrium.

Theorem 4.6. *Suppose that $M_{\mathbf{H}}$ has a smooth structure which makes the projection π smooth and assume the hybrid flow $\Psi^{\mathbf{H}}$ is smooth with respect to that structure. Then \mathbf{H} admits no Zeno executions or equivalently, there are no Zeno states in $M_{\mathbf{H}}$.*

Proof. Fix a smooth structure on $M_{\mathbf{H}}$ which turns π into a smooth map, and assume that there exists a Zeno state $z \in M_{\mathbf{H}}$ for some point $x_0 \in M_{\mathbf{H}}$. Since the hybrid flow $\Psi^{\mathbf{H}}$ is assumed to be smooth, it is generated by a smooth vector field Y on $M_{\mathbf{H}}$. The projection π is smooth, so $Y(x) = T\pi(X_i(p))$, for any point p such that $\pi(p) = x$, where X_i is the vector field on the domain containing x . By Proposition 4.4, $X_i(p) \neq 0$, for all $p \in \pi^{-1}(z)$ with $p \in D_i$. Observe that on $\overline{G} \cup \overline{R}$, π is not one-to-one. However, π does have the property that $T\pi(v) \neq 0$, for every vector nonzero v tangent to D . Therefore, $Y(z) \neq 0$. By the Flow Box Theorem applied to a neighborhood of z , $\Psi^{\mathbf{H}}$ is the trivial horizontal flow (equivalent to the flow of the vector field $\partial/\partial x_1$ on \mathbb{R}^n), which implies that $z = \Psi_{\tau_\infty(x)}^{\mathbf{H}}(x)$. But then $t \mapsto \Psi_t^{\mathbf{H}}(x)$ is clearly defined beyond the Zeno time $\tau_\infty(x)$, which is contrary to our assumption. \square

In general it may not be easy to check whether, given \mathbf{H} , the hybridfold $M_{\mathbf{H}}$ is smooth. Even if it were, non-smoothness of the hybrid flow itself may cause Zeno to occur (cf., *BB(2)*). However, assuming that smoothness of the hybridfold can be verified, the following result provides an easily verifiable criterion for smoothness of $\Psi^{\mathbf{H}}$.

Theorem 4.7. *Suppose that $M_{\mathbf{H}}$ is smooth in the same sense as in the previous theorem, and for every $e = (i, j) \in E$, there exists a smooth extension \tilde{R}_e of R_e such that X_i and X_j are \tilde{R}_e -related on $\overline{G(e)}$. That is, for every $p \in \overline{G(e)}$,*

$$T\tilde{R}_e(X_i(p)) = X_j(\tilde{R}_e(p)). \quad (4)$$

Then the hybrid flow is smooth.

Proof. Define a vector field Y on $M_{\mathbf{H}}$ as follows. If $x \in M_{\mathbf{H}}$, then $x = \pi(p)$ for some $p \in D_i$. Set

$$Y(x) = T\pi(X_i(p)).$$

We will show that Y is well defined.

If p is not in $\overline{G} \cup \overline{R}$ (i.e., p/\sim is a single point), then there is no ambiguity in the definition of $Y(x)$.

If $p \in \overline{G} \cup \overline{R}$, then the ambiguity arises because p is identified with some p' via an extended reset map. Assume that $p \in \overline{G(e)}$ for some $e \in E$ and let $p' = \tilde{R}_e(p)$. Since $\pi \circ \tilde{R}_e = \pi$, (4) and the chain rule yield

$$\begin{aligned} T\pi(X_j(p')) &= T\pi T\tilde{R}_e(X_i(p)) \\ &= T(\pi \circ \tilde{R}_e)(X_i(p)) \\ &= T\pi(X_i(p)). \end{aligned}$$

Therefore, Y is well defined.

Next we show that Y is smooth. Let X be the vector field on the total domain D which coincides with X_i on D_i , for all $i \in Q$. That is, X is the union of X_i 's. Since D_i 's are mutually disjoint and each X_i is smooth, X is smooth. By definition of Y , the vector fields X and Y are π -related. Therefore, since X and π are smooth, so is Y .

