Set Target Aggregation of Multiple Mechanical Systems

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Abstract— In this paper, we study the set target aggregation problem of multiple mechanical systems. Each system is modeled by a Lagrangian dynamical equation and observes a convex set as its local target. The objective of the group is to reach an aggregation towards these target sets. We propose a set target aggregation algorithm that is constructed based on each mechanical system's own target sensing and the exchange of its information with local neighbors. With necessary connectivity for both fixed and switching communication topologies, multiple mechanical systems are shown to converge to the intersection of all the local target sets while the vectors of generalized coordinate derivatives are driven to zero. Simulations are given to validate the theoretical results.

I. INTRODUCTION

A large amount of results have been obtained towards distributed control of multi-agent systems during the last several decades (see e.g., [1]–[3]). The key idea is to realize a collective task for the overall system by using only local information interactions [4]–[6]. Such an algorithm naturally requires that the individual system is equipped with communication units and therefore the analysis on the influence of the communication link failure and communication data loss is important [2], [7]. Also, for the agent dynamics, both continuous-time and discrete-time models were studied and deep understanding was obtained including convergence speed, delay robustness, nonlinear convexity, general linear dynamics and so on [8]–[12].

The single integrator model assumption has been widely used in the literature to derive mathematically beautiful results. On the other hand, in order to satisfy real application requirements, the study on the distributed control of multiple Lagrangian systems has attracted extensive attention recently. Here, a Lagrangian model can be used to describe mechanical systems including autonomous vehicles, mobile robots, robotic manipulators, and rigid bodies [13]–[16]. In particular, the author of [17] proposed distributed model-independent consensus algorithms for multiple Lagrangian systems in a leaderless setting. The rendezvous algorithm was proposed for multiple nonholonomic vehicles in [18]

while finite-time coordinated tracking algorithms were presented in [19] over graphs that are quasi-strongly connected. In order to obtain zero-error coordinated tracking result, a sliding mode based strategy was proposed in [20] for the leader-follower tracking problem. A similar problem was also studied in [21], [22], where continuous control algorithms were proposed. The authors of [23] established containment, group dispersion and group cohesion behaviors for multiple Lagrangian systems, where both the cases of constant and time-varying leaders' velocities were considered. The authors of [24] applied the coordination algorithms to the shape and formation control.

In this paper, we study the set target aggregation problem of multiple mechanical systems driven by Lagrangian dynamical equation and each agent observes a convex set as its local target. We propose a set target aggregation algorithm that is constructed based on each agent's own target sensing and exchange of its information with local neighbors. We show that multiple Lagrangian systems converge to the intersection of all the local target sets while the vectors of generalized coordinate derivatives of all the agents are driven to zero for both cases of fixed communication topology and switching communication topology provided that some necessary connectivity conditions are satisfied.

The remainder of the paper is organized as follows. In Section II, we give some basic notation and definitions on convex analysis, graph theory, and Dini derivatives. We then formulate the problem in Section III. The main results are presented in Sections IV and V for the cases of fixed and switching communication topologies, respectively. The simulations are followed in Section VI and a brief concluding remark is given in Section VII.

II. PRELIMINARIES

A. Convex Analysis

Denote $\|\cdot\|$ the Euclidean norm. For any nonempty set $\mathcal{S} \subseteq \mathbb{R}^m$, we use $\|x\|_{\mathcal{S}} = \inf_{y \in \mathcal{S}} \|x - y\|$ to describe the distance between $x \in \mathbb{R}^m$ and $\mathcal{S}$. A set $\mathcal{S} \subset \mathbb{R}^m$ is called convex if $(1 - \zeta)x + \zeta y \in \mathcal{S}$ when $x \in \mathcal{S}$, $y \in \mathcal{S}$, and $0 \leq \zeta \leq 1$.

