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Singular Perturbation Approximation of Semistable Linear Systems

Takayuki Ishizaki[†1], Henrik Sandberg[2], Karl Henrik Johansson[2], Kenji Kashima[3], Jun-ichi Imura[1], Kazuyuki Aihara[4]

Abstract— **This paper proposes a singular perturbation approximation for semistable linear systems. In particular, we derive a novel expression of error systems in the Laplace domain. As a result, we obtain an $\mathcal{H}_2$-error bound in terms of the sum of eigenvalues of an index matrix, which coincides with a controllability gramian of the state-derivative. Furthermore, we show that the singular perturbation model appropriately preserves the semistability of the original system and also guarantees the stability of the error system. The efficiency of the proposed method is shown through a numerical example of a Markov chain model approximation.**

I. Introduction

Along with recent technical development, the architecture of systems covered by the control community has tended to become more complex and larger in scale. For example, in weather prediction simulations, the system dimension possibly reaches hundreds of thousands. Therefore, it is crucial to develop model approximation methods to reduce system complexity. As one of possible approaches, model reduction methods have been extensively developed in literature, and they have been used as an efficient tool for analyzing and/or synthesizing large-scale systems; see [1], [2] for an overview.

For various purposes, a number of model reduction frameworks have been proposed in past decades. For instance, model reduction methods based on a notion of the principal component analysis, which includes the balanced truncation [3], [4] and the Hankel norm approximation [5], are well known. A major advantage of this kinds of methods is that an error bound is available in terms of, e.g., the $\mathcal{H}_\infty$-, or Hankel norm. Furthermore, moment matching methods, which include the Krylov projection methods, are also well known [6], [7]. This kind of methods aims to suppress the discrepancy of the response for specific input signals, and has an advantage that they can be implemented in a computationally efficient manner. However, unlike the aforementioned methods, no error bound is available, except an optimal $\mathcal{H}_2$-reduction procedure has been developed in the past few years [8].

[†]Research Fellow of the Japan Society for the Promotion of Science
[1]Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology; 2-12-1, Meguro, Tokyo, Japan: {ishizaki, imura}@mei.titech.ac.jp
[2]School of Electrical Engineering, Automatic Control, Royal Institute of Technology (KTH), SE-100 44 Stockholm, Sweden: {hsan, kallej}@ee.kth.se
[3]Department of Systems Innovation, Graduate School of Engineering Science, Osaka University; 1-3, Machikaneyama, Toyonaka, Osaka, Japan: kashima@sys.es.osaka-u.ac.jp
[4]Institute of Industrial Science, University of Tokyo; 4-6-1 Komaba, Meguro ward, Tokyo, Japan: aihara@sat.t.u-tokyo.ac.jp

In addition to the two kinds of methods above, singular perturbation-based approximation has been developed as one of frameworks to reduce the complexity of dynamical systems. Actually, it is known that this method preserves certain important properties such as stability and steady-state distribution under appropriate conditions. However, the applicability of the classical singular perturbation theory is still limited. This is because the classical theory is based on the premise that systems of interest are intrinsically separated into subsystems having different time scales [9], [10]. It should be remarked that most of the existing error analyses are carried out based on this time scale separation, which often leads to the asymptotic analysis in the time domain [11], [12].

In view of this, we attempt to develop a novel singular perturbation-based model reduction method for semistable linear systems. Our mathematical formulation does not require a priori time scale separation. Instead, we introduce a *pre-conditioning* coordinate transformation to separate time scales. In this formulation, we analyze how the coordinate selection affects the stability and approximation error. Unlike the classical theory, our error analysis is based on deriving a novel expression of error systems in the Laplace domain, which provides an insight to properly quantifying the approximation quality. This work is a generalization of our work in [13], where we have developed a singular perturbation approximation for stable systems. This generalization enables us to deal with more broader class of systems including Markov chain systems and multi-agent systems arising in consensus problems. It should be remarked that the approximation of semistable systems is more challenging task than that of stable systems. This is because the (semi)stability of approximants does not necessarily imply the stability of the error system. In this paper, we prove the stability of the error system by exactly preserving the semistable eigenspaces of the original system into those of the singular perturbation model.

The rest of this paper is organized as follows: In Section II, we mathematically formulate a singular perturbation approximation to be investigated. In Section III, we provide main results of this paper, which include the derivation of a novel error system expression in the Laplace domain, the stability analysis of the error system, and the approximation error analysis in terms of the $\mathcal{H}_2$-norm. Furthermore, based on this theoretical development, we provide an algorithm to implement the proposed approximation method systematically. In Section IV, we show the efficiency of the proposed approximation through a numerical example, where the reduction

of a Markov chain model is considered. Finally, concluding remarks are provided in Section V.