Smoothness of $\Psi^{\mathbf{H}}$ now follows directly from the fact that Y generates it, i.e.,

$$\left. \frac{d}{dt} \right|_0 \Psi_t^{\mathbf{H}}(x) = Y(x),$$

for all $x \in M_{\mathbf{H}}$. □

Example 4.8. Consider $BB(2)$. Here we have

$$X_1(1, x_1, x_2) = (x_2, -g)^T = X_2(2, x_1, x_2), \quad \tilde{R}_{(i,j)}(i, x_1, x_2) = (j, x_1, -cx_2),$$

where $(i, j) = (1, 2)$ or $(2, 1)$. Therefore,

$$T\tilde{R}_{(1,2)}(X_1) = (x_2, cg)^T \neq X_2,$$

so the hybrid flow for $BB(2)$ is not smooth, as expected, since $BB(2)$ is Zeno.

Example 4.9. It is not difficult to check that in the case of $T^2(\alpha)$, (4) is satisfied for every α . Thus $T^2(\alpha)$ does not accept Zeno executions, as was already shown above.

Corollary 4.10. *If $M_{\mathbf{H}}$ is smooth in the sense of Theorem 4.6 and \mathbf{H} satisfies condition (4), then \mathbf{H} accepts no Zeno executions.*

A Zeno state is an equilibrium of a sort. So we can define its stable and unstable sets in a standard way.

Definition 4.11. Let z be an isolated Zeno state and let $U \subset M_{\mathbf{H}}$ be its neighborhood. The set

$$Z_U^s(z) = \{x \in U : \omega(x) = \{z\} \text{ and the forward execution of } x \text{ stays in } U\}$$

is called the local stable set of z .

The local unstable set $Z_U^u(z)$ of z is defined as the stable set of z relative to the reverse hybrid system. We also define the global stable set of z by

$$Z^s(z) = \{\Psi_t^{\mathbf{H}}(x) : x \in Z_U^s(z), t \leq 0, t \in J(x)\},$$

where U is any neighborhood of z in $M_{\mathbf{H}}$. Thus $Z^s(z)$ is the union of all backward iterates of the local stable set of z . The global unstable set of z is defined as the union of all forward iterates of the local unstable set of z . For instance, if $\mathbf{H} = WT, BB$ or $BB(m)$, then $Z^s(\mathbf{0})$ is a neighborhood of $\mathbf{0}$ (in fact, equals $M_{\mathbf{H}}$). We will see in the next section that this is not a coincidence. Namely, in dimension two, the global stable set of an isolated Zeno state is a neighborhood of the Zeno state. In higher dimensions, this is not the case, as the following simple example illustrates.

Example 4.12. The idea is to take the water tank example and add another direction along which the system is unstable. The system decouples into its (x_1, x_2) - and x_3 -part. More precisely, we have (cf., Example 2.2) $n = 3$, $k = 2$, $E = \{(1, 2), (2, 1)\}$, $D_i = \{i\} \times C \times \mathbb{R}$, where $C = [0, \infty) \times [0, \infty)$,

$$X_1(1, x_1, x_2, x_3) = (w - v_1, -v_2, 0)^T, \quad X_2(2, x_1, x_2, x_3) = (-v_1, w - v_2, 0)^T,$$

where $\max(v_1, v_2) < w < v_1 + v_2$,

$$G(1, 2) = \{(1, x_1, x_2, x_3) \in D_1 : x_2 = 0\},$$

$$G(2, 1) = \{(2, x_1, x_2, x_3) \in D_2 : x_1 = 0\},$$

and

$$R_{(1,2)}(1, x_1, 0, x_3) = (2, x_1, 0, 2x_3), \quad R_{(2,1)}(2, 0, x_2, x_3) = (1, 0, x_2, 2x_3).$$

The hybridfold of the described system is homeomorphic to \mathbb{R}^3 . The only Zeno state is $\mathbf{0}$ and its stable and unstable set are the x_1x_2 -plane and the x_3 -axis respectively. Executions starting from points with $x_3 = 0$ spiral around the x_3 -axis and converge to $\mathbf{0}$ in finite time. Executions starting from points with $x_3 \neq 0$ spiral around the x_3 -axis, but escape to infinity in finite time.