Let $\mathcal{S}$ be a convex set. The convex projection of any $x \in \mathbb{R}^m$ onto $\mathcal{S}$ is denoted by $P_{\mathcal{S}}(x) \in \mathcal{S}$ satisfying $\|x - P_{\mathcal{S}}(x)\| = \|x\|_{\mathcal{S}}$. We also know that $\|x\|_{\mathcal{S}}^2$ is continuously differentiable for all $x \in \mathbb{R}^m$, and its gradient can be explicitly obtained by [25]:

$$\nabla \|x\|_{\mathcal{S}}^2 = 2(x - P_{\mathcal{S}}(x)). \tag{1}$$

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Also, it is trivial to see that

$$(P_{\mathcal{S}}(x) - x)^{\mathrm{T}}(P_{\mathcal{S}}(x) - y) \leq 0, \quad \forall y \in \mathcal{S}, \qquad (2)$$

and

$$\|P_{\mathcal{S}}(x) - P_{\mathcal{S}}(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^m. \qquad (3)$$

B. Graph theory

An undirected graph $\mathcal{G}$ consists of a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is a finite, nonempty set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of unordered pairs of nodes. An arc $(j, i) \in \mathcal{E}$ denotes that node i can obtain information from node j. An undirected graph $\mathcal{G}$ is defined such that for any two nodes i and j, $(j, i) \in \mathcal{E}$ if and only if $(i, j) \in \mathcal{E}$. All neighbors of node i are denoted $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}\}$. A path is a sequence of arcs of the form $(i, j), (j, k), \ldots$. An undirected graph $\mathcal{G}$ is said to be connected if each node has an undirected path to any other node.

The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ associated with the graph $\mathcal{G}$ is defined such that a_{ij} is positive if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We also assume that $a_{ij} = a_{ji}$, for all $i, j \in \mathcal{V}$ for the undirected graph in this paper. The weighted Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n \times n}$ associated with $\mathcal{A}$ is defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$.

C. Dini derivatives and Matrosov's theorem

Let $D^+ V(x(t), t)$ be the upper Dini derivative of $V(x(t), t)$ with respect to t, i.e.,

$$D^+ V(x, t) = \limsup_{\tau \to 0^+} \frac{V(x(t + \tau), t + \tau) - V(x(t), t)}{\tau}.$$

The following lemma will be used for our analysis later.

Lemma 1. *([26]) Suppose for each $i \in \mathcal{V}$, $V_i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Let $V(x, t) = \max_{i \in \mathcal{V}} V_i(x, t)$, and let $\overline{\mathcal{V}}(t) = \{i \in \mathcal{V} : V_i(x(t), t) = V(x(t), t)\}$ be the set of indices where the maximum is reached at time t. Then*

$$D^+ V(x(t), t) = \max_{i \in \overline{\mathcal{V}}(t)} \dot{V}_i(x(t), t).$$

The following lemma is a variant of Matrosov's theorem stated in [27], which will be very useful in analyzing uniformly asymptotically stable of the equilibrium of a non-autonomous system.

Lemma 2 (Matrosov's theorem). *Given the system*

$$\dot{x} = f(t, x), \qquad (4)$$

where $f(t, 0) = 0$ and $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x. Let $V(t, x)$ and $\psi(t, x)$ be continuously differentiable functions on domain $[t_0, \infty) \times \mathcal{D}$ and satisfy the following conditions:

1) *$V(t, x)$ is positive definite and decrescent.*
2) *$\dot{V}(t, x) \leq W(x) \leq 0$, where $W(x)$ is continuous.*
3) *$|\psi(t, x)|$ is bounded.*
4) *The function $\dot{\psi}(t, x)$ is continuous in both arguments and $\dot{\psi}(t, x) = g(x, \chi(t))$, where g is continuous in both of its arguments, $\chi(t)$ is also continuous and bounded.*

5) *There exists a class $\mathcal{K}$ function μ such that $|\dot{\psi}(t, x)| \geq \mu(\|x\|)$, $\forall x \in \mathcal{M}$, where $\mathcal{M} = \{x | W(x) = 0\}$.*

Then, the equilibrium of (4) is uniformly asymptotically stable on $\mathcal{D}$.

III. PROBLEM FORMULATION

Suppose that there are n agents in the group, labelled by $\mathcal{V} = \{1, 2, \ldots, n\}$. The system dynamics of the agents are described by the Lagrangian equations

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i = \tau_i, \ i = 1, 2, \ldots, n, \qquad (5)$$

where $q_i \in \mathbb{R}^m$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{m \times m}$ is the $m \times m$ inertia (symmetric) matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i$ is the Coriolis and centrifugal terms, and $\tau_i \in \mathbb{R}^m$ is the control force. The dynamics of a Lagrangian system satisfies the following properties [28]:

1. $M_i(q_i)$ is positive definite and is bounded for any $q_i \in \mathbb{R}^m$.
2. $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric.
3. $C_i(q_i, \dot{q}_i)$ is bounded with respect to q_i and linearly bounded with respect to $\dot{q}_i$. More specifically, there is positive constant k_C such that $\|C_i(q_i, \dot{q}_i)\| \leq k_C \|\dot{q}_i\|$.

