Nonlinear Consensus Protocols With Applications to Quantized Communication and Actuation

Jieqiang Wei, Xinlei Yi, Henrik Sandberg, and Karl Henrik Johansson, *Fellow, IEEE*

***Abstract*—Nonlinearities are present in all real applications. Two types of general nonlinear consensus protocols are considered in this paper, namely, the systems with nonlinear communication and actuator constraints. The solutions of the systems are understood in the sense of Filippov to handle the possible discontinuity of the controllers. For each case, we prove the asymptotic stability of the systems with minimal assumptions on the nonlinearity, for both directed and undirected graphs. These results extend the literature to more general nonlinear dynamics and topologies. As applications of established theorems, we interpret the results on quantized consensus protocols.**

***Index Terms*—Networks of autonomous agents, nonlinear systems, nonsmooth analysis, stability.**

I. Introduction

DISTRIBUTED consensus is a benchmark problem in the study of multiagent systems. For continuous-time models, many control protocols have been proposed to solve asymptotic or finite-time consensus problems [4], [6], [23]. In addition to the well-studied linear consensus problem (e.g., [18], [22]), the nonlinear scenario has attracted many researchers' attention.

Nonlinearities are present in any real systems. In this paper, we consider a first-order nonlinear multiagent system for various topologies and with minimal assumptions on the nonlinearity, which make the stability analysis harder to conduct.

One source of nonlinearities is communication constraints. In [9] and [11], the authors considered the case when only quantized agent states can be exchanged across the communication links. Building on the assumption that the communication graph is undirected, asymptotic convergence of all Krasovskii solutions to practical consensus was provided, i.e., convergence to a neighborhood of the consensus point. In [16], the authors considered a consensus protocol with nonlinear communication constraints, where the nonlinear functions are assumed to be piecewise continuous, strictly increasing, and sign preserving. In this case, precise consensus can be achieved. However, the stronger assumption will put limits on the application of the results; notice that quantizer that is not strictly increasing fails to satisfy the assumption in [16].

Another source of nonlinearities is actuator constraints. Two cases are commonly seen in the literature. The first case is that the actuation is a nonlinear function of the sum of the agent state differences. In [6], the author studied such a case when the nonlinearity is a sign function. In [26], the authors considered a more general model by replacing the sign function by any sign-preserving functions. Sufficient conditions to guarantee asymptotic consensus of all Filippov solutions are provided. In [13], the authors considered a system, which can be seen as the discretized version of the system in [26] with quantizers and the time-varying topology. The second case is that the actuation is the sum of nonlinear functions of the agent state differences. In [14], the authors considered this case with Lipschitz continuous functions under a switching topology. In [8], [12], and [28], the authors considered the situation that the nonlinear functions are uniform quantizers. More precisely, in [8], the authors considered quantized protocols within the framework of hybrid dynamical systems. In [12], the author considered the same system as [8] and proved the stability for undirected graphs using the notion of the Filippov solution. In a more recent work [28], the authors proposed self-triggered rules to avoid continuous communications between agents based on the same model. In [26], the authors considered sign-preserving nonlinear possibly discontinuous functions and gave sufficient conditions to guarantee asymptotic consensus under undirected graphs. It is worth to mention that nonlinear actuators can be useful to fulfill some specific control objective, e.g., finite-time convergence [4], [25].

In this paper, we develop a general framework for nonlinear consensus protocols. With respect to earlier literature, our contribution is twofold. On one hand, we propose two general frameworks, which incorporate communication and actuator constraints, respectively. In these frameworks, the nonlinearities are only assumed to be increasing, which include a sign function, quantizer as special cases. The differential inclusion

Manuscript received February 18, 2018; revised May 14, 2018; accepted July 11, 2018. Date of publication July 26, 2018; date of current version May 28, 2019. This work was supported in part by the Knut and Alice Wallenberg Foundation, in part by the Swedish Research Council, and in part by the Swedish Foundation for Strategic Research. Recommended by Associate Editor D. Nesic. *(Corresponding author: Jieqiang Wei.)*

The authors are with the Autonomic Complex Communication Networks, Signals and Systems Linnaeus Centre, School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden (e-mail: jieqiang@kth.se; xinleiy@kth.se; hsan@ee.kth.se; kallej@kth.se).

Digital Object Identifier 10.1109/TCNS.2018.2860461

of the dynamical system, which evidently is an extension of the systems with continuous control protocol, is considered. The notion of Filippov is employed to understand the solution of the system. Sufficient conditions are given to guarantee the convergence. On the other hand, we consider these frameworks defined on directed graphs, by which we extend some existing results for consensus, e.g., in [9], [11], and [12], to weaker topological conditions, namely directed graphs. More precisely, for systems with nonlinear communication, we prove that the states of the agents converge to practical consensus for heterogeneous nonlinearity if the graph is strongly connected and for homogeneous case if the graph contains a directed spanning tree. As an application, we extend the results in [9] and [11] from undirected graphs to directed ones. For the systems with nonlinear actuation, we provide the sufficient conditions, under the assumption that nonlinear functions are odd and nondecreasing, to guarantee asymptotic convergence to the practical consensus set for all Filippov solutions when the underlying graph is undirected, directed ring, and directed spanning tree, respectively. Moreover, we show the result cannot be extended to more general topologies by examples. Again, as a specific application, we extend the results in [12] to directed graphs.

The reasons we choose the Filippov solution are the twofolds. First, for many nonlinear consensus protocols with a discontinuous controller, the classical and Carathéodory solutions do not exist. In [9], it is proven that these solutions do not exist for quantized consensus protocols. So, considering generalized solutions is unavoidable. Furthermore, for systems that can guarantee the existence of a classical solution, the results in this paper can be applied in a straightforward manner. Second, the Filippov set-valued map, compared to Krasovskii's, eliminates possible misbehavior of the right-hand side of the differential equation on sets of zero measure [3]. Moreover, as will be proved in the Appendix, the Filippov and Krasovskii solutions are equivalent for the systems in Sections III and V-A.

The structure of this paper is as follows. In Section II, we introduce some background material. In Section III, we consider consensus protocols with nonlinear communication among the agents. Section IV is devoted to the case when the actuator is nonlinear. In Section V, we apply the results in Sections III and IV to quantized consensus protocols. Finally, this paper is wrapped up with the conclusion in Section VI.

Notations: With $\mathbb{R}_-$, $\mathbb{R}_+$, and $\mathbb{R}_{\geqslant 0}$, we denote the sets of negative, positive, and nonnegative real numbers, respectively. The ith row and jth column of a matrix M are denoted as $M_{i,\cdot}$ and $M_{\cdot,j}$, respectively. For simplicity, let $M_{\cdot,j}^\top$ denote $(M_{\cdot,j})^\top$. The vectors $\rho_1, \rho_2, \ldots, \rho_n$ denote the canonical basis of $\mathbb{R}^n$. We denote $\mathbb{1}_n$ and 0_n as the column vectors containing only ones and zeros in $\mathbb{R}^n$. The cardinality of a set A is denoted $|A|$.

II. Preliminaries

In this section, we briefly review some essentials from graph theory [2] and give some properties of Filippov solutions [10].

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph with node set $\mathcal{V} = \{v_1, \ldots, v_n\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. An edge of $\mathcal{G}$ is an ordered pair $e_{ij} = (v_i, v_j)$, and we write $\mathcal{I} = \{1, 2, \ldots, n\}$. The adjacency elements a_{ij} are defined as $a_{ij} > 0$ if and only if $e_{ji} \in \mathcal{E}$. Moreover, $a_{ii} = 0$ for all $i \in \mathcal{I}$. For undirected graphs, $A = A^\top$.

The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. For each node v_i, its in-degree is defined as $\deg_{\text{in}}(v_i) = \sum_{j=1}^n a_{ij}$. The degree matrix of the digraph $\mathcal{G}$ is a diagonal matrix Δ where $\Delta_{ii} = \deg_{\text{in}}(v_i)$. The *graph Laplacian* is defined as $L = \Delta - A$. This implies $L\mathbb{1}_n = 0_n$.

A directed path from node v_i to node v_j is a sequence of edges from $\mathcal{E}$ such that the first edge starts from v_i, the last edge ends at v_j, and every edge starts where the previous edge ends. A graph is called *strongly connected* if for every two nodes v_i and v_j, there is a directed path from v_i to v_j. A subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', A')$ of $\mathcal{G}$ is called a *directed spanning tree* for $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$, $\mathcal{E}' \subseteq \mathcal{E}$, and, for every node $v_i \in \mathcal{V}'$, there is exactly one v_j such that $e_{ji} \in \mathcal{E}'$, except for one node, which is called the root of the spanning tree. Furthermore, we call a node $v \in \mathcal{V}$ a *root* of $\mathcal{G}$ if there is a directed spanning tree of $\mathcal{G}$ with v as a root. In other words, if v is a root of $\mathcal{G}$, then there is a directed path from v to every other node in the graph. A digraph is *a directed ring* if for every node v_i, there exists exactly one v_j such that $e_{ij} \in \mathcal{E}$ and there exists exactly one v_k such that $e_{ki} \in \mathcal{E}$.

The *incidence matrix* of a digraph is denoted as $B \in \mathbb{R}^{n \times m}$, with $B_{ij} = -1$ if the jth edge is toward vertex i, and equal to 1 if the jth edge is originating from vertex i, and 0 otherwise. For undirected graphs, the incidence matrix can be defined with an arbitrary orientation.

Lemma 1 (See [17, Lemma 2]): The Laplacian matrix L of a strongly connected digraph $\mathcal{G}$ satisfies that zero is an algebraically simple eigenvalue of L and there is a $w \in \mathbb{R}_+^n$ such that $w^\top L = 0$ and $\mathbb{1}_n^\top w = 1$. Moreover, the symmetric part of $L^\top \text{diag}(w)$ is positive semidefinite.