Open question: When is a Zeno state hyperbolic? That is, under which conditions are the global stable and unstable set of a Zeno state smooth, transversely intersecting manifolds of complementary dimension? Note that this means there is no “center manifold” passing through the Zeno state.

5 Equivalence of hybrid systems

In this section we discuss the following question: when are two hybrid systems qualitatively the same? For that purpose we borrow the notion of equivalence from the classical theory of dynamical systems [31]. Roughly speaking, two dynamical systems are (topologically) equivalent if their phase portraits look qualitatively (or topologically) the same. Similarly, two hybrid systems are equivalent if their hybrid flows are equivalent. We now make this more precise.

Definition 5.1. *Two hybrid systems \mathbf{H}_1 and \mathbf{H}_2 are said to be topologically equivalent, denoted by $\mathbf{H}_1 \approx \mathbf{H}_2$, if there exists a homeomorphism $h : M_{\mathbf{H}_1} \rightarrow M_{\mathbf{H}_2}$ which sends orbits of $\Psi^{\mathbf{H}_1}$ to orbits of $\Psi^{\mathbf{H}_2}$, preserving their orientation.*

If $M_{\mathbf{H}_1}$ and $M_{\mathbf{H}_2}$ happen to be smooth manifolds of class C^r ($r \geq 1$) and h is a C^r diffeomorphism, then \mathbf{H}_1 and \mathbf{H}_2 are said to be C^r -equivalent.

As usual, by the *orbit* of a point x under a (local) flow $\{\phi_t\}$ we mean the set of points $\phi_t(x)$ for all t for which $\phi_t(x)$ is defined. We usually think of h as a change of coordinates; so two hybrid systems are topologically equivalent if their hybrid flows are the same up to a continuous coordinate change. Note that equivalence does not necessarily preserve the time parameter t . If it does, it is called *conjugacy*.

Example 5.2. WT is topologically equivalent to BB . This can be seen in the following way. Assume M_{WT} is embedded in \mathbb{R}^3 in such a way that its “origin” coincides with the point $(0, 0, 0)$ and M_{WT} lies entirely in the upper half space \mathbb{R}_+^3 and each vertical line intersects it exactly once. Let P be the plane $x_3 = 0$ and let $h : M_{WT} \rightarrow P$ be the orthogonal projection. Then h is a homeomorphism which sends orbits of Ψ^{WT} to the orbits of the flow Φ in Fig. 16.

By smoothing Φ along the y -axis, we obtain that it is topologically equivalent to a (smooth) spiral sink at the origin, e.g., the flow of the linear vector field $X(x, y) = (-x - y, x - y)^T$.

Similarly, we can show that also Ψ^{BB} is topologically equivalent to the flow of the same X . Since topological equivalence is transitive, we obtain that $WT \approx BB$, as claimed. We will see later that in dimension two this picture is typical.

Example 5.3. $T^2(1)$ is not equivalent to $T^2(\sqrt{2})$. Even though the hybrid fold for both hybrid systems is the same (the 2-torus), every orbit of $T^2(1)$ is closed, while every orbit of $T^2(\sqrt{2})$ is dense in T^2 . Since equivalence always sends closed orbits to closed orbits, the statement above follows immediately.

Ideally, one would like to be able to classify all hybrid systems up to topological equivalence. Unfortunately, this attempt fails even for smooth dynamical systems on compact boundaryless manifolds of dimension greater

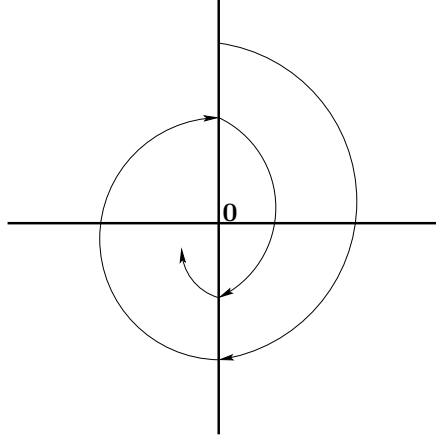


Figure 16: Flow in \mathbb{R}^2 topologically equivalent to Ψ^{WT} .

than two, as can be seen in the standard dynamics literature (for instance, [31]). However, it turns out that it is possible to obtain a complete picture of the local behavior of 2-dimensional hybrid flows near an isolated Zeno state. For completeness, we include an easy observation for 1-dimensional hybrid flows.