We consider the set target aggregation problem for a group of Lagrangian systems. Each agent $i \in \mathcal{V}$ observes its own target set X_i. The objective is to ensure that the generalized coordinate derivatives of all the agents converge to zero and their generalized coordinates achieve agreement, while the destination of each agent is constrained by its target set. At each time, we assume that each agent observes the boundary points of its target set and obtain the relative distance information between the target set and itself. Also, the state information of each agent are exchanged by equipping each agent with simple and cheap communication unit. In the following, we give some necessary assumptions on set X_i, $i \in \mathcal{V}$ and define the set target aggregation problem.

Assumption 3.1. $X_1, X_2, \ldots, X_n$ are closed convex sets.

Assumption 3.2. $X_0 = \bigcap_{i=1}^n X_i$ is nonempty and bounded.

Definition 1. *Multi-agent system (5) is said to achieve set target aggregation if*

1) $\lim_{t \to \infty} \|q_i(t)\|_{X_0} = 0, \quad \forall i \in \mathcal{V}$,
2) $\lim_{t \to \infty} (q_i(t) - q_j(t)) = 0, \quad \forall i, j \in \mathcal{V}$,
3) $\lim_{t \to \infty} \dot{q}_i(t) = 0, \quad \forall i \in \mathcal{V}$.

Remark 1. *In fact, this set target aggregation problem under convexity assumptions was a classical problem in optimization, where projected consensus algorithm was a standard solution [29]. This algorithm was then generalized to distributed versions via consensus dynamics in [30], [31]. However, all these existing algorithms are designed for agents with first-order dynamics, and are therefore not applicable to the problem studied in the current paper.*

IV. FIXED COMMUNICATION TOPOLOGY

In this section, the communication topology is assumed to be fixed. The following target aggregation algorithm is proposed for all $i \in \mathcal{V}$,

$$\tau_i = -k\dot{q}_i - \sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j) - (q_i - P_{X_i}(q_i)), \quad (6)$$

where $k > 0$ denotes generalized coordinate derivative damping, a_{ij} is (i,j)th entry of $\mathcal{A}$ associated with $\mathcal{G}$ defined in Section II-B. Note that a_{ij} is a positive constant if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise.

Theorem 1. *Suppose Assumptions 3.1 and 3.2 hold. The multi-agent system (5) with (6) achieves set target aggregation in the sense of Definition 1 if the fixed communication topology $\mathcal{G}$ is connected.*

Proof: In what follows, we will use Lemma 2 to prove Theorem 1. Consider the following Lyapunov function:

$$V = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} M_i(q_i)\dot{q}_i + \frac{1}{2}\sum_{i=1}^{n} \|q_i - P_{X_i}(q_i)\|^2$$
$$+ \frac{1}{4}\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} a_{ij}\|q_i - q_j\|^2. \quad (7)$$

Let $\tilde{q}$ denote a column stack vector of all $q_i - q_j$, where $i < j$ and $a_{ij} \neq 0$. It then follows that V is positive definite and decrescent with respect to $\tilde{q}$, $q_i - P_{\mathcal{X}_i}(q_i)$, $i \in \mathcal{V}$, and $\dot{q}_i$, $i \in \mathcal{V}$ since the communication graph $\mathcal{G}$ is connected. This verifies the first condition of Lemma 2. The derivative of V along (5) with (6) is

$$\dot{V} = \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}\left(\frac{1}{2}\dot{M}_i(q_i)\dot{q}_i + M_i(q_i)\ddot{q}_i\right) + \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}(q_i - P_{X_i}(q_i))$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)^{\mathrm{T}}(\dot{q}_i - \dot{q}_j)$$
$$= \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}\left(-k\dot{q}_i - (q_i - P_{\mathcal{X}_i}(q_i)) - \sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)\right)$$
$$+ \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}(q_i - P_{X_i}(q_i)) + \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}\sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)$$
$$= -\sum_{i=1}^{n} k\dot{q}_i^{\mathrm{T}}\dot{q}_i =: W \leq 0,$$

where we have used (1) to derive the first equality, the fact that $a_{ij} = a_{ji}$ and the second property of Lagrangian dynamics to derive the second equality. The above inequality verifies the second condition of Lemma 2.