NOTATION The following notation is to be used. $\mathbb{R}$: the set of real numbers; $\mathrm{im}(M)$: the image of a matrix M; $\mathrm{tr}(M)$: the trace of a matrix M; $\|M\|$: the induced 2-norm of a matrix M, i.e., the maximum singular value of M; $\mathcal{V} \perp \mathcal{W}$: the orthogonality of the spaces between $\mathcal{V}$ and $\mathcal{W}$, i.e., $v^{\mathsf{T}} w = 0$ holds for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$; $\mathrm{diag}(M_1, \ldots, M_n)$: the block diagonal matrix whose diagonal blocks are composed of matrices $M_1, \ldots, M_n$.

The negative (semi)definiteness of a matrix $M \in \mathbb{R}^{n\times n}$, which is not necessarily symmetric, is denoted by $M \prec O_n$ ($M \preceq O_n$), i.e., $x^{\mathsf{T}} M x < 0$ ($x^{\mathsf{T}} M x \leq 0$) holds for all $x \neq 0 \in \mathbb{R}^n$. Similarly, the positive definiteness is denoted by $M \succ O_n$.

A matrix A (resp. a transfer matrix G) is said to be *semistable* if its eigenvalues (poles) are all in the closed left-half plane, and all eigenvalues (poles) on the imaginary axis are simple. In particular, A (resp. G) is said to be *stable* if all eigenvalues (poles) are in the open left-half plane.

For transfer matrices, we use the notation of

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI_n - A)^{-1}B + D.$$

The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G and the $\mathcal{H}_2$-norm of a stable strictly proper transfer matrix G are defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|$$
$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)G^{\mathsf{T}}(-j\omega))d\omega\right)^{\frac{1}{2}},$$

respectively.

II. Preliminaries

In this section, we mathematically formulate a singular perturbation approximation to be investigated. Let us consider a linear system

$$\Sigma : \left\{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}\right. \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m_y\times n}$ and $D \in \mathbb{R}^{m_y\times m_u}$. In the literature on the singular perturbation theory, it is often assumed that the system (1) intrinsically has several different time scales [9], [10]. In contrast to this, we do not impose such an assumption, but introduce a coordinate transformation to decouple time scales. More specifically, considering the coordinate transformation of Σ by a unitary matrix $[P, \overline{P}] \in \mathbb{R}^{n\times n}$ with $P \in \mathbb{R}^{n\times\hat{n}}$ and $\overline{P} \in \mathbb{R}^{n\times(n-\hat{n})}$, we obtain

$$\tilde{\Sigma} : \left\{\begin{array}{rl} \left[\begin{array}{c} \dot{\xi} \\ \dot{\eta} \end{array}\right] &= \left[\begin{array}{cc} P^{\mathsf{T}}AP & P^{\mathsf{T}}A\overline{P} \\ \overline{P}^{\mathsf{T}}AP & \overline{P}^{\mathsf{T}}A\overline{P} \end{array}\right]\left[\begin{array}{c} \xi \\ \eta \end{array}\right] + \left[\begin{array}{c} P^{\mathsf{T}}B \\ \overline{P}^{\mathsf{T}}B \end{array}\right]u \\ y &= \left[\begin{array}{cc} CP & C\overline{P} \end{array}\right]\left[\begin{array}{c} \xi \\ \eta \end{array}\right] + Du. \end{array}\right. \tag{2}$$

To reduce the dynamical dimension of $\tilde{\Sigma}$, let us impose $\dot{\eta} \equiv 0$, which means that the behavior of η is to be algebraically determined by ξ and u. In other words, the static state $\hat{\eta}$, which denotes the approximant of η, is constrained by the algebraic equation

$$\hat{\eta} = -(\overline{P}^{\mathsf{T}}A\overline{P})^{-1}\overline{P}^{\mathsf{T}}AP\hat{\xi} - (\overline{P}^{\mathsf{T}}A\overline{P})^{-1}\overline{P}^{\mathsf{T}}Bu \tag{3}$$

where $\hat{\xi}$ is the approximant of ξ and $\overline{P}^{\mathsf{T}}A\overline{P}$ is assumed to be non-singular. This approximation is intuitively reasonable when the convergence rate of η is sufficiently faster than that of ξ.

Substituting (3) into the equation with respect to $\dot{\xi}$, we have the singular perturbation model

$$\hat{\Sigma}_{\mathrm{sp}} : \left\{\begin{array}{l} \dot{\hat{\xi}} = \hat{A}\hat{\xi} + \hat{B}u \\ \hat{y} = \hat{C}\hat{\xi} + \hat{D}u \end{array}\right. \tag{4}$$

where

$$\begin{aligned} \hat{A} &:= P^{\mathsf{T}}AP - P^{\mathsf{T}}A\Pi AP \in \mathbb{R}^{\hat{n}\times\hat{n}} \\ \hat{B} &:= (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)B \in \mathbb{R}^{\hat{n}\times m_u} \\ \hat{C} &:= C(P - \Pi AP) \in \mathbb{R}^{m_y\times\hat{n}} \\ \hat{D} &:= D - C\Pi B \in \mathbb{R}^{m_y\times m_u} \end{aligned} \tag{5}$$

and

$$\Pi := \overline{P}(\overline{P}^{\mathsf{T}}A\overline{P})^{-1}\overline{P}^{\mathsf{T}} \in \mathbb{R}^{n\times n}. \tag{6}$$