In the remainder of this section, we discuss Filippov solutions. Let X be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $2^{\mathbb{R}^n}$ denote the collection of all subsets of $\mathbb{R}^n$. The *Filippov set-valued map* of X, denoted $\mathcal{F}[X] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, is defined as

$$\mathcal{F}[X](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \overline{\text{co}} \{X(B(x,\delta) \backslash S)\} \tag{1}$$

where $B(x, \delta)$ is the open ball centered at x with radius $\delta > 0$, S is a subset of $\mathbb{R}^n$, μ denotes the Lebesgue measure, and $\overline{\text{co}}$ denotes the convex closure. If X is continuous at x, then $\mathcal{F}[X](x)$ contains only the point $X(x)$.

Lemma 2: For an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, the Filippov set-valued map satisfies the following.

1) $\mathcal{F}[\varphi](x) = [\varphi(x^-), \varphi(x^+)]$, where $\varphi(x^-), \varphi(x^+)$ are the left and right limits of φ at x, respectively.
2) For any $x_1 < x_2$, and $\nu_i \in \mathcal{F}[\varphi](x_i), i = 1, 2$, we have $\nu_1 \leqslant \nu_2$.
3) $\mathcal{F}[\varphi](x) = \{\varphi(x)\}$ for almost all x.

Proof: These items can be seen as a straightforward deduction from [20, Th. 1(1)], the definition of increasing functions, and the fact that monotone functions are continuous almost everywhere, respectively. □

With a slight abuse of the notation, we denote $\varphi(x^+)$ $(\varphi(x^-)) : \mathbb{R} \to \mathbb{R}$. This should not be confused in the context.

Lemma 3 (See [27, Lemma 3. p. 365]): For an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x^+)$ $(\varphi(x^-))$ is defined everywhere and is right (left) continuous for all x.

A *Filippov solution* of the differential equation $\dot{x}(t) = X(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^n$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[X](x(t)) \tag{2}$$

for almost all $t \in [0, t_1]$. A Filippov solution $t \mapsto x(t)$ is *maximal* if it cannot be extended forward in time, that is, if $t \mapsto x(t)$ is not the result of the truncation of another solution with a larger interval of definition. Since Filippov solutions are not necessarily unique, we need to specify two types of invariant sets. A set $\mathcal{R} \subset \mathbb{R}^n$ is called *weakly invariant* for (2) if, for each $x_0 \in \mathcal{R}$, at least one maximal solution of (2) with initial condition x_0 is contained in $\mathcal{R}$. Similarly, $\mathcal{R} \subset \mathbb{R}^n$ is called *strongly invariant* for (2) if, for each $x_0 \in \mathcal{R}$, every maximal solution of (2) with initial condition x_0 is contained in $\mathcal{R}$. For more details, see [7] and [10].

If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then its *generalized gradient* $\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\partial f(x) := \mathrm{co}\left\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to x, x_i \notin S \cup \Omega_f\right\} \tag{3}$$

where ∇ denotes the gradient operator, $\Omega_f \subset \mathbb{R}^n$ denotes the set of points, where f fails to be differentiable, and $S \subset \mathbb{R}^n$ is a set of Lebesgue measure zero that can be arbitrarily chosen to simplify the computation. The resulting set $\partial f(x)$ is independent of the choice of S [5].

Given a set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, the *set-valued Lie derivative*$\mathcal{L}_{\mathcal{F}} f : \mathbb{R}^n \to 2^{\mathbb{R}}$ of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to $\mathcal{F}$ at x is defined as

$$\mathcal{L}_{\mathcal{F}} f(x) := \{a \in \mathbb{R} \mid \text{there exists } \nu \in \mathcal{F}(x) \text{ such that } \zeta^\top \nu = a \text{ for all } \zeta \in \partial f(x)\}. \tag{4}$$

If $\mathcal{F}$ takes convex and compact values, then, for each x, $\mathcal{L}_{\mathcal{F}} f(x)$ is a closed and bounded interval, possibly empty [7].

Finally, we recall that he following functions are regular[1] and Lipschitz continuous

$$V(x) := \max_{i\in\mathcal{I}} x_i, \qquad W(x) := -\min_{i\in\mathcal{I}} x_i \tag{5}$$

which will be used to prove stability for some dynamical systems in this paper, using [7, Th. 2].

III. Multiagent Systems With Nonlinear Communications

In this section, we consider a network of n agents with a communication topology given by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. Agent i receives information from agent j if and only if there is an edge from node v_j to node v_i in $\mathcal{G}$. Unlike the linear consensus protocol, here, a nonlinear map of the states is available to the agents. More precisely, we consider the nonlinear consensus protocol

$$\dot{x} = -Lf(x) \tag{6}$$

where $f(x) = [f_1(x_1), \dots, f_n(x_n)]^\top$ and $f_i : \mathbb{R} \to \mathbb{R}$ satisfies the following assumption.

Assumption 1: The function $f_i : \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying $\lim_{x_i\to+\infty} f_i(x_i) > 0$ and $\lim_{x_i\to-\infty} f_i(x_i) < 0$.

Note that we do *not* assume continuity of the function f_i. Examples of functions satisfying Assumption 1 include the sign function and various quantization functions.

Lemma 4: Suppose the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(x) = [f_1(x_1), \dots, f_n(x_n)]^\top$ and each f_i satisfies Assumption 1; then, the Filippov set-valued map obeys

$$\mathcal{F}[f](x) = \mathop{\times}_{i=1}^{n} \mathcal{F}[f_i](x_i).$$

Proof: First, by Lemma 2, we have $\times_{i=1}^n \mathcal{F}[f_i](x_i) = \times_{i=1}^n [f_i(x_i^-), f_i(x_i^+)]$. Second, by [20, Th. 1(1)] and assumptions of this lemma, we have $\mathcal{F}[f](x) = \mathrm{co}\{E\}$, where $E = \{e \in \mathbb{R}^n \mid e_i \in \{f_i(x_i^-), f_i(x_i^+)\}, \forall i \in \mathcal{I}\}$. Notice that $|E| \leqslant 2^n$. Hence, the conclusion follows by the observation that $\mathrm{co}\{E\} = \times_{i=1}^n [f_i(x_i^-), f_i(x_i^+)]$. □

In order to handle possible discontinuities, we understand the solution of (6) in the Filippov sense, i.e., we consider the differential inclusion

$$\begin{aligned} \dot{x} &\in \mathcal{F}[-Lf(x)](x) && (7)\\ &= -L\mathcal{F}[f](x) && (8)\\ &= -L \mathop{\times}_{i=1}^{n} \mathcal{F}[f_i](x_i) && (9)\\ &=: \mathcal{K}_1(x) && (10) \end{aligned}$$

where (8) and (9) are implied by [20, Th. 1(5)] and Lemma 4. The existence of Filippov solutions can be guaranteed by assuming monotonicity of f_i, which implies that f_i is locally essentially bounded. Furthermore, we assume the maximal solution of (10) exists for any initial condition. Denote

$$\mathcal{D}_1 = \left\{x \in \mathbb{R}^n \mid \exists \sigma \in \mathbb{R} \text{ s.t. } \sigma \mathbb{1}_n \in \mathop{\times}_{i=1}^{n} \mathcal{F}[f_i](x_i)\right\}. \tag{11}$$

More precise illustrations of the set $\mathcal{D}_1$ and the set $\mathcal{D}_2$ defined in (16) can be found in Remark 3.

Remark 1: One closely related notion of the Filippov solution is the Krasovskii solution, e.g., [3]. Since the Krasovskii set-valued map is the same as the Filippov one for any monotone function, we can show that Krasovskii and Filippov solutions of (6) are equivalent. More detailed discussion can be found in the Appendix.

Proposition 5: If f_i satisfies Assumption 1, the set $\mathcal{D}_1$ is closed.

Proof: For any sequence $\{y^k\} \subset \mathbb{R}^n$ satisfying $\lim_{k\to\infty} y^k = x$ and $y^k \in \mathcal{D}_1, k = 1, 2, \dots$, we shall show that $x \in \mathcal{D}_1$. Since $\mathbb{R}^n$ can be divided into finite orthants, there exists a subsequence of $\{y^k\}$, denoted as $\{y^{k_\ell}\}$, which satisfies that all the elements in the sequence $\{y^{k_\ell} - x\}$ belong to one orthant

[1] The definition of a regular function can be found in [5], and we recall that any convex function is regular.

of $\mathbb{R}^n$. In other words, $y_i^{k_\ell}$ converge to x_i from one side, i.e., $y_i^{k_\ell} < x_i$ or $y_i^{k_\ell} > x_i$. To simplify the notation, we assume the sequence $\{y^k - x\}$ itself belongs to one orthant of $\mathbb{R}^n$.

Note that $y^k \in \mathcal{D}_1$ implies that $\cap_{i=1}^n \mathcal{F}[f_i](y_i^k) \neq \varnothing$. For the case $y_i^k > x_i$, we have $f_i(y_i^{k-}) \geqslant f_i(x_i^-)$, $f_i(y_i^{k+}) \geqslant f_i(x_i^+)$ and $\lim_{k\to\infty} f_i(y_i^{k+}) = f_i(x_i^+)$, which is based on Lemma 3. Hence, we have

$$\left[\lim_{k\to\infty} f_i\left(y_i^{k-}\right), \lim_{k\to\infty} f_i\left(y_i^{k+}\right)\right] \subset \left[f_i\left(x_i^-\right), f_i\left(x_i^+\right)\right].$$

Similarly, for the case $y_i^k < x_i$, we also get that result. Then, $\cap_{i=1}^n \mathcal{F}[f_i](x_i) \neq \varnothing$, i.e., $x \in \mathcal{D}_1$. □

Next, we establish asymptotic convergence of system (10) with respect to $\mathcal{D}_1$.