Pick any $q_0 \in \mathcal{X}_0$. Such a q_0 exists due to the fact that $\mathcal{X}_0$ is nonempty and bounded. Define $\psi = \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} M_i(q_i)(q_i - q_0)$. It follows that q_i, $\dot{q}_i$, for all $i \in \mathcal{V}$ are bounded from the first property of Lagrangian dynamics and the fact that $\dot{V} \leq 0$. Combining with the fact that q_0 is bounded, we know that $|\psi|$ is bounded. This verifies the third condition

of Lemma 2. We then get that the derivative of ψ is

$$\dot{\psi} = \sum_{i=1}^{n} \ddot{q}_i^{\mathrm{T}} M_i(q_i)(q_i - q_0) + \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} \dot{M}_i(q_i)(q_i - q_0)$$
$$+ \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} M_i(q_i)\dot{q}_i$$
$$= -\sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} C_i^{\mathrm{T}}(q_i)(q_i - q_0) - \sum_{i=1}^{n} k\dot{q}_i^{\mathrm{T}}(q_i - q_0)$$
$$- \sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}\sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)$$
$$- \sum_{i=1}^{n}(q_i - P_{\mathcal{X}_i}(q_i))^{\mathrm{T}}(q_i - q_0)$$
$$+ \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} \dot{M}_i(q_i)(q_i - q_0) + \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}} M_i(q_i)\dot{q}_i.$$

It is trivial to see that the form of $\dot{\psi}$ satisfies the fourth condition of Lemma 2 based on the facts that q_0 is a constant and $P_{X_i}(q_i)$ is a function of q_i. It follows that on the set $\mathcal{M} = \{q_i \in \mathbb{R}^m, \dot{q}_i \in \mathbb{R}^m, \forall i \in \mathcal{V} \mid \dot{q}_i = 0, \forall i \in \mathcal{V}\}$, $\dot{\psi}$ becomes

$$\dot{\psi} = -\sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}} \sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)$$
$$- \sum_{i=1}^{n}(q_i - P_{X_i}(q_i))^{\mathrm{T}}(q_i - q_0).$$

We know that $\sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}\sum_{j \in \mathcal{N}_i} a_{ij}(q_i - q_j)$
$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} a_{ij}\|q_i - q_j\|^2 \geq 0$ by noting that $a_{ij} = a_{ji}$. Also, we know from (2) that for all $i \in \mathcal{V}$, $(P_{X_i}(q_i) - q_0)^{\mathrm{T}}(q_i - P_{\mathcal{X}_i}(q_i)) \geq 0$. It then follows that

$$(q_i - q_0)^{\mathrm{T}}(q_i - P_{X_i}(q_i)) = \|q_i - P_{X_i}(q_i)\|^2$$
$$+ (P_{X_i}(q_i) - q_0)^{\mathrm{T}}(q_i - P_{X_i}(q_i))$$
$$\geq \|q_i - P_{X_i}(q_i)\|^2.$$

This shows that on the set $\mathcal{M}$,

$$\dot{\psi} \leq -\frac{1}{2}\sum_{i=1}^{n}\sum_{j \in \mathcal{N}_i} a_{ij}\|q_i - q_j\|^2 - \sum_{i=1}^{n}\|q_i - P_{X_i}(q_i)\|^2.$$

Therefore, $|\dot{\psi}|$ is positive definite with respect to $\tilde{q}$ and $q_i - P_{X_i}(q_i)$, $i \in \mathcal{V}$. This verifies the last condition of Lemma 2. It thus follows from Lemma 2 that $\lim_{t \to \infty} \dot{q}_i(t) = 0$, $\lim_{t \to \infty}(q_i(t) - P_{X_i}(q_i(t))) = 0$, $\forall i \in \mathcal{V}$, and $\lim_{t \to \infty} \tilde{q}(t) = 0$. Therefore, we know that $\lim_{t \to \infty}(q_i(t) - q_j(t)) = 0$, $\forall i,j \in \mathcal{V}$ from the fact that $\mathcal{G}$ is connected. It then follows that for all $i \in \mathcal{V}$ and $l \in \mathcal{V}$