Note that this Π does not depend on the basis selected for the projection $\overline{P}$. This follows from the fact that

$$\Pi = \overline{P}H(H^{\mathsf{T}}\overline{P}^{\mathsf{T}}A\overline{P}H)^{-1}H^{\mathsf{T}}\overline{P}^{\mathsf{T}}$$

holds for any unitary matrix $H \in \mathbb{R}^{(n-\hat{n})\times(n-\hat{n})}$. This implies that the singular perturbation model $\hat{\Sigma}_{\mathrm{sp}}$ in (4) depends only on the choice of P. In the following section, we analyze how the choice of the projection P affects the approximation error between Σ and $\hat{\Sigma}_{\mathrm{sp}}$.

III. Singular Perturbation Analysis

A. Factorization of Error System

In the rest of this paper, we define the transfer matrix of Σ in (1) by

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{7}$$

Furthermore, we define the set of projectors

$$\mathcal{P}^{n\times\hat{n}} := \{P \in \mathbb{R}^{n\times\hat{n}} : P^{\mathsf{T}}P = I_{\hat{n}}, \quad \hat{n} \leq n\}. \tag{8}$$

In this notation, the approximant of G is defined as follows:

Definition 1: Consider a transfer matrix G in (7). The *singular perturbation approximant of G associated with $P \in \mathcal{P}^{n\times\hat{n}}$* is defined by

$$\hat{G}(s; P) := \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \tag{9}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are given by (5).

Obviously, the quality of the approximant $\hat{G}$ depends on the determination of $P \in \mathcal{P}^{n\times\hat{n}}$. In literature on the singular

perturbation theory, most of error analyses are carried out either in the time domain by using the asymptotic analysis [11], [12], or based on the premise of the balanced realization [14]. In contrast to this, we analyze the approximation error in the Laplace domain without relying on the balanced realization. The following theorem shows a tractable representation of the error system in the Laplace domain:

Theorem 1: Given a transfer matrix G in (7), let $\hat{G}$ be the singular perturbation approximant of G in (9) associated with $P \in \mathcal{P}^{n\times\hat{n}}$. Then

$$G(s) - \hat{G}(s;P) = \Xi(s;P)\overline{P}\,\overline{P}^{\mathsf{T}} X(s) \tag{10}$$

holds, where

$$\Xi(s;P) := \left[\begin{array}{c|c} \hat{A} & P^{\mathsf{T}}A\Pi \\ \hline \hat{C} & C\Pi \end{array}\right],\ X(s) := \left[\begin{array}{c|c} A & B \\ \hline A & B \end{array}\right] \tag{11}$$

with $\hat{A}$ and $\hat{C}$ defined in (5).

Proof: Denote the error system by

$$G(s) - \hat{G}(s;P) := \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right]$$

where $A_e = \mathrm{Diag}(\hat{A}, A),\ B_e = [\hat{B}^{\mathsf{T}}, B^{\mathsf{T}}]^{\mathsf{T}},\ C_e = [-\hat{C}, C]$ and $D_e = -\hat{D}+D$. Considering the similarity transformation of the error system with

$$T = \left[\begin{array}{cc} I_{\hat{n}} & -(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} \\ 0 & I_n \end{array}\right],$$
$$T^{-1} = \left[\begin{array}{cc} I_{\hat{n}} & (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} \\ 0 & I_n \end{array}\right],$$

we have

$$\begin{aligned} &TA_eT^{-1} = \left[\begin{array}{cc} \hat{A} & \mathcal{A} \\ 0 & A \end{array}\right],\ TB_e = \left[\begin{array}{c} -P^{\mathsf{T}}A\Pi\overline{P}\,\overline{P}^{\mathsf{T}}B \\ B \end{array}\right] \\ &C_eT^{-1} = \left[\begin{array}{cc} -\hat{C} & \mathcal{C} \end{array}\right],\quad D_e = C\Pi B \end{aligned} \tag{12}$$

where

$$\begin{aligned} \mathcal{A} &= \hat{A}(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} - (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}}A \\ \mathcal{C} &= -\hat{C}(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} + C. \end{aligned}$$

To make the following argument clear, we first assume that

$$\hat{A}(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} = (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)A \tag{13}$$

holds. Under this assumption, we have

$$\mathcal{A} = (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)(I_n - PP^{\mathsf{T}})A = -P^{\mathsf{T}}A\Pi\overline{P}\,\overline{P}^{\mathsf{T}}A.$$