Theorem 6: Consider the differential inclusion (10) with f_i satisfying Assumption 1. Suppose the underlying topology $\mathcal{G}$ is strongly connected. Then, all the Filippov solutions converge to $\mathcal{D}_1$ asymptotically.

Proof: Consider the Lyapunov function $V_1(x) = w^\top F(x)$, where $w \in \mathbb{R}_+^n$ is given by Lemma 1 and $F(x) = [F_1(x_1), \ldots, F_n(x_n)]$ with $F_i(x_i) = \int_0^{x_i} f_i(\tau)d\tau$. It can be verified that $V_1 \in \mathcal{C}^0$ and V_1 is convex, which implies that V_1 is regular. Moreover, by the monotonicity of f_i, we have $\partial F_i(x_i) = [f_i(x_i^-), f_i(x_i^+)] = \mathcal{F}[f_i](x_i)$. Hence, V_1 is locally Lipschitz continuous. Moreover, by Assumption 1, the function V_1 is radially unbounded. Indeed, $\lim_{x_i\to\infty} f_i(\tau)d\tau = \infty$.

Let Ψ_1 be defined as

$$\Psi_1 = \left\{t \geqslant 0 \mid \text{both } \dot{x}(t) \text{ and } \frac{d}{dt}V_1(x(t)) \text{ exist}\right\}. \quad (12)$$

Since x is absolutely continuous and V_1 is locally Lipschitz, we can let $\Psi_1 = \mathbb{R}_{\geqslant 0} \backslash \bar{\Psi}_1$, where $\bar{\Psi}_1$ is a Lebesgue measure zero set. By [1, Lemma 1], we have

$$\frac{d}{dt}V_1(x(t)) \in \mathcal{L}_{\mathcal{K}_1} V_1(x(t)) \quad (13)$$

for all $t \in \Psi_1$, and, hence, that the set $\mathcal{L}_{\mathcal{K}_1} V_1(x(t))$ is nonempty for all $t \in \Psi_1$. For $t \in \bar{\Psi}_1$, we have that $\mathcal{L}_{\mathcal{K}_1} V_1(x(t))$ is empty; hence, $\max \mathcal{L}_{\mathcal{K}_1} V_1(x(t)) < 0$. In the rest of the proof, we only consider $t \in \Psi_1$. Moreover, in the proofs of the remaining theorems in this paper, we always focus on a subset of $\mathbb{R}_{\geqslant 0}$ on which the set-valued Lie derivative of the corresponding Lyapunov functions is not empty.

The gradient of V_1 is given as

$$\partial V_1(x) = \text{co}\left\{\text{diag}(w)\nu \mid \nu \in \bigtimes_{i=1}^n \mathcal{F}[f_i](x_i)\right\}. \quad (14)$$

Next, we shall prove $\max \mathcal{L}_{\mathcal{K}_1} V_1(x(t)) \leqslant 0$. By definition, $\forall a \in \mathcal{L}_{\mathcal{K}_1} V_1(x(t))$, we have that $\exists u \in \times_{i=1}^n \mathcal{F}[f_i](x_i)$ such that

$$a = -u^\top L^\top \text{diag}(w)\nu \quad (15)$$

for all $\nu \in \times_{i=1}^n \mathcal{F}[f_i](x_i)$, i.e., $\text{diag}(w)\nu \in \partial V_1(x)$. A special case is that $\nu = u$, which implies that $a \leqslant 0$ by Lemma 1. Hence, we have $\max \mathcal{L}_{\mathcal{K}_1} V_1(x(t)) \leqslant 0$.

Finally, notice that $a = 0$ if and only if $\times_{i=1}^n \mathcal{F}[f_i](x_i) \cap \text{span}\{\mathbb{1}_n\} \neq \varnothing$. Hence, by the fact that $\mathcal{D}_1$ is closed, we have $\overline{\{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x)\}} = \mathcal{D}_1$. By [7, Th. 2], all the Filippov trajectories converge into the largest weakly invariant set contained in $\overline{\{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x)\}}$. Hence, the conclusion holds. □

In the previous theorem, the topology was assumed to be strongly connected. Next, we shall relax this assumption to digraphs containing a directed spanning tree, which for linear consensus protocol is known to be a necessary and sufficient condition for consensus. However, for the digraph cases, we make an additional assumption that the agents are homogeneous, i.e., $f_i = f_j$ for any $i, j \in \mathcal{I}$.

Theorem 7: Suppose the nonlinear functions in (6) are given as $f(x) = [\bar{f}(x_1), \bar{f}(x_2), \ldots, \bar{f}(x_n)]$, where $\bar{f}$ satisfies Assumption 1. If the underlying digraph $\mathcal{G}$ contains a directed spanning tree, then all Filippov solutions of (10) asymptotically converge to

$$\mathcal{D}_2 = \left\{x \in \mathbb{R}^n \mid \exists \sigma \in \mathbb{R} \text{ s.t. } \sigma \mathbb{1}_n \in \bigtimes_{i=1}^n \mathcal{F}\left[\bar{f}\right](x_i)\right\}. \quad (16)$$

Proof: In this case, the differential inclusion (10) can be written as

$$\dot{x} \in -L \bigtimes_{i=1}^n \mathcal{F}\left[\bar{f}\right](x_i) =: \mathcal{K}_2(x). \quad (17)$$

This proof is divided into five steps.

1) Denote the set $\mathcal{I}_r = \{i \in \mathcal{I} \mid v_i \text{ is a root of } \mathcal{G}\}$ and the subvectors x_r and x_f of x corresponding to $\mathcal{I}_r$ and $\mathcal{I} \backslash \mathcal{I}_r$, respectively. Since the subgraph corresponding to the roots is strongly connected and the states of the roots are not affected by the other agents, system (17) can be written as

$$\dot{x}_r \in -L_r \bigtimes_{i=1}^n \mathcal{F}\left[\bar{f}\right](x_i), \quad i \in \mathcal{I}_r \quad (18)$$

$$\dot{x}_f \in -L_f \bigtimes_{i=1}^n \mathcal{F}\left[\bar{f}\right](x_i), \quad i \in \mathcal{I} \quad (19)$$

where L_r is the Laplacian matrix of the subgraph corresponding to the roots. By applying Theorem 6, x_r converges to

$$\left\{x_r \mid \exists a \text{ s.t. } a \in \mathcal{F}[\bar{f}](x_i)\ \forall i \in \mathcal{I}_r\right\}. \quad (20)$$

2) In this item and the following item 3, we shall prove that the functions $V(x(t))$ and $W(x(t))$, given as in (5), are not increasing along the trajectories $x(t)$ of the system (17). We start with V in this part. We only focus on a subset of $\mathbb{R}_{\geqslant 0}$ on which the set-valued Lie derivative of V is not empty. Let $x(t)$ be a trajectory of (17) and define

$$\alpha(x(t)) = \{k \in \mathcal{I} \mid x_k(t) = V(x(t))\}. \quad (21)$$

Denote $\overline{x}(t) = x_i(t)$ for $i \in \alpha(x(t))$. The generalized gradient of V is given as [5, Example 2.2.8]

$$\partial V(x(t)) = \text{co}\{\rho_k \in \mathbb{R}^n \mid k \in \alpha(x(t))\}. \quad (22)$$

Similar to the proof of Theorem 6, we focus on the set Ψ_2 such that $\mathcal{L}_{\mathcal{K}_2} V(x(t)) \neq \varnothing$ for $t \in \Psi_2$ and $\mu(\mathbb{R}_{\geqslant 0} \backslash \Psi_2) = 0$. For $t \in \Psi_2$, let $a \in \mathcal{L}_{\mathcal{K}_2} V(x(t))$. By definition, there exists a $\nu^a \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ such that

$a = (-L\nu^a)^\top \cdot \zeta$ for all $\zeta \in \partial V(x(t))$. Consequently, by choosing $\zeta = \rho_k$ for $k \in \alpha(x(t))$, we observe that ν^a satisfies

$$-L_{k,\cdot}\nu^a = a \qquad \forall k \in \alpha(x(t)). \tag{23}$$

Next, we show that $\max \mathcal{L}_{\mathcal{K}_2} V(x(t)) \leqslant 0$ for all $t \in \Psi_2$ by considering two possible cases: $\mathcal{I}_r \subseteq \alpha(x(t))$ and $\mathcal{I}_r \nsubseteq \alpha(x(t))$.

If $\mathcal{I}_r \subseteq \alpha(x(t))$, there are two subcases. First, $|\mathcal{I}_r| = 1$, i.e., there is only one root, denoted as v_i. Then, $L_{i,\cdot} = 0$; hence, $L_{i,\cdot}\nu = 0$ for any $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$. By the observation (23), we have $\mathcal{L}_{\mathcal{K}_2} V(x(t)) = \{0\}$. Second, $|\mathcal{I}_r| \geqslant 2$. By the fact that the subgraph spanned by the roots is strongly connected, there exists $w_i > 0$ for $i \in \mathcal{I}_r$ such that $\sum_{i\in\mathcal{I}_r} w_i L_{i,\cdot} = 0_n$, which implies that

$$\sum_{i\in\mathcal{I}_r} w_i L_{i,\cdot}\nu = 0 \tag{24}$$

for any $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$. Again, by (23), we have $\mathcal{L}_{\mathcal{K}_2} V(x(t)) = \{0\}$.

If $\mathcal{I}_r \nsubseteq \alpha(x(t))$, there exists $i \in \mathcal{I}_r \setminus \alpha(x(t))$. We define $\alpha'(\nu)$ as

$$\alpha'(\nu) = \left\{ i \in \alpha(x(t)) \mid \nu_i = \max_{j\in\alpha(x(t))} \nu_j \right\} \tag{25}$$

for any $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$. From item 2 of Lemma 2, for any $j \in \alpha'(\nu)$, we know that $\nu_j = \max \nu_i$, thus $L_{j,\cdot}\nu \geqslant 0$. By the fact that the choice of ν is arbitrary in $\times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ and the observation (23), we have $\mathcal{L}_{\mathcal{K}_2} V(x(t)) \subset \mathbb{R}_{\leqslant 0}$.