$$\|q_i - P_{X_l}(q_i)\| \leq \|q_i - q_l\| + \|q_l - P_{X_l}(q_l)\|$$
$$+ \|P_{X_l}(q_l) - P_{X_l}(q_i)\|$$
$$\leq 2\|q_i - q_l\| + \|q_l - P_{X_l}(q_l)\|, \quad (8)$$

where we have used (3). This implies that $\lim_{t \to \infty}(q_i(t) - P_{X_l}(q_i(t))) = 0$, $\forall i \in \mathcal{V}$ and $l \in \mathcal{V}$. Therefore, $\lim_{t \to \infty}\|q_i(t)\|_{X_0} = 0$, $\forall i \in \mathcal{V}$. This shows that set target aggregation is achieved in the sense of Definition 1. $\qquad \square$

V. SWITCHING COMMUNICATION TOPOLOGY

One issue of introducing the communication unit is the possible communication link failure. Such issue becomes even more important when we consider the real applications including controlling multiple autonomous vehicles in the environments with limited power. Therefore, it is necessary to consider the case of switching communication topology. We associate the switching communication topology with a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, where $\sigma : [t_0, +\infty) \rightarrow \mathcal{P}$ is a piecewise constant function and $\mathcal{P}$ is finite set of all possible graphs. $\mathcal{G}_{\sigma(t)}$ remains constant for $t \in [t_l, t_{l+1})$, $l = 0, 1, \ldots$ and switches at $t = t_l$, $l = 1, \ldots$. In addition, we assume that $\inf_l(t_{l+1} - t_l) \geq \tau_d > 0$, $l = 1, \ldots$, where τ_d is a constant and this dwell time assumption is extensively used in the analysis of switched systems [32]. The joint graph of $\mathcal{G}_{\sigma(t)}$ during time interval $[t_1, t_2)$ is defined by $\mathcal{G}([t_1, t_2)) = \bigcup_{t \in [t_1, t_2)} \mathcal{G}(t) = (\mathcal{V}, \bigcup_{t \in [t_1, t_2)} \mathcal{E}(t))$. Moreover, j is a neighbor of i at time t when $(j, i) \in \mathcal{E}_{\sigma(t)}$, and $\mathcal{N}(\sigma(t))$ represents the set of agent i's neighbors at time t.

Definition 2. $\mathcal{G}_{\sigma(t)}$ *is uniformly jointly connected if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is connected for any $t \geq t_0$.*

The existence of the switching communication topology complicates the problem significantly. In order to simplify the problem, we assume that the exact information of system dynamical parameters are available and propose the following algorithm

$$\tau_i = C_i(q_i, \dot{q}_i)\dot{q}_i + M_i(q_i)u_i, \qquad \forall i \in \mathcal{V}, \qquad (9)$$

where

$$u_i = -k\dot{q}_i - \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(q_i - q_j) - (q_i - P_{X_i}(q_i)),$$

$k > 0$ denotes generalized coordinate derivative damping, and $a_{ij}(p)$ is the weight of arc (j, i) associated with graph $\mathcal{G}_p$, for all $p \in \mathcal{P}$.

By introducing (9) into (5), we have

$$\ddot{q}_i = -k\dot{q}_i - \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(q_i - q_j) - (q_i - P_{X_i}(q_i)). \tag{10}$$

We next focus on the closed-loop system (10).

Lemma 3. *Suppose that Assumptions 3.1 and 3.2 hold and choose k large enough. The states of the multi-agent system (5) with (9) achieve local target aggregation, i.e., $\lim_{t \to \infty} \|q_i(t)\|_{X_i} = 0$, and $\lim_{t \to \infty} \dot{q}_i(t) = 0$, for all $i \in \mathcal{V}$.*

Proof: By picking any $q_0 \in X_0$, we propose the following Lyapunov function:

$$V = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}\dot{q}_i + \sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}\dot{q}_i + \frac{k}{2}\sum_{i=1}^{n} \|q_i - q_0\|^2$$
$$+ \frac{1}{2}\sum_{i=1}^{n} \|q_i - P_{X_i}(q_i)\|^2,$$

where we choose $k > 1$ to guarantee V is positive definite. The derivative of V along (10) is