Furthermore, we have

$$\mathcal{C} = -C\left\{(I_n - \Pi A)(I_n - \overline{P}\,\overline{P}^{\mathsf{T}}) - I_n\right\} = C\Pi A.$$

Thus, the block structure of (12) implies that the error system $G - \hat{G}$ is given by

$$\begin{aligned} &\hat{C}(sI_{\hat{n}} - \hat{A})^{-1}P^{\mathsf{T}}A\Pi\overline{P}\,\overline{P}^{\mathsf{T}}A(sI_n - A)^{-1}B + C\Pi B \\ &\quad + \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}P^{\mathsf{T}}A\Pi\overline{P}\,\overline{P}^{\mathsf{T}}B + C\Pi A(sI_n - A)^{-1}B. \end{aligned}$$

By noting that $\Pi = \Pi\overline{P}\,\overline{P}^{\mathsf{T}}$ holds, the factorization of (10) is verified.

What remains to be shown is (13). To this end, it suffices to show that

$$P^{\mathsf{T}}(I_n - A\Pi)A\left\{P(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}} - I_{\hat{n}}\right\} = 0$$

where $\hat{A} = P^{\mathsf{T}}(I_n - A\Pi)AP$ is substituted. By using $\Pi P = 0$, the left-hand side is equal to

$$\begin{aligned} &P^{\mathsf{T}}(I_n - A\Pi)A(PP^{\mathsf{T}} - I_{\hat{n}}) \\ &\quad = P^{\mathsf{T}}(I_n - A\Pi)A\overline{P}\,\overline{P}^{\mathsf{T}} = P^{\mathsf{T}}(A\overline{P} - A\overline{P})\overline{P}^{\mathsf{T}} = 0 \end{aligned}$$

which proves (13). ∎

The factorization of the error system shown in Theorem 1 provides an interesting insight, that is, the singular perturbation approximation works well if the norm of $\overline{P}^{\mathsf{T}}X$ is sufficiently small, where $\overline{P}$ is an orthogonal complement of P. It is also interesting to note that X in (11) coincides with the transfer matrix from u to $\dot{x}$ of the original system Σ.

B. Stability Analysis of Error System

Next, we investigate the relation between the choice of P and the stability of the error system. In the rest of this paper, we assume that the original system Σ in (1) is semistable and, for simplicity, A is diagonalizable.

1) Case of Stable Systems: First of all, we state the following fundamental lemma to ensure the (semi)stability preservation of the approximant:

Lemma 1: For any $A \in \mathbb{R}^{n\times n}$ and $P \in \mathcal{P}^{n\times\hat{n}}$, $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ in (5) is factorized as

$$\hat{A} = (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)A(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)^{\mathsf{T}} \tag{14}$$

where $\Pi \in \mathbb{R}^{n\times n}$ is defined by (6). In addition, if $A \prec O_n$ (resp. $A \preceq O_n$) holds, then $\hat{A} \prec O_{\hat{n}}$ ($\hat{A} \preceq O_{\hat{n}}$) holds.

Proof: Using the relation of

$$(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}}\{(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)PP^{\mathsf{T}}\}^{\mathsf{T}} = I_{\hat{n}} \tag{15}$$

for (13), we obtain

$$\begin{aligned} \hat{A} &= (P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)APP^{\mathsf{T}}(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)^{\mathsf{T}} \\ &= \left[(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)A(I_n - \overline{P}\,\overline{P}^{\mathsf{T}})\right](P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)^{\mathsf{T}} \\ &= \left[(P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)A\right](P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi)^{\mathsf{T}}, \end{aligned}$$

which proves (14). Note that (15) implies that

$$\mathrm{rank}(Z) = \hat{n},\quad Z := P^{\mathsf{T}} - P^{\mathsf{T}}A\Pi.$$

This is proven by the fact that if we assume $\mathrm{rank}(Z) < \hat{n}$, then

$$\mathrm{rank}(ZPP^{\mathsf{T}}) \le \min(\mathrm{rank}(Z), \mathrm{rank}(PP^{\mathsf{T}})) < \hat{n},$$

which contradicts $\mathrm{rank}(ZPP^{\mathsf{T}}) = \mathrm{rank}(I_{\hat{n}}) = \hat{n}$. Therefore, if $A \prec O_n$ ($A \preceq O_n$), then $\hat{A} \prec O_{\hat{n}}$ ($\hat{A} \preceq O_{\hat{n}}$). ∎

Lemma 1 shows that the negative (semi)definiteness of A, which is a stronger (semi)stability condition, is preserved through the singular perturbation approximation. Obviously, the error system $G - \hat{G}$ in (10) is stable if both $A \prec O_n$ and $\hat{A} \prec O_{\hat{n}}$ hold. Note that for any stable matrix $A \in$

$\mathbb{R}^{n\times n}$, $V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1} \prec O_n$ holds for the Cholesky factor $V_{\frac{1}{2}}$ of a Lyapunov function such that

$$V = V_{\frac{1}{2}}^{\mathsf{T}} V_{\frac{1}{2}} \succ O_n, \quad VA + A^{\mathsf{T}} V \prec O_n.$$

This fact ensures the existence of a *pre-conditioning* coordinate transformation to make stable A negative definite.