Moreover, we investigate the set $\{x \mid 0 \in \mathcal{L}_{\mathcal{K}_2} V(x)\}$ in a more detailed manner. Denoting

$$\mathcal{E}_{\alpha(x)} = \{e_{ij} \in \mathcal{E} \mid j \in \alpha(x)\} \tag{26}$$

we next show that $0 \in \mathcal{L}_{\mathcal{K}_2} V(x)$ if and only if $\exists \nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ such that $\nu_i = \nu_j$ for any $e_{ij} \in \mathcal{E}_{\alpha(x)}$, which is equivalent to $\mathcal{F}[\bar{f}](x_i) \cap \mathcal{F}[\bar{f}](x_j) \neq \varnothing$ for all $e_{ij} \in \mathcal{E}_{\alpha(x)}$. The sufficient part is straightforward; in fact, we can take $\nu_i = \nu_j = f(\overline{x}^-)$ for any $e_{ij} \in \mathcal{E}_{\alpha(x)}$. Then, $0 \in \mathcal{L}_{\mathcal{K}_2} V(x)$. The necessary part can be proved as follows. Since $0 \in \mathcal{L}_{\mathcal{K}_2} V(x)$, there exists $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ such that $L_{j,\cdot}\nu = 0$ for any $j \in \alpha(x)$. Then, this ν satisfies that $\alpha'(\nu) = \alpha(x)$. Indeed, if $\alpha'(\nu) \subsetneq \alpha(x)$, then for any $j \in \alpha'(\nu)$ with $e_{ij} \in \mathcal{E}$ and $i \notin \alpha'(\nu)$, $L_{j,\cdot}\nu < 0$. Hence, $\alpha'(\nu) = \alpha(x)$. Furthermore, by using the same argument, we have for any $e_{ij} \in \mathcal{E}$ satisfying $i \notin \alpha(x)$ and $j \in \alpha(x)$, $f(\overline{x}^-) \in \mathcal{F}[\bar{f}](x_i)$.

3) For the Lyapunov functions W as given in (5), denote

$$\beta(x(t)) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\} \tag{27}$$

and $x_i(t) = \underline{x}(t)$ for $i \in \beta(x(t))$, and $\mathcal{E}_{\beta(x(t))} = \{e_{ij} \in \mathcal{E} \mid j \in \beta(x(t))\}$. By similar derivations, we find that $\max \mathcal{L}_{\mathcal{K}_2} W(x(t)) \leqslant 0$ and $0 \in \mathcal{L}_{\mathcal{K}_2} W(x(t))$ if and only if $\exists \nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ such that $\nu_i = \nu_j$ for any $e_{ij} \in \mathcal{E}_{\beta(x(t))}$, which is equivalent to $\mathcal{F}[\bar{f}](x_i) \cap \mathcal{F}[\bar{f}](x_j) \neq \varnothing$ for all $e_{ij} \in \mathcal{E}_{\beta(x(t))}$.

4) So far, we have that $V(x(t))$ and $W(x(t))$ are not increasing along the trajectories $x(t)$ of the system (17). Hence, the trajectories are bounded and remain in the set $[\underline{x}(0), \overline{x}(0)]^n$ for all $t \geqslant 0$. Therefore, for any $N \in \mathbb{R}_+$, the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant for (17). By [7, Th. 2], we have that all solutions of (17) starting in S_N converge to the largest weakly invariant set M contained in

$$S_N \cap \overline{\{x \in \mathbb{R}^n : 0 \in \mathcal{L}_{\mathcal{K}_2} V(x)\}} \cap \overline{\{x \in \mathbb{R}^n : 0 \in \mathcal{L}_{\mathcal{K}_2} W(x)\}}. \tag{28}$$

5) We have proved the asymptotic stability of the system. Next, we will prove that the set $\mathcal{D}_2$ is strongly invariant, and for any $x_0 \notin \mathcal{D}_2$, all the solutions satisfying $x(0) = x_0$ converge to $\mathcal{D}_2$. Notice that $\mathcal{D}_2$ is closed by the same argument as the one in Proposition 5.

We start with the strong invariance of $\mathcal{D}_2$. Notice that by the monotonicity of $\bar{f}$, we can reformulate $\mathcal{D}_2$ as

$$\mathcal{D}_2 = \left\{x \mid \mathcal{F}\left[\bar{f}\right](\underline{x}) \cap \mathcal{F}\left[\bar{f}\right](\overline{x}) \neq \varnothing\right\}. \tag{29}$$

For any $x_0 \in \mathcal{D}_2$, we proved that any trajectories starting from x_0, $V(x(t))$, and $W(x(t))$ are not increasing. Hence, $\overline{x}(t) \leqslant \overline{x}_0$ and $\underline{x}(t) \geqslant \underline{x}_0$ for all $t \geqslant 0$, which, by Lemma 2, implies that $\mathcal{F}[\bar{f}](\underline{x}(t)) \cap \mathcal{F}[\bar{f}](\overline{x}(t)) \neq \varnothing$ for all t and $x(t)$ satisfying $x(0) = x_0$. Then, $x(t) \in \mathcal{D}_2$ for all $t \geqslant 0$, which implies that $\mathcal{D}_2$ is strongly invariant.

Next, we show that for any $x_0 \notin \mathcal{D}_2$, all solutions satisfying $x(0) = x_0$ converge to $\mathcal{D}_2$. We will prove it by contradiction. Indeed, we assume that there exists $x_0 \notin \mathcal{D}_2$ and one solution $\tilde{x}(t)$ satisfying $\tilde{x}(0) = x_0$ that does not converge to $\mathcal{D}_2$. Since the set $\mathcal{D}_2$ is strongly invariant, we have $\tilde{x}(t) \notin \mathcal{D}_2$ for all $t \geqslant 0$. Then, $\mathcal{F}[\bar{f}](\underline{\tilde{x}}) \cap \mathcal{F}[\bar{f}](\overline{\tilde{x}}) = \varnothing$, where

$$\overline{\tilde{x}} = \lim_{t\to\infty} V(\tilde{x}(t)),\ \underline{\tilde{x}} = -\lim_{t\to\infty} W(\tilde{x}(t)).$$

Hence, there exists a constant $C > 0$, such that $d(\mathcal{F}[\bar{f}](\underline{\tilde{x}}), \mathcal{F}[\bar{f}](\overline{\tilde{x}})) > C$, where $d(S_1, S_2) = \inf_{y_1\in S_1, y_2\in S_2} d(y_1, y_2)$ is the *distance* between two sets S_1 and S_2. For any $i, j \in \mathcal{I}$ with $i \neq j$, there exists a vector $w^{ij} \in \mathbb{R}^n$ such that ${w^{ij}}^\top L = (\rho_i - \rho_j)^\top$. For each pair $i, j \in \mathcal{I}$, we choose one w^{ij} and collect all the w^{ij} for $i, j \in \mathcal{I}$ in the set Ω. Notice that there are only finite number of vectors in Ω. Then, for any $t, i \in \alpha(\tilde{x}(t))$ and $j \in \beta(\tilde{x}(t))$, we have $\overline{\tilde{x}}(t) \geqslant \overline{\tilde{x}}$ and $\underline{\tilde{x}}(t) \leqslant \underline{\tilde{x}}$. Moreover, since $\tilde{x}(t)$ is uniformly bounded, there exists a constant τ (does not depend on t) such that for any $s \in [t, t+\tau]$

$$w(s)^\top \dot{x}(s) > \frac{C}{2} \tag{30}$$

where $w : \mathbb{R} \to \Omega$ is piecewise constant and $w(s) = w^{ij}$ with $i \in \alpha(t), j \in \beta(t)$ for $s \in [t, t+\tau]$. Note that for any $T \geqslant 0$, the function $w(s)^\top \dot{x}(s)$ is Lebesgue integrable on $[0, T]$, and by (30), we have

$$\int_0^\top w(s)^\top \dot{x}(s) ds > \frac{C}{2} T \tag{31}$$

which diverges as $T \to \infty$. This is a contradiction to the fact that $w(s)$ is globally bounded, and for any $T < \infty$ and $i \in \mathcal{I}$,

$\int_0^\top \dot{x}_i(s)ds$ is bounded. Hence, we have that for any $x_0 \notin \mathcal{D}_2$, all the solutions satisfying $x(0) = x_0$ will converge to $\mathcal{D}_2$. Here ends the proof. □

Remark 2: From the proof of Theorem 7, we know that the maximal components of the trajectories of the system (17) are not increasing, while the minimal ones are not decreasing. Hence, (17) is a nonlinear positive system (e.g., [21]), i.e., with positive initial conditions, the trajectories will be positive for all the time. However, system (10) with all nonlinear functions different is not necessarily a positive system.

Remark 3: Theorem 7 has precise interpretations in some specific scenarios.

For the linear consensus protocol $\bar{f}(x_i) = x_i$, trivially $\mathcal{F}[f_i](x_i) = \{x_i\}$ and set $\mathcal{D}_2 = \text{span}\{\mathbb{1}\}$. This coincides with the result in [24].

For the case that $\bar{f}$ is strictly increasing, by the fact that $\mathcal{F}[\bar{f}](y_1) \cap \mathcal{F}[\bar{f}](y_2) = \varnothing$ for any $y_1 \neq y_2$, we have $\mathcal{D}_2 = \text{span}\{\mathbb{1}\}$. Denote the subgraph spanned by the roots of $\mathcal{G}$ as $\mathcal{G}_r$, the corresponding Laplacian matrix as L_r, and $x_r \in \mathbb{R}^{|\mathcal{I}_r|}$ as the vector containing the root components of x. Since $\mathcal{G}_r$ is strongly connected, there exists a positive vector $\xi \in \mathbb{R}^{|\mathcal{I}_r|}$ such that $\xi^\top L_r = 0_{|\mathcal{I}_r|}$, e.g., [15]. Then, we can check that $\xi^\top \dot{x}_r = \{0\}$. Hence, by Theorem 7, x_r converges to $\frac{\xi^\top x_r(0)}{\xi^\top \mathbb{1}}$ asymptotically. This coincides with the result in [15].