Theorem 5.4. *Assume \mathbf{H} is a regular hybrid system.*

- (a) *If $\dim \mathbf{H} = 1$, then \mathbf{H} has no Zeno states.*
- (b) *Suppose $\dim \mathbf{H} = 2$, \mathbf{H} is without branching, and $z \in M_{\mathbf{H}}$ is an isolated Zeno state. Then there is a neighborhood U of z in $M_{\mathbf{H}}$ and a neighborhood V of $\mathbf{0}$ in M_{WT} such that $\Psi^{\mathbf{H}}|_U$ is topologically equivalent to $\Psi^{WT}|_V$.*

Proof. (a) Topologically, $M_{\mathbf{H}}$ must be either the circle or an interval (bounded or unbounded, open, closed or half-closed). Suppose $M_{\mathbf{H}}$ contains a Zeno state for some $x \in M_{\mathbf{H}}$. Then $\chi(x)$ must visit at least one domain infinitely often, so by construction, $M_{\mathbf{H}}$ must be a circle. Therefore, $\chi(x)$ must go around this circle and hence must be the whole $M_{\mathbf{H}}$. This implies that its execution time must be infinite, which is a contradiction.

(b) Let z be the Zeno state for some $x_0 \in M_{\mathbf{H}}$. Then $\tau_{\infty}(x_0) > 0$. Denote by χ_0 the execution starting from x_0 . Assume

$$\pi^{-1}(z) = \{z_1, \dots, z_m\},$$

and $z_j \in D_j$ (if this is not the case, reorder the domains). Let $\tilde{\chi}_0 = \pi^{-1}(\chi_0)$ and assume $\tilde{\chi}_0$ visits the domains D_1, \dots, D_l infinitely often, for some $1 \leq$

$l \leq m$ (if necessary, reorder the domains). Then $\tilde{\chi}_0$ must accumulate at z_1, \dots, z_l . Suppose that starting from D_1 , it goes through D_2, \dots, D_l in that order. This means that $e_j = (j, j+1)$, for $j = 1, \dots, l-1$, are all allowed transitions, i.e., $e_j \in E$. Denote

$$A_j = \text{im } R_{e_{j-1}}, \quad B_j = G(e_j).$$

It is easy to see that $z_j \in \overline{A_j} \cap \overline{B_j}$ ($1 \leq j \leq l-1$). At this point, after passing through D_1, \dots, D_{l-1} , and completing its evolution through D_l , $\tilde{\chi}_0$ has to switch to one of the domains D_1, \dots, D_l . We show that it must switch to D_1 . To see this, observe that at z_j (as at any other boundary point of any guard or image of a reset), A_j , B_j , ∂A_j , and ∂B_j all have to look like manifolds of appropriate dimensions. Therefore, since $\dim D_j = 2$, there is no guard in D_j , other than B_j , containing z_j in its boundary, and there is no reset, other than $R_{(j-1,j)}$ whose image contains z_j in its boundary. Assume now that from D_{l-1} , $\tilde{\chi}_0$ switches to D_r , for some $r \neq 1$. Then, again since $\tilde{\chi}_0$ accumulates at z_1, \dots, z_l , and by the previous observation about the structure of A_j 's and B_j , after completing its evolution through D_r , it will be forced to switch to D_{r+1} , etc., therefore avoiding D_1 altogether in the future, which is a contradiction.

Therefore, we can, without loss, assume that $\tilde{\chi}_0$ visits $D_1, D_2, \dots, D_l, D_1, D_2, \dots$, respectively.

Next, the basic observation is: since $X_j(z_j) \neq 0$, for every j , no trajectory of X_j can reach z_j in finite time. Otherwise, it would prevent χ_0 from making infinitely many turns around z . Details follow.

1. We claim that for each j there exists V_j , a neighborhood of z_j in D_j , such that every execution starting in $A_j \cap V_j - \{z_j\}$ reaches $B_j \cap V_j - \{z_j\}$.