2) Case of Semistable Systems: It should be remarked that the semistability of the approximant does not imply the stability of the error system. To guarantee its stability, we need further consideration on semistable eigenspaces. We define the eigenspace $\mathcal{E}$ of A associated with all eigenvalues on the imaginary axis by

$$v \in \mathcal{E}(A) \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{R} \text{ s.t. } Av = j\lambda v.$$

Furthermore, we define the space spanned by all $v \in \mathcal{E}$ as

$$\mathcal{V}(A) = \sum_{\forall v \in \mathcal{E}(A)} \mathrm{im}(v).$$

Using this notation, we state the following lemma:

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$ and $P \in \mathcal{P}^{n\times \hat{n}}$ be given. If $\mathcal{V}(A) \subseteq \mathrm{im}(P)$ holds, then $P^{\mathsf{T}} v \in \mathcal{V}(\hat{A})$ holds for any $v \in \mathcal{V}(A)$, where $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ in (5). Similarly, if $\mathcal{V}(A^{\mathsf{T}}) \subseteq \mathrm{im}(P)$ holds, then $P^{\mathsf{T}} v \in \mathcal{V}(\hat{A}^{\mathsf{T}})$ holds for any $v \in \mathcal{V}(A^{\mathsf{T}})$.

Proof: If $\mathcal{V}(A) \subseteq \mathrm{im}(P)$, then for any $v \in \mathcal{V}(A)$

$$PP^{\mathsf{T}} v = v, \quad \overline{P}^{\mathsf{T}} v = 0$$

hold, where $\overline{P}$ denotes an orthogonal complement of P. Thus, there exists $\lambda \in \mathbb{R}$ such that $\hat{A}(P^{\mathsf{T}} v) = j\lambda (P^{\mathsf{T}} v)$, which implies $P^{\mathsf{T}} v \in \mathcal{V}(\hat{A})$. The second claim is also proven by the same manner. ■

Lemma 2 provides a condition to exactly preserve the particular left and right eigenspaces upon the approximation in (4). For convenience of notation, we define the set

$$\mathcal{A}^{n\times n} := \{A \in \mathbb{R}^{n\times n} : A \preceq O_n, \ \mathcal{V}(A) = \mathcal{V}(A^{\mathsf{T}})\}. \quad (16)$$

The following fact is fundamental for the arguments below:

Lemma 3: For any semistable $A \in \mathbb{R}^{n\times n}$, there exists a non-singular matrix $V \in \mathbb{R}^{n\times n}$ such that $V^{-1} A V \in \mathcal{A}^{n\times n}$ holds.

Proof: We denote the dimension of $\mathcal{V}(A)$ by ν. By eigendecomposition, there exist $V = [V_1, V_2] \in \mathbb{R}^{n\times n}$ and $V^{-1} = [U_1^{\mathsf{T}}, U_2^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ such that

$$U_1 A V_1 + (U_1 A V_1)^{\mathsf{T}} = 0, \ U_2 A V_2 = \mathrm{diag}(\lambda_1 \ldots, \lambda_{n-\nu})$$

for the diagonal blocks of

$$V^{-1} A V = \begin{bmatrix} U_1 A V_1 & 0 \\ 0 & U_2 A V_2 \end{bmatrix}.$$

Note that $U_1 A V_1$ is a skew symmetric matrix that satisfies $\mathcal{V}(U_1 A V_1) = \mathcal{V}((U_1 A V_1)^{\mathsf{T}})$. Hence, the claim follows. ■

This lemma shows that there always exists a coordinate transformation that changes the semistability into the negative semidefiniteness and matches the particular left and right eigenspaces. This match of the left and right eigenspaces is useful for preserving the eigenspaces, and will play a nice role for constructing a qualified approximant. Unfortunately, finding such coordinate transformation for general semistable matrices relies on the eigendecomposition, which possibly requires heavy computational load for large-scale systems. However, for irreducible Metzler matrices, we do not need the full eigendecomposition; see Remark 1.

Combining all lemmas above, we obtain the following theorem that ensures the stability of the error system:

Theorem 2: Given a transfer matrix G in (7), let $\hat{G}$ be the singular perturbation approximant of G in (9) associated with $P \in \mathcal{P}^{n\times\hat{n}}$. Assume that $A \in \mathcal{A}^{n\times n}$. If $\mathcal{V}(A) \subseteq \mathrm{im}(P)$ holds, then the error system $G - \hat{G}$ is stable.