For the case $\bar{f}$ is a uniform quantizer, Theorem 7 extends the result in [9] to directed topologies. We shall elaborate on this in Section V.

The stability of system (6) under more general assumptions than the ones in Theorem 7, namely, that $f_i \neq f_j, i \neq j$ but still the graph is directed, is an open problem. More precisely, additional assumptions are needed for f_i in order to guarantee that V and W are nonincreasing.

IV. Multiagent Systems With Nonlinear Actuators

In this section, we consider the case when the actuators of the agents are nonlinear, instead of the nonlinear communications as in the previous section. Specifically, we consider the following nonlinear consensus protocol:

$$\dot{x}_i = -\sum_{e_{ji} \in \mathcal{E}} a_{ij} g_{ij}(x_i - x_j) \tag{32}$$

where $g_{ij} : \mathbb{R} \to \mathbb{R}$ satisfy Assumption 1. The existence of Filippov solutions can be guaranteed similarly as in Section III. We assume that the maximal solution exists for all initial condition.

In this section, we consider three different topologies, namely, connected undirected graph, directed ring graph, and directed spanning tree. We will show, at the end of the section, that the result cannot be extended to general digraphs.

First, we assume that the underlying graph $\mathcal{G}$ is undirected. Besides Assumption 1, we assume that $g_{ij} = g_{ji}$ for all $i, j \in \mathcal{I}$ such that there exists $e_{ij} \in \mathcal{E}$, which is reasonable since there is no orientation on each edge. Recall that m denotes the number of edges. We denote the edges $e_1, \ldots, e_m$ (instead of e_{ij}), and the corresponding weights $a_1, \ldots, a_m$. Furthermore, if g_{ij}s are odd, we can write the system (32) in a vectorized form as

$$\dot{x} = -Bg(B^\top x) =: -Bh(x) \tag{33}$$

where $g(x) = [a_1 g_1(x_1), a_2 g_2(x_2), \ldots, a_m g_m(x_m)]$, $h(x) = g(B^\top x)$, and B is the incidence matrix. The procedure described in this paragraph is explained in the following example.

Example 1: Consider system (32) defined on the undirected ring graph with three nodes v_1, v_2, v_3. Then, system (32) is given as

$$\begin{aligned}\dot{x}_1 &= -g_{12}(x_1 - x_2) - g_{13}(x_1 - x_3)\\ \dot{x}_2 &= -g_{21}(x_2 - x_1) - g_{23}(x_2 - x_3)\\ \dot{x}_3 &= -g_{31}(x_3 - x_1) - g_{32}(x_3 - x_2).\end{aligned}$$

Relabel the edges e_{12}, e_{23}, e_{31} as e_1, e_2, e_3, respectively. Then, by the assumptions that $g_{ji} = g_{ij}$, we can also relabel the nonlinear function as g_1, g_2, g_3 according to the edges. This allows us to rewrite the dynamic as

$$\begin{aligned}\dot{x}_1 &= -g_1(x_1 - x_2) - g_3(x_1 - x_3)\\ \dot{x}_2 &= -g_1(x_2 - x_1) - g_2(x_2 - x_3)\\ \dot{x}_3 &= -g_3(x_3 - x_1) - g_2(x_3 - x_2)\end{aligned}$$

which is in the compact form as (33).

Theorem 8: Suppose the underlying graph is a connected undirected graph. If the functions g_i satisfy Assumption 1 and are odd, then all the Filippov solutions of (32) asymptotically converge into

$$\mathcal{H}_1 = \left\{x \in \mathbb{R}^n \mid 0_m \in \bigtimes_{i=1}^{m} \mathcal{F}[g_i]\left(B_{\cdot,i}^\top x\right)\right\}. \tag{34}$$

Proof: Notice that we do not assume that g_i is continuous. By (33) and [20, Th. 1], the differential inclusion (32) satisfies

$$\begin{aligned}\dot{x} &\in -B\mathcal{F}[h](x) &(35)\\ &\subset -B \bigtimes_{i=1}^{m} a_i \mathcal{F}[g_i](B_{\cdot,i}^\top x) =: \mathcal{K}_3(x). &(36)\end{aligned}$$

Consider the Lyapunov function $V_3(x) = \frac{1}{2}x^\top x$ and only a subset of $\mathbb{R}_{\geqslant 0}$ on which the set-valued Lie derivative of V_3 is not empty. The set-valued Lie derivative $\mathcal{L}_{\mathcal{K}_3} V_3(x)$ is given as

$$\left\{a \in \mathbb{R} \mid a = -x^\top B\nu, \nu \in \bigtimes_{i=1}^{m} a_i \mathcal{F}[g_i]\left(B_{\cdot,i}^\top x\right)\right\}.$$

In this case, $\mathcal{L}_{\mathcal{K}_3} V_3(x) \neq \varnothing$ for all $t > 0$.

By the fact that g_i is monotone and $g_i(0) = 0$ (since g_i is an odd function), we have

$$\mathcal{F}[g_i](y_i) \subset \begin{cases}\mathbb{R}_{\geqslant 0}, & \text{if } y_i > 0\\ \mathbb{R}_{\leqslant 0}, & \text{if } y_i < 0.\end{cases} \tag{37}$$

Hence, ν_i and $(B^\top x)_i$ have the same sign for any $\nu \in \times_{i=1}^m a_i \mathcal{F}[g_i](B_{\cdot,i}^\top x)$ and $i \in \mathcal{I}$. This implies that $\max \mathcal{L}_{\mathcal{K}_3} V_3(x) \leqslant 0$. By [7, Th. 2], all solutions of (36) converge to the largest weakly invariant set M contained in

$$\overline{\{x \in \mathbb{R}^n : 0 \in \mathcal{L}_{\mathcal{K}_3} V_3(x)\}}. \tag{38}$$

Notice that $0 \in \mathcal{L}_{\mathcal{K}_3} V_3(x)$ if and only if $0_m \in \times_{i=1}^m \mathcal{F}[g_i](B_{:,i}^\top x)$, and the conclusion holds.

The remaining task is to show that $\mathcal{H}_1$ is closed. First, we notice that $\mathcal{H}_1 \neq \varnothing$. Indeed, since the functions g_i are odd, then $0 \in \mathcal{F}[g_i](0)$, i.e., $\text{span}\{\mathbb{1}_n\} \subset \mathcal{H}_1$. Second, since $\mathcal{F}[g_i](z_i) = [g_i(z_i^-), g_i(z_i^+)]$ and $g_i(z_i^-)(g_i(z_i^+))$ is left (right) continuous (see Lemma 3), there exists $\lambda_i > 0$, such that $\{z_i, | \, 0 \in \mathcal{F}[g_i](z_i)\} = [-\lambda_i, \lambda_i]$. In conclusion, we have $\mathcal{H}_1 = \{x \mid |x_k - x_\ell| \leqslant \lambda_i, \forall e_i = (x_k, x_\ell) \in \mathcal{E}\}$, which is closed. □

Remark 4: If the function g_i is strictly sign preserving, namely, $y g_i(y) > 0$ if $y \neq 0$, the set $\mathcal{H}_1 = \text{span}\{\mathbb{1}\}$ when $\mathcal{G}$ is connected. Therefore, Theorem 8 indicates that the states of the agents asymptotically converge to consensus. In fact, finite-time consensus can be proved for some special cases, e.g., $g_i = \text{sign}$ [4].

If g_i is not strictly sign preserving, for example, if g_i is a uniform quantizer, the points in set $\mathcal{H}_1$ are sometimes referred as practical consensus. This coincides with the results in the literature, e.g., [12].

Remark 5: Theorem 8 is different from [26, Th. 14] in the sense that sign preservation (see [26, Def. 1]) is not assumed for the functions g_i in Theorem 8. Hence, precise consensus cannot be concluded in general from Theorem 8.

Before we present the next result, we employ an example to show that if the functions g_{ij} are not odd, the trajectories of (32) can be unbounded.

Example 2: Consider the system (32) defined on the undirected graph with two nodes v_1, v_2 and one edge e_1. Furthermore, assume $g_1 = \varphi$ with

$$\varphi(x) = \begin{cases} 0, & \text{if } x > 0 \\ -1, & \text{if } x \leqslant 0. \end{cases} \tag{39}$$

Then, the closed-loop system can be written as

$$\begin{aligned} \dot{x}_1 &= -\varphi(x_1 - x_2) \\ \dot{x}_2 &= -\varphi(x_2 - x_1). \end{aligned} \tag{40}$$

With a slight abuse of the notation, we denote

$$\varphi(Lx) := \begin{bmatrix} \varphi(x_1 - x_2) \\ \varphi(x_2 - x_1) \end{bmatrix} \tag{41}$$

where L is the Laplacian matrix of the graph. Notice that since φ is not an odd function, the system *cannot* be written in the form (33). Moreover, for any $x_0 \in \text{span}\{\mathbb{1}_2\}$, the Filippov set-valued map

$$\mathcal{F}[-\varphi(Lx)] = \overline{\text{co}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{42}$$

which implies that $x(t) = x_0 + \frac{1}{2}\mathbb{1}_2 t$ is a Filippov solution. Hence, the trajectories can diverge. A simulation with $x_0 = [0.5, 0]^\top$ is given in Fig. 1.