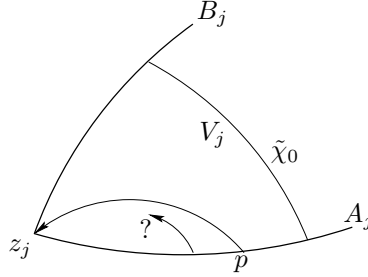


Figure 17: The execution starting at p cannot reach z_j .

To prove this, let V_j be the region in D_j bounded by A_j , B_j and a single arc of $\tilde{\chi}_0$, and let $p \in A_j \cap V_j - \{z_j\}$ be an arbitrary point; see Fig. 17. The execution $\chi(p)$ cannot intersect $\tilde{\chi}_0$, so it must reach $\overline{B_j} \cap V_j$. If it passes through z_j before it reaches $B_j \cap V_j - \{z_j\}$, then *every* execution starting from a point in $\overline{A_j}$ between p and z_j must pass through z_j . This is impossible

since, according to Proposition 4.4, z_j is not an equilibrium of X_j (here we also used the fact that X_j points inside D_j along A_j). Therefore, $\chi(p)$ reaches $B_j \cap V_j - \{z_j\}$.

Let $V = \bigcup V_j$ and $U = \pi(V)$.

2. We now investigate the only two possibilities:

Case 1 : ∂D_j is smooth at z_j . Then it follows immediately from (A6) and Proposition 4.4 that X_j is *tangent* to ∂D_j at z_j . Therefore, by 1., in a smooth flow box around z_j , the local picture is as in part (A) of Fig. 18. (Recall that X_j extends smoothly to a neighborhood of D_j .)

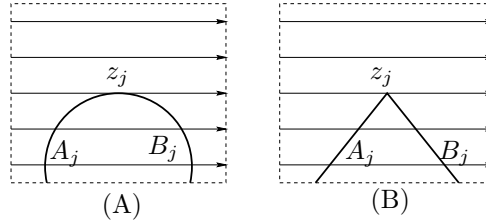


Figure 18: Local picture around z_j .

Case 2 : ∂D_j is not smooth at z_j . Because of 1., it is not difficult to see that the local picture around z_j looks like part (B) of Fig. 18.

3. In fact, in both cases, up to a continuous change of coordinates, the local picture around z_j looks like part (B) of Fig. 18, but with A_j and B_j being straight line segments. To construct a topological equivalence between $\Psi^{\mathbf{H}}$ near z_j and Ψ^{WT} near $\mathbf{0}$, subdivide D_1 of WT into $l - 1$ subdomains D'_1, \dots, D'_{l-1} by $l - 2$ rays from the origin. Let $D'_l = D_2$. Define a hybrid system WT_l by: the domains are D'_1, \dots, D'_l , the vector fields X'_j are the restrictions of the vector fields of WT to the corresponding new domains, and the resets are identity maps.

It is easily seen that X_j on V_j is topologically equivalent to X'_j on D'_j . Call the conjugating homeomorphism h_j . Glue the h_j 's together to obtain a homeomorphism h between $\Psi^{\mathbf{H}}$ on U and Ψ^{WT_l} on a neighborhood of $\mathbf{0}$. Since Ψ^{WT_l} is clearly equivalent to Ψ^{WT} , the proof of the theorem is complete. \square

6 Concluding remarks

We study a class of hybrid systems which can be identified with piecewise smooth continuous-time dynamical systems on piecewise smooth manifolds. This is the class of “regular hybrid systems with no branching”. In our framework, which includes many known examples and makes the hybrid systems

under consideration manageable for analysis, we investigate the Zeno phenomenon, show that it is due to the lack of smoothness, and classify it locally in dimension two.

This article is an initial step towards a global geometric theory of hybrid systems. Some work in the same direction can be found in [12, 21, 34, 35], where we studied structural stability, stability of equilibria, and stability of closed orbits of hybrid systems. We are interested in global geometric and topological properties of a larger class of hybrid systems, including non-deterministic ones, examples of “chaotic” behavior in dimension two (which would further distinguish hybrid systems from smooth 2-dimensional flows) and different ways of studying stability of invariant sets, including Zeno ones. We postpone the discussion of these and other question to our forthcoming articles.

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