Proof: To prove the claim, it suffices to show that both Ξ and $\overline{P}^{\mathsf{T}} X$ in (10) are stable. Obviously, $\overline{P}^{\mathsf{T}} X$ is semistable due to the assumption of $A \preceq O_n$. Furthermore, we notice that $\overline{P}^{\mathsf{T}} A v = 0$ holds for all $v \in \mathcal{V}(A) \subseteq \mathrm{im}(P)$. This implies that the eigenspace of the observability pair $(A, \overline{P}^{\mathsf{T}} A)$ associated with all eigenvalues on imaginary axis is unobservable. Thus, $\overline{P}^{\mathsf{T}} X$ is stable.

Similarly, we consider the eigenspace of the controllability pair $(\hat{A}, P^{\mathsf{T}} A \Pi)$. Note that the semistability of Ξ is ensured by Lemma 1. From Lemma 2 and the assumption of $\mathcal{V}(A) = \mathcal{V}(A^{\mathsf{T}})$, we verify that $P^{\mathsf{T}} v \in \mathcal{V}(\hat{A}^{\mathsf{T}})$ holds for any $v \in \mathcal{V}(A^{\mathsf{T}}) \subseteq \mathrm{im}(P)$. Thus, we have

$$(P^{\mathsf{T}} v)^{\mathsf{T}} P^{\mathsf{T}} A \Pi = v^{\mathsf{T}} P P^{\mathsf{T}} A \Pi = v^{\mathsf{T}} A \Pi = 0,$$

which means the uncontrollability of the eigenspace. Hence, Ξ is stable as well. ■

Theorem 2 provides a sufficient condition to ensure the stability of the error system. The proof of this theorem is based on the fact that the semistable eigenspace of the original system is exactly preserved into that of the singular perturbation model.

Remark 1: For any semistable irreducible Metzler matrix $A \in \mathbb{R}^{n\times n}$, there exists a positive diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $D^{-1} A D \in \mathcal{A}^{n\times n}$ holds [15]. In addition, the diagonal entries of D can be constructed by using the left and right Frobenius eigenvectors of A, which is efficiently found by the existing algorithms such as the power method [1]. Thus, a coordinate transformation to achieve $A \in \mathcal{A}^{n\times n}$ can be efficiently found for positive systems including electric circuit systems and Markov chain systems [16].

C. Construction of Approximant

In this subsection, we construct an approximant with a good approximation quality. Recall that the singular perturbation approximation works well if the norm of $\overline{P}^{\mathsf{T}} X$ is sufficiently small. The following lemma provides an expression of the $\mathcal{H}_2$-norm of $\overline{P}^{\mathsf{T}} X$:

Lemma 4: Consider the transfer matrix X in (11) and assume that $A \in \mathcal{A}^{n\times n}$. Let $[V, \overline{V}] \in \mathbb{R}^{n\times n}$ be a unitary matrix such that $\mathrm{im}(V) = \mathcal{V}(A)$ holds. Define

$$\Phi := \overline{V}^{\mathsf{T}} A \overline{V} \phi \overline{V}^{\mathsf{T}} A^{\mathsf{T}} \overline{V} \in \mathbb{R}^{(n-\nu)\times(n-\nu)} \quad (17)$$

where ν denotes the dimension of $\mathcal{V}(A)$ and

$$\phi := \int_0^\infty e^{\overline{V}^\mathsf{T} A \overline{V} \tau} \overline{V}^\mathsf{T} B (e^{\overline{V}^\mathsf{T} A \overline{V} \tau} \overline{V}^\mathsf{T} B)^\mathsf{T} d\tau. \quad (18)$$

If $\mathrm{im}([V,B]) \subseteq \mathrm{im}(P)$ holds, then the $\mathcal{H}_2$-norm of $\overline{P}^\mathsf{T} X$ is bounded and is given by

$$\|\overline{P}^\mathsf{T} X(s)\|_{\mathcal{H}_2} = \sqrt{\mathrm{tr}(\overline{P}^\mathsf{T} \overline{V} \Phi \overline{V}^\mathsf{T} \overline{P})}. \quad (19)$$

Proof: Using the relations $\overline{P}^\mathsf{T} B = 0$, $\overline{P}^\mathsf{T} A V = 0$ and $\overline{P}^\mathsf{T} = \overline{P}^\mathsf{T} \overline{V}\, \overline{V}^\mathsf{T}$, we have

$$\begin{aligned}
\overline{P}^\mathsf{T} X(s) &= \overline{P}^\mathsf{T} A (sI_n - A)^{-1} B + \overline{P}^\mathsf{T} B \\
&= \overline{P}^\mathsf{T} A [V, \overline{V}] (sI_n - \tilde{A})^{-1} [V, \overline{V}]^{-1} B \\
&= [0, \overline{P}^\mathsf{T} A \overline{V}] (sI_n - \tilde{A})^{-1} \begin{bmatrix} V^\mathsf{T} B \\ \overline{V}^\mathsf{T} B \end{bmatrix} \\
&= \overline{P}^\mathsf{T} A \overline{V} \tilde{X}(s) \\
&= \overline{P}^\mathsf{T} \overline{V}\, \overline{V}^\mathsf{T} A \overline{V} \tilde{X}(s)
\end{aligned}$$