In fact, by analyzing the Lyapunov function $|x_1 - x_2|$, we can show that the trajectory of (40) converges to the consensus space $\{x \in \mathbb{R}^2 \mid x_1 = x_2\}$ in finite time and then slides on it. Hence, in this example, the consensus space and the solutions $x(t) = \eta(t)\mathbb{1}_2$ with nonconstant η are regarded as a sliding surface and sliding solutions, respectively. In some applications, the diverging sliding solutions are undesired; thus, extra conditions on the nonlinearities are needed to eliminate these trajectories (see Remark 6).

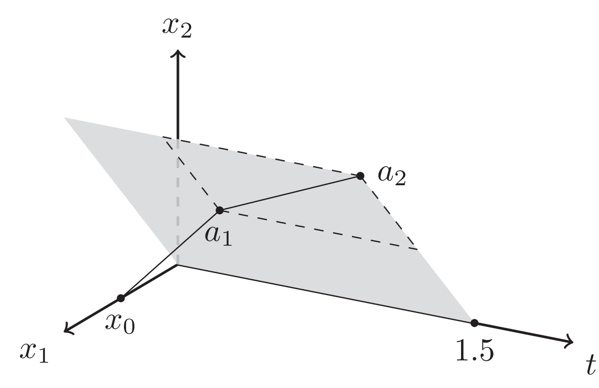


Fig. 1. Simulation of system (40) with initial condition $x_0 = [\frac{1}{2}, 0]^\top$. The gray surface is the consensus space. The trajectory hits the consensus space at $\frac{1}{2}\mathbb{1}_2$ when $t = \frac{1}{2}$, i.e., the point a_1. For $t \in [\frac{1}{2}, \infty)$, the solution is $(\frac{1}{4} + \frac{1}{2}t)\mathbb{1}_2$, which diverges as $t \to \infty$.

We shall consider extensions of Theorem 8 from undirected graphs to digraphs. We start with directed rings. By relabeling the edges as $e_1 = (v_n, v_1), e_2 = (v_1, v_2), \ldots, e_n = (v_{n-1}, v_n)$, the dynamical system (32) can be written in the vectorized form

$$\dot{x} = -g\left(-B^\top x\right) \tag{43}$$

where B is the incidence matrix of the ring and g is the same as in (33).

Theorem 9: Suppose the underlying graph is a ring and g_i satisfies Assumption 1. Then, all the Filippov solutions of (32) asymptotically converge to

$$\mathcal{H}_2 = \left\{ x \in \mathbb{R}^n \mid 0_n \in \mathop{\times}_{i=1}^{n} \mathcal{F}[g_i]\left(-B_{:,i}^\top x\right) \right\} \tag{44}$$

if

1) $|\mathcal{I}| = 2$ and g_i is odd for any $e_i \in \mathcal{E}$, or
2) $|\mathcal{I}| \geqslant 3$ and $g_i(0) = 0, \forall e_i \in \mathcal{E}$ and $\exists e_j \in \mathcal{E}$ such that $\mathcal{F}[g_j](0) = \{0\}$.

Proof: The Filippov differential inclusion corresponding to (43) is given as

$$\dot{x} \in \mathcal{F}\left[-g\left(B^\top x\right)\right](x) =: \mathcal{K}_4(x). \tag{45}$$

Consider the candidate Lyapunov functions V and W given as in (5). Similar to the proof of Theorem 6, we only consider $t \in \Psi_4$ such that $\mathcal{L}_{\mathcal{K}_4} V(x(t))$ and $\mathcal{L}_{\mathcal{K}_4} W(x(t))$ are nonempty, and $\mu(\mathbb{R}_{\geqslant 0} \setminus \Psi_4) = 0$. In order to show $\max \mathcal{L}_{\mathcal{K}_4} V(x(t)) \leqslant 0$, we consider the following two cases.

1) If $x \notin \text{span}\{\mathbb{1}\}$, there exists $i \in \alpha(x)$, which in defined as (21), and $\{j\} = N_i$ such that $x_j < x_i$. Since g_i is monotone and $g_i(0) = 0$, we have (37) holds. Hence, $\max \mathcal{L}_{\mathcal{K}_4} V(x(t)) \leqslant 0$.
2) If $x \in \text{span}\{\mathbb{1}\}$, $\mathcal{K}_4(x)$ can be analyzed in details as follows.

a) If $|\mathcal{I}| = 2$ and g_i is odd for any $e_i \in \mathcal{E}$, then

$$\mathcal{F}[-g(B^\top x)](x) = \overline{\text{co}} \left\{ \begin{bmatrix} a_1 g_1(0^+) \\ a_2 g_2(0^-) \end{bmatrix}, \begin{bmatrix} a_1 g_1(0^-) \\ a_2 g_2(0^+) \end{bmatrix} \right\}.$$

By the fact that g_i is odd, the set $\mathcal{F}[g(B^\top x)](x) \cap \text{span}\{\mathbb{1}_2\} = [0,0]^\top$.

b) if $|\mathcal{I}| \geqslant 3$ and $g_i(0) = 0, \forall e_i \in \mathcal{E}$ and there exists $e_j \in \mathcal{E}$ such that $\mathcal{F}[g_j](0) = \{0\}$, without loss of generality, we assume $\mathcal{F}[g_1](0) = \{0\}$. For any $x \in \text{span}\{\mathbb{1}_n\}$, we have $\nu_1 = 0$ for any $\nu \in \mathcal{F}[-g(B^\top x)](x)$.

In both cases, the set-valued Lie derivative $\mathcal{L}_{\mathcal{K}_4} V(x(t)) = \{0\}$.

So far, we proved that $\max \mathcal{L}_{\mathcal{K}_4} V(x(t)) \leqslant 0$. And $\max \mathcal{L}_{\mathcal{K}_4} W(x(t)) \leqslant 0$ can be proved in the same manner. This implies that the system (45) is Lyapunov stable.[2] Next, we shall show to which set the trajectories converge.

Consider the coordination transformation $z = -B^\top x$. By [20, Th. 1], we have that

$$\begin{aligned} \dot{z} &= -B^\top \dot{x} \\ &\subset -B^\top \mathcal{F}\left[-g\left(-B^\top x\right)\right](x) \\ &\subset -B^\top \mathop{\times}_{i=1}^{n} a_i \mathcal{F}[-g_i]\left(-B_{\cdot,i}^\top x\right) \\ &= B^\top \mathop{\times}_{i=1}^{n} a_i \mathcal{F}[g_i](z_i). \end{aligned} \tag{46}$$

Again, since $-B^\top$ is the Laplacian matrix of the ring graph, we have that the differential inclusion of z is the same as (10). Hence, by Theorem 6, the trajectories $z(t)$ converge to $\{z \in \mathbb{R}^n \mid \exists c \in \mathbb{R} \text{ s.t. } c\mathbb{1} \in \times_{i=1}^n a_i \mathcal{F}[g_i](z_i)\}$. Moreover, by the fact that $\mathbb{1}^\top z = 0$ and (37), we have $c = 0$. This implies that the trajectories $x(t)$ of (45) converge to $\mathcal{H}_2$. Here, the closedness of $\mathcal{H}_2$ follows the same arguments as in the proof of Theorem 8. □

Remark 6: For condition 1 in Theorem 9, Example 2 can be employed to show the necessity of having odd function g_i. For condition 2, [26, Example 16], which consider the case $g_i = \text{sign}, \forall e_i \in \mathcal{E}$, shows the necessity of existence $e_i \in \mathcal{E}$ s.t. $\mathcal{F}[g_i](0) = \{0\}$ to avoid an unbounded solution like in Example 2.

For the rest of this section, we consider the case that the underlying graph is a directed spanning tree.

Corollary 10: Consider the dynamical system (32) defined on a directed spanning tree. Suppose that $g_{ij} = \bar{g}$ satisfies Assumption 1 and $\bar{g}(0) = 0$; the weights $a_{ij} = a, \forall e_{ji} \in \mathcal{E}$. Then, all the Filippov solutions asymptotically converge to

$$\mathcal{H}_3 = \left\{ x \in \mathbb{R}^n \mid \exists \sigma \in \mathcal{F}[\bar{g}](0) \text{ s.t. } \sigma \mathbb{1}_{n-1} \in \mathop{\times}_{i=1}^{n-1} \mathcal{F}[\bar{g}]\left(-B_{\cdot,i}^\top x\right) \right\}. \tag{47}$$

[2]Notice that in this paper, we do not assume the nonlinear functions to be *sign preserving*, as defined in [26, Def. 1], so exact consensus cannot be expected.

Proof: Denote the root of $\mathcal{G}$ as v_1. Since the state of the root is constant, the differential inclusion corresponding to (32) can be written as

$$\dot{x} \in \mathcal{F}\begin{bmatrix} 0 \\ -\bar{g}(-B^\top x) \end{bmatrix}(x) =: \mathcal{K}_5(x) \tag{48}$$

where, without loss of generality, we take the weight $a = 1$, and for general weight, we denote $a g_{ij}$ as $\bar{g}$.

Since the Laplacian matrix of the tree is given by

$$-L = \begin{bmatrix} 0_n^\top \\ B^\top \end{bmatrix} \tag{49}$$

it can be verified by (1) that

$$\mathcal{K}_5(x) = \mathcal{F}[-\bar{g}(Lx)](x). \tag{50}$$

Similarly to Theorem 9, we shall first show that $\max \mathcal{L}_{\mathcal{K}_5} V(x(t)) \leqslant 0$ and $\max \mathcal{L}_{\mathcal{K}_5} W(x(t)) \leqslant 0$, where V and W given as in (5), for $t \in \Psi_5$ satisfying that $\mathbb{R}_{\geqslant 0} \setminus \Psi_5$ is a Lebesgue measure zero set. In order to show $\max \mathcal{L}_{\mathcal{K}_4} V(x(t)) \leqslant 0$, we consider the following two cases.