where $\tilde{X}(s) := (sI_{n-\nu} - \overline{V}^\mathsf{T} A \overline{V})^{-1} \overline{V}^\mathsf{T} B$ is strictly proper and stable, and

$$\tilde{A} := [V, \overline{V}]^{-1} A [V, \overline{V}] = \begin{bmatrix} V^\mathsf{T} A V & 0 \\ 0 & \overline{V}^\mathsf{T} A \overline{V} \end{bmatrix}.$$

Note that ϕ is the controllability gramian of $\tilde{X}$. Thus

$$\|\overline{P}^\mathsf{T} X(s)\|_{\mathcal{H}_2}^2 = \mathrm{tr}(\overline{P}^\mathsf{T} \overline{V} \Phi \overline{V}^\mathsf{T} \overline{P})$$

follows. ∎

Note that if A is stable, namely, if $\mathcal{V}(A) = \emptyset$, then $\overline{V}$ is reduced to I_n. This means that if the original Σ is stable, ϕ in (18) coincides with the usual controllability gramian of Σ, and Φ in (17) is reduced to $A\phi A^\mathsf{T}$, which coincides with the controllability gramian of the state-derivative. Based on Lemma 4, we obtain the following theorem to regulate the approximation quality:

Theorem 3: Given a transfer matrix G in (7), let $\hat{G}$ be the singular perturbation approximant of G in (9) associated with $P \in \mathcal{P}^{n \times \hat{n}}$. Assume that $A \in \mathcal{A}^{n \times n}$. Let $[V, \overline{V}] \in \mathbb{R}^{n \times n}$ be a unitary matrix such that $\mathrm{im}(V) = \mathcal{V}(A)$ holds. Furthermore, let $[W, \overline{W}] \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ be a unitary matrix such that

$$\Phi \overline{W} = \overline{W} \mathrm{diag}(\lambda_1, \ldots, \lambda_L)$$

holds for $\Phi \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ in (17), where ν denotes the dimension of $\mathcal{V}(A)$. If

$$\mathrm{im}(P) = \mathrm{im}([V, \overline{V} W, B]) \quad (20)$$

holds, then $\hat{G}$ in (9) satisfies

$$\|\hat{G}(s;P) - G(s)\|_{\mathcal{H}_2} \le \|\Xi(s;P)\|_{\mathcal{H}_\infty} \sqrt{\sum_{l=1}^{L} \lambda_l} \quad (21)$$

where Ξ is defined in (11).

Proof: Using Theorem 1, we have

$$\|\hat{G}(s;P) - G(s)\|_{\mathcal{H}_2} \le \|\Xi(s;P)\|_{\mathcal{H}_\infty} \|\overline{P}^\mathsf{T} X(s)\|_{\mathcal{H}_2}.$$

Note that (20) is sufficient for $\mathrm{im}([V,B]) \subseteq \mathrm{im}(P)$. Thus, from Lemma 4, we verify that the $\mathcal{H}_2$-norm of $\overline{P}^\mathsf{T} X$ is given by (19). For its value, we have

$$\mathrm{tr}(\overline{P}^\mathsf{T} \overline{V} \Phi \overline{V}^\mathsf{T} \overline{P}) \le \mathrm{tr}(\overline{W}^\mathsf{T} \overline{V}^\mathsf{T} \overline{V} \Phi \overline{V}^\mathsf{T} \overline{V}\, \overline{W}) = \sum_{l=1}^{L} \lambda_l$$

where $\overline{V}^\mathsf{T} \overline{V} = I_{n-\nu}$ and the first inequality follows from the Cauchy interlacing theorem with $\mathrm{im}(\overline{P}) \subseteq \mathrm{im}(\overline{W})$. ∎

This theorem shows that the singular perturbation approximation works well if the sum of eigenvalues of Φ that are *neglected* through the approximation is small enough. This is similar to the balanced truncation [1], where an $\mathcal{H}_\infty$-error bound is related to the sum of neglected Hankel singular values. On the other hand, it should be remarked that our singular perturbation approximation takes into account only the controllability for the order reduction while the balanced truncation deals with the controllability and observability simultaneously.