1) If $1 \in \alpha(x)$, defined in (21), then by the fact that $\mathcal{F}[\bar{g}(0)](x) = \{0\}$, we have $\mathcal{L}_{\mathcal{K}_5} V(x(t)) = \{0\}$.
2) If $1 \notin \alpha(x)$, then there exists $i \in \alpha(x)$ and $\{j\} = N_i$ such that $x_j \leqslant x_i$. Then, by (37) and $\mathcal{F}[\bar{g}(Lx)](x) \subset \times_{i=1}^n \mathcal{F}[\bar{g}(L_{i,\cdot}x)](x)$, we have $\mathcal{L}_{\mathcal{K}_5} V(x(t)) \subset \mathbb{R}_{\leqslant 0}$.

So far, we have proved that $\max \mathcal{L}_{\mathcal{K}_5} V(x(t)) \leqslant 0$. The same conclusion holds for W. This implies that the set $\mathcal{H}_3$ is Lyapunov stable for system (48). Next, we shall show $\mathcal{H}_3$ is in fact asymptotically stable.

Consider the coordinate transformation $z = Lx$. Then, z satisfies the following differential inclusion:

$$\begin{aligned} \dot{z} &\in L\mathcal{K}_5(x) \\ &\subset L\left(\{0\} \times \mathop{\times}_{i=1}^{n-1} \mathcal{F}[-\bar{g}]\left(-B_{\cdot,i}^\top x\right)\right) \\ &\subset -L\left(\mathcal{F}[\bar{g}](0) \times \mathop{\times}_{i=1}^{n-1} \mathcal{F}[\bar{g}]\left(-B_{\cdot,i}^\top x\right)\right) \end{aligned} \tag{51}$$

where the first $\subset$ is implied by [20, Th. 1(3) (4)], and the second one is implied by $\{0\} \subset \mathcal{F}[\bar{g}](0)$, which can be seen from the assumption that $\bar{g}(0) = 0$ and $\bar{g}$ is monotone. So far, we have

$$\dot{z} \in -L\left(\mathop{\times}_{i=1}^{n} \mathcal{F}[\bar{g}](z_i)\right) \tag{52}$$

which is in the same form as (10). Hence, by Theorem 7 and the fact $z_1 \equiv 0$, the conclusion holds. Here, the closedness of $\mathcal{H}_3$ follows the same arguments as in the proof of Theorem 8.

Here, we interpret Corollary 10 by an example. □

Example 3: If the function $\bar{g}$ is strictly increasing, $\bar{g}(0) = 0$, and is continuous at the origin, then trivially $\mathcal{F}[\bar{g}](0) = \{0\}$. Furthermore, we have $\mathcal{F}[\bar{g}](y) \subset \mathbb{R}_+$ if $y > 0$ and $\mathcal{F}[\bar{g}](y) \subset \mathbb{R}_-$ if $y < 0$. Hence, Corollary 10 implies that the states of the agents asymptotically converge to consensus.

For general $\bar{g}$, precise consensus cannot be guaranteed, for instance, if $\bar{g}$ is a uniform quantizer (see Corollary 12).

Remark 7: For general directed graphs, the trajectories will *not* converge to the sets given in Theorem 9 and Corollary 10. An example is given in the following section.

V. Applications to Quantized Consensus

In this section, we shall discuss the results in Sections III and IV for a special type of nonlinear consensus algorithms, namely, quantized consensus algorithms. Quantization is common in applications (e.g., [8], [9], [12], [19]) and may describe imperfect information exchange or communication constraints. There are three main types of quantizers, namely, uniform, asymmetric, and logarithmic quantizers

$$q^u(z) = \left\lfloor \frac{z}{\Delta} + \frac{1}{2} \right\rfloor \Delta \tag{53}$$

$$q^a(z) = \left\lfloor \frac{z}{\Delta} \right\rfloor \Delta \tag{54}$$

$$q^l(z) = \begin{cases} \operatorname{sign}(z) \exp\left(q^u\left(\ln(|z|)\right)\right), & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases} \tag{55}$$

respectively, where Δ is a positive constant.

In this section, we replace the nonlinear functions in system (6) and (32) by the aforementioned quantizers. By doing this, we extend some existing results about quantized consensus algorithms from undirected to directed graphs.

A. Quantized Communication

Consider a multiagent system with quantized communicated data

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\left(q_j(x_j) - q_i(x_i)\right) \tag{56}$$

where $q_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n$, denote one of (possibly different) the quantizers (53)–(55). If $x \in \mathbb{R}^n$, we denote with some abuse of notation $q(x) = (q_1(x_1), \ldots, q_n(x_n))^\top$. Hence, the dynamics (56) can be written in the vector form as

$$\dot{x} = -Lq(x). \tag{57}$$

Consider directed graphs. Let the quantizer be uniform, i.e., $q_i = q^u \ \forall i \in \mathcal{I}$. Then, system (57) can be written as

$$\dot{x} = -Lq^u(x). \tag{58}$$

In this case, the set $\mathcal{D}_2$ defined in (16) is given as

$$\{x \in \mathbb{R}^n \mid \exists k \in \mathbb{Z} \text{ such that } k\Delta\mathbb{1}_n \in \mathcal{F}[q^u](x)\} \tag{59}$$

which is equivalent to

$$\mathcal{Q} := \Big\{x \in \mathbb{R}^n \mid \exists k \in \mathbb{Z} \text{ s.t.} \left(k - \frac{1}{2}\right)\Delta \leqslant x_i \leqslant \left(k + \frac{1}{2}\right)\Delta \ \forall i \in \mathcal{I}\Big\}. \tag{60}$$

With quantized communication, precise consensus cannot be achieved in general. We say that the state variables of the agents converge to *practical consensus*, if $x(t) \to \mathcal{Q}$ as $t \to \infty$.

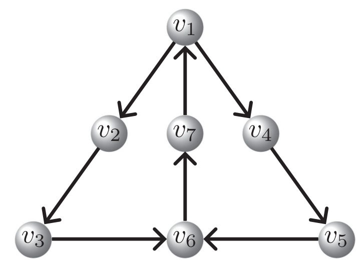


Fig. 2. Strongly connected digraph used in Examples 4.

Now, we can extend the result in [9], based on Theorem 7, from undirected to directed graphs.

Corollary 11: Consider the system (58) defined on a directed graph. If the graph contains a spanning tree, all the Filippov solutions asymptotically converge to $\mathcal{Q}$.

Remark 8: When the underlying graph is strongly connected or undirected, Theorem 6 implies the stability of system (57) when each q_i takes any of the form q^u, q^a, or q^l.

B. Quantized Actuation

Consider a multiagent system with quantized actuation

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} q(x_j - x_i). \tag{61}$$

By specifying the quantizer q to be uniform quantizer q^u, we have the set $\mathcal{H}_1$ in (34) can be rewritten as

$$\mathcal{P} := \left\{x \in \mathbb{R}^n \mid -\frac{1}{2}\Delta \leqslant x_i - x_j \leqslant \frac{1}{2}\Delta \ \forall e_{ij} \in \mathcal{E}\right\}. \tag{62}$$

Then, Theorems 8 and 9 and Corollary 10 lead to the following result.

Corollary 12: Consider the system (61) with uniform quantizer q^u; then, all the Filippov solutions asymptotically converge to the set $\mathcal{P}$ if

1) $\mathcal{G}$ is undirected, or
2) $\mathcal{G}$ is a directed ring or a directed spanning tree.

Proof: This corollary is a direct application of the results in Section IV, since q^u is odd and continuous at the origin, which implies that $\mathcal{F}[q^u](0) = \{0\}$. □

Remark 9: The undirected case corresponding to Corollary 12 was presented in [12].

In the following example, we show that the extension *cannot* be made to more general directed graphs containing cycles.

Example 4: Consider the multiagent system (32) defined on the digraph in Fig. 2. Furthermore, assume $g_{ij} = q^u$ with quantizer constant $\Delta = 1$. Given the initial condition $x_0 = [0, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0]^\top$, it can be verified that $x(t) = x_0, \forall t > 0$, is a Filippov solution. However, this solution does *not* belong to the set $\mathcal{P}$ in (62). In fact, $|x_3 - x_6| = |x_5 - x_6| > \frac{1}{2}\Delta$.

If the quantizer in (61) is replaced by the asymmetric q^a, diverging sliding solutions will appear as shown in the following example.

Example 5: Consider the system (61) with asymmetric quantizer q^a defined on the directed ring graph with two nodes. Since $\mathcal{F}[q^a](0) = \mathcal{F}[\varphi](0)$, where φ is defined in (39), for any

$x \in \mathrm{span}\{\mathbb{1}_2\}$, the Filippov set-valued map $\mathcal{F}[q^a(-Lx)](x) = \mathcal{F}[\varphi(-Lx)](x)$ is given as (42). Hence, for any $x_0 \in \mathrm{span}\{\mathbb{1}_2\}$, $x(t) = x_0 + \frac{1}{2}\mathbb{1}t$ is a Filippov solution. However, for system (61) defined on a directed ring graph, if there exists an edge e_{ij} on which the quantizer is q^u or q^l, the solution will be bounded.

VI. Conclusion

In this paper, we considered two general nonlinear consensus protocols, namely, multiagent systems with nonlinear communication or actuator constraints. For both cases, we proved asymptotic convergence to a consensus set for various topologies. More precisely, for the case with nonlinear communication, we considered the undirected graphs and directed graphs with a spanning tree. For the case with nonlinear actuation, we considered undirected graphs, directed rings, and directed spanning trees. Finally, we applied the results to quantized consensus protocols. Interesting problems for the future include switching topologies, and robustness studies.

Appendix

In this appendix, we discuss the equivalence between the Filippov and Krasovskii solutions of system (10).

First, we introduce the Krasovskii set-valued map and the corresponding Krasovskii solutions. The *Krasovskii set-valued map* of X, denoted $\mathcal{K}[X] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, is defined as

$$\mathcal{K}[X](x) \triangleq \bigcap_{\delta > 0} \overline{\mathrm{co}}\,\{X(B(x,\delta))\}. \tag{63}$$

Notice that different from Filippov set-valued map (1), here, we do not exclude any zero-measure subset S. If X is continuous at x, then $\mathcal{K}[X](x)$ contains only the point $X(x)$.

In the following proposition, we show a special case that the Krasovskii and Filippov set-valued maps are coincident.

Proposition 13: Suppose the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(x) = [f_1(x_1), \ldots, f_n(x_n)]^\top$ and each f_i satisfies Assumption 1; then, the Krasovskii set-valued map obeys

$$\mathcal{K}[f](x) = \mathop{\times}_{i=1}^{n} \mathcal{F}[f_i](x_i).$$

Proof: By definition (63) and Lemma 2, we have that for each component f_i satisfying Assumption 1

$$\mathcal{K}[f_i](x_i) = [f_i(x_i^-), f_i(x_i^+)]$$

which is equivalent to the Filippov set-valued map $\mathcal{F}[f_i](x_i)$. Then, by using the same arguments as in Lemma 4, we conclude that for the vector-valued function f, we have

$$\begin{aligned}\mathcal{K}[f](x) &= \mathop{\times}_{i=1}^{n} [f_i(x_i^-), f_i(x_i^+)] \\ &= \mathop{\times}_{i=1}^{n} \mathcal{F}[f_i](x_i).\end{aligned}$$

□

A *Krasovskii solution* of the differential equation $\dot{x}(t) = X(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^n$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{K}[X](x(t)) \tag{64}$$

for almost all $t \in [0, t_1]$. Now, the Krasovskii solutions of (6) are the solutions of the following differential inclusion:

$$\begin{aligned}\dot{x} &\in \mathcal{K}[-Lf(x)](x) \\ &= -L\mathcal{K}[f](x) \\ &= -L\mathcal{F}[f](x)\end{aligned}$$

where the first equality is based on [3, Proposition 11], and the second one is based on Proposition 13. Hence, the Krasovskii and Filippov solutions of (6) are equivalent.

References

[1] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions," *ESAIM: Control, Optim. Calculus Variations*, vol. 4, pp. 361–376, 1999.

[2] N. Biggs, *Algebraic Graph Theory*. Cambridge, U.K.: Cambridge Math. Library, Cambridge Univ. Press, 1993.

[3] F. Ceragioli, "Discontinuous ordinary differential equations and stabilization," Ph.D. dissertation, Univ. Firenze, Florence, Italy, 2000.

[4] G. Chen, F. L. Lewis, and L. Xie, "Finite-time distributed consensus via binary control protocols," *Automatica*, vol. 47, no. 9, pp. 1962–1968, 2011.

[5] F. H. Clarke, *Optimization and Nonsmooth Analysis* (ser. Classics in Applied Mathematics). Philadelphia, PA, USA: SIAM, 1990.

[6] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[7] J. Cortés, "Discontinuous dynamical systems," *IEEE Control Syst.*, vol. 28, no. 3, pp. 36–73, Jun. 2008.

[8] D. V. Dimarogonas and K. H. Johansson, "Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control," *Automatica*, vol. 46, no. 4, pp. 695–700, 2010.

[9] C. De Persis F. Ceragioli, and P. Frasca, "Discontinuities and hysteresis in quantized average consensus," *Automatica*, vol. 47, no. 9, pp. 1916–1928, 2011.

[10] A. F. Filippov and F. M. Arscott, *Differential Equations with Discontinuous Righthand Sides: Control Systems (ser. Mathematics and its Applications)*. New York, NY, USA: Springer, 1988.

[11] P. Frasca, "Continuous-time quantized consensus: Convergence of Krasovskii solutions," *Syst. Control Lett.*, vol. 61, no. 2, pp. 273–278, 2012.

[12] M. Guo and D. V. Dimarogonas, "Consensus with quantized relative state measurements," *Automatica*, vol. 49, no. 8, pp. 2531–2537, 2013.

[13] A. Kashyap, T. Başar, and R. Srikant, "Quantized consensus," *Automatica*, vol. 43, no. 7, pp. 1192–1203, 2007.

[14] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous time coupled nonlinear systems," *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 288–307, 2007.

[15] B. Liu, W. Lu, and T. Chen, "Consensus in networks of multiagents with switching topologies modeled as adapted stochastic processes," *SIAM J. Control Optim.*, vol. 49, no. 1, pp. 227–253, 2011.

[16] B. Liu, W. Lu, and T. Chen, "Consensus in continuous-time multiagent systems under discontinuous nonlinear protocols," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 2, pp. 290–301, Feb. 2015.

[17] W. Lu, F. M. Atay, and J. Jost, "Synchronization of discrete-time dynamical networks with time-varying couplings," *SIAM J. Math. Anal.*, vol. 39, no. 4, pp. 1231–1259, 2008.

[18] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Proc. 43rd IEEE Conf. Decis. Control*, 2004, pp. 3998–4003.

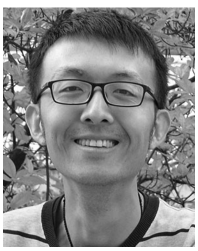
[19] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, "On distributed averaging algorithms and quantization effects," in *Proc. 47th IEEE Conf. Decis. Control*, 2008, pp. 4825–4830.

[20] B. Paden and S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. CAS-34, no. 1, pp. 73–82, Jan. 1987.

[21] A. Rantzer, "Distributed control of positive systems," in *Proc. 50th IEEE Conf. Decis. Control/Eur. Control Conf.*, 2011, pp. 6608–6611.

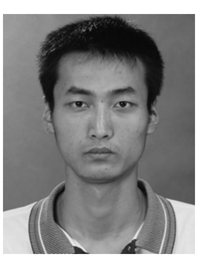
[22] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[23] W. Ren, R. W. Beard, and E. M. Atkins, “A survey of consensus problems in multi-agent coordination,” in *Proc. Amer. Control Conf.*, Jun. 2005, pp. 1859–1864.
[24] R. O. Saber and R. M. Murray, “Consensus protocols for networks of dynamic agents,” in *Proc. IEEE Amer. Control Conf.*, 2003, vol. 2, pp. 951–956.
[25] J. Wei, B. Besselink, J. Wu, H. Sandberg, and K. H. Johansson, “Finite-time consensus protocols for multi-dimensional multi-agent systems,” 2017. [Online]. Available: https://arxiv.org/abs/1705.02189
[26] J. Wei, A. R. F. Everts, M. K. Camlibel, and A. J. van der Schaft, “Consensus problems with arbitrary sign-preserving nonlinearities,” *Automatica*, vol. 83, pp. 226–233, 2017.
[27] J. Yeh. *Lectures on Real Analysis*. Singapore: World Scientific, 2000.
[28] X. Yi, J. Wei, and K. H. Johansson, “Self-triggered control for multi-agent systems with quantized communication or sensing,” in *Proc. 55th IEEE Conf. Decis. Control*, 2016, pp. 2227–2232.



Jieqiang Wei was born in Qufu, China, in May 1987. He received the B.Sc. degree in mathematics from Qufu Normal University, Jining, China, in 2008, the M.Sc. degree in mathematics from Beijing Normal University, Beijing, China, in 2011, and the Ph.D. degree from the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen, The Netherlands, in 2016.

He is currently a Postdoctoral Researcher with the Autonomic Complex Communication Networks, Signals and Systems Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden. His research interests include networked control systems, optimization, nonlinear systems and control, and stochastic systems.



Xinlei Yi received the B.S. degree in mathematics from the China University of Geoscience, Wuhan, China, in 2011, and the M.S. degree in mathematics from Fudan University, Shanghai, China, in 2014. He is currently working toward the Ph.D. degree in automatic control with the KTH Royal Institute of Technology, Stockholm, Sweden.

His current research interests include multiagent systems and distributed optimization.

Henrik Sandberg received the M.Sc. degree in engineering physics and the Ph.D. degree in automatic control from Lund University, Lund, Sweden, in 1999 and 2004, respectively.

He is currently a Professor with the Department of Automatic Control, KTH Royal Institute of Technology, Stockholm, Sweden. From 2005 to 2007, he was a Postdoctoral Scholar with the California Institute of Technology, Pasadena, CA, USA. In 2013, he was a Visiting Scholar with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA. He has also held visiting appointments with the Australian National University and the University of Melbourne, Australia. His current research interests include security of cyber-physical systems, power systems, model reduction, and fundamental limitations in control.

Dr. Sandberg was a recipient of the Best Student Paper Award from the IEEE Conference on Decision and Control in 2004, an Ingvar Carlsson Award from the Swedish Foundation for Strategic Research in 2007, and the Consolidator Grant from the Swedish Research Council in 2016. He was on the Editorial Board of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He is an Associate Editor for the International Federation of Automatic Control journal *Automatica*.

Karl Henrik Johansson (F’13) received the M.Sc. and Ph.D. degrees from Lund University, Lund, Sweden, in 1992 and 1997, respectively.

He is the Director of the Stockholm Strategic Research Area ICT The Next Generation and a Professor with the School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden. He has held visiting positions with the University of California, Berkeley, California Institute of Technology, Nanyang Technological University, HKUST Institute of Advanced Studies, and Norwegian University of Science and Technology. His research interests include networked control systems, cyber-physical systems, and applications in transportation, energy, and automation.

Dr. Johansson is a member of the IEEE Control Systems Society Board of Governors, the International Federation of Automatic Control (IFAC) Executive Board, and the European Control Association Council. He is a recipient of several best paper awards and other distinctions. He has been awarded a Distinguished Professor with the Swedish Research Council and Wallenberg Scholar. He is a recipient of the Future Research Leader Award from the Swedish Foundation for Strategic Research and the triennial Young Author Prize from the IFAC. He is Fellow of the Royal Swedish Academy of Engineering Sciences and an IEEE Distinguished Lecturer.