Based on the theoretical analysis in Theorem 3, we provide the following algorithm to find $P \in \mathcal{P}^{n \times \hat{n}}$ such that $\lambda_i < \epsilon$ hold for all $i \in \{1, \ldots, L\}$ in (21):

(a) Find a coordinate transformation such that $A \in \mathcal{A}^{n \times n}$ holds, and construct a unitary matrix $[V, \overline{V}] \in \mathbb{R}^{n \times n}$ such that $\mathrm{im}(V) = \mathcal{V}(A)$ holds.
(b) Prescribe a threshold ϵ, and let a null matrix W.
(c) Calculate the index matrix Φ in (17) from (A, B), and find all eigenpairs (λ_i, w_i) of Φ for $i \in \{1, \ldots, n-\nu\}$.
(d) For each $i \in \{1, \ldots, n-\nu\}$, update $W \leftarrow [W, w_i]$ if $\lambda_i \ge \epsilon$.
(e) Find $P \in \mathcal{P}^{n \times \hat{n}}$ such that (20) holds by the Gram-Schmidt process.

In this proposed algorithm, the threshold ϵ can be used as a parameter to regulate the approximation error.

IV. Numerical Example

We show the efficiency of the proposed method through a numerical example. In what follows, we deal with a 50th dimensional continuous-time Markov chain model, whose state transition diagram is depicted in Fig. 1. Its dynamics is given as

$$\begin{cases} \dot{x} = Ax, & x(0) = x_0 \\ y = Cx \end{cases} \quad (22)$$

where $x_0 \in \mathbb{R}^{50}$ is an initial state and

$$A = \begin{bmatrix} -2 & 1 & & 1 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{bmatrix}, \quad C = [1, 0, \ldots, 0]. \quad (23)$$

It is readily verified that $A \preceq O_{50}$ and $\mathcal{V}(A) = \mathcal{V}(A^\mathsf{T}) = [1, \ldots, 1]^\mathsf{T}$ hold, and thus $A \in \mathcal{A}^{50\times 50}$ is satisfied.

In order to approximate the system behavior for *any* initial state x_0, we apply the dual counterpart of Theorem 3, namely we approximate the state-to-output mapping defined by (A, C). More specifically, we use Theorem 3 by replacing

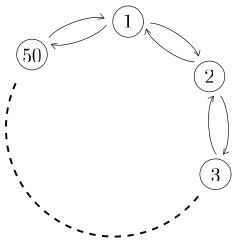

Fig. 1. State Transition Diagram of Markov Chain.

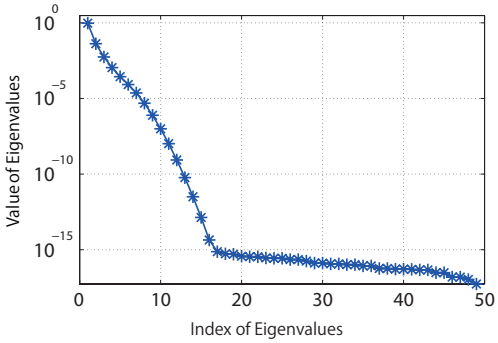

Fig. 2. Eigenvalues of Φ.

the pair (A, B) with $(A^\mathsf{T}, C^\mathsf{T})$ in (23). The semi-log plot of the eigenvalues of $\Phi \in \mathbb{R}^{49\times 49}$ in (17) is shown in Fig. 2. From this figure, we can see that many of the eigenvalues are much smaller than the maximal (the first) eigenvalue.

Next, we show that the threshold ϵ in the proposed algorithm is used as a parameter to regulate the approximation error. By assigning ϵ as 10^{-2}, 10^{-3} and 10^{-5}, we obtain 4-, 6- and 9-dimensional singular perturbation models, respectively. With the initial state $x_0 = e_{10}$, which denotes the 10th canonical unit vector, the output trajectories of the original system (the solid line) and the singular perturbation models (the broken lines) are shown in Fig. 3. This figure shows that the output discrepancy between the original system and the singular perturbation model reduces with decreasing the value of ϵ. As shown in this numerical example, the threshold ϵ can be used as a design criterion to regulate the approximation error.

V. Conclusion

In this paper, we have proposed a singular perturbation approximation for semistable linear systems. The proposed method is a generalization of our method in [13], where a singular perturbation approximation for stable systems is developed by using the reciprocal transformation. The proposed method provides a singular perturbation model, whose approximation quality in terms of the $\mathcal{H}_2$-norm is regulated in a systematic manner. The efficiency of the proposed method has been shown through an example of Markov chain approximation.

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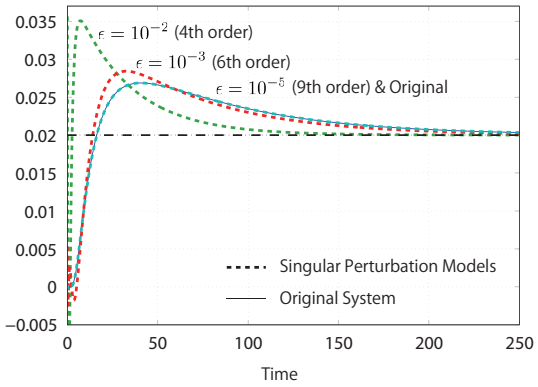

Fig. 3. Initial Value Responses.

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