$$\dot{V} = \sum_{i=1}^{n} \dot{q}_i^{\mathrm{T}}\left(-k\dot{q}_i - \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(q_i - q_j) \right.$$
$$- (q_i - P_{X_i}(q_i)) + \sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}(-k\dot{q}_i$$
$$\left. - \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(q_i - q_j) - (q_i - P_{X_i}(q_i)) \right)$$
$$+ \sum_{i=1}^{n}\|\dot{q}_i\|^2 + k\sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}\dot{q}_i + \sum_{i=1}^{n}\dot{q}_i^{\mathrm{T}}(q_i - P_{X_i}(q_i))$$
$$= -(k-1)\sum_{i=1}^{n}\|\dot{q}_i\|^2 - \sum_{i=1}^{n}\dot{q}_i^{\mathrm{T}}\sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))$$
$$\times (q_i - q_j) - \sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}(q_i - P_{X_i}(q_i))$$
$$- \sum_{i=1}^{n}(q_i - q_0)^{\mathrm{T}}\sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(q_i - q_j)$$
$$\leq -\begin{bmatrix} \bar{q} & \dot{q} \end{bmatrix}\begin{bmatrix} \mathcal{L}_\sigma & \frac{\mathcal{L}_\sigma}{2} \\ \frac{\mathcal{L}_\sigma}{2} & (k-2)I_n \end{bmatrix}\begin{bmatrix} \bar{q} \\ \dot{q} \end{bmatrix}$$
$$- \sum_{i=1}^{n}\|q_i\|_{X_i}^2 - \sum_{i=1}^{n}\|\dot{q}_i\|^2,$$

where $\bar{q}_i = q_i - q_0$, $\bar{q} = [\bar{q}_1^{\mathrm{T}}, \bar{q}_2^{\mathrm{T}}, \ldots, \bar{q}_n^{\mathrm{T}}]^{\mathrm{T}}$, $q = [q_1^{\mathrm{T}}, q_2^{\mathrm{T}}, \ldots, q_n^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{L}_\sigma$ is a weighted Laplacian matrix associated with $\mathcal{G}_\sigma$ defined in Section II-B, and we have used the fact that $(q_i - q_0)^{\mathrm{T}}(q_i - P_{X_i}(q_i)) \geq \|q_i - P_{X_i}(q_i)\|^2$. We know that all the eigenvalues of $\mathcal{L}_p$ are non-positive and real, for all $p \in \mathcal{P}$. Therefore, if k is chosen such that $k > 2 + \frac{1}{4}\max_{p \in \mathcal{P}}\{\lambda_{\max}(\mathcal{L}_p)\}$, or equivalent, $k > 2 + \frac{(n-1)a^*}{2}$, we can show that $\begin{bmatrix} \mathcal{L}_p & \frac{\mathcal{L}_p}{2} \\ \frac{\mathcal{L}_p}{2} & (k-2)I_n \end{bmatrix}$ is positive semi-definite, for all $p \in \mathcal{P}$, where $a^* = \max_{p \in \mathcal{P}}\max_{i,j \in \mathcal{V}} a_{ij}(p)$. Under this condition, it then follows that

$$\dot{V} \leq -\sum_{i=1}^{n}\|q_i\|_{X_i}^2 - \sum_{i=1}^{n}\|\dot{q}_i\|^2 \leq 0. \tag{11}$$

Therefore, q_i and $\dot{q}_i$, $\forall i \in \mathcal{V}$ are bounded. We also know that (11) implies that

$$\int_{t_0}^{\infty}\left(\sum_{i=1}^{n}\|q_i(t)\|_{X_i}^2 + \sum_{i=1}^{n}\|\dot{q}_i(t)\|^2 \right)\mathrm{d}t$$

is bounded by $V(t_0)$. In addition, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\|q_i(t)\|_{X_i}^2 + \sum_{i=1}^{n}\|\dot{q}_i(t)\|^2\right) = 2\sum_{i=1}^{n}\left((q_i - P_{X_i}(q_i))^{\mathrm{T}}\dot{q}_i + \dot{q}_i^{\mathrm{T}}\ddot{q}_i\right)$. Therefore, from (10) and the facts that q_i and $\dot{q}_i$, $\forall i \in \mathcal{V}$ are bounded, we know that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\|q_i(t)\|_{X_i}^2 + \sum_{i=1}^{n}\|\dot{q}_i(t)\|^2\right)$ is bounded, $\forall t \geq t_0$. Then, based on Barbalat's lemma, we can show that $\sum_{i=1}^{n}\|q_i(t)\|_{X_i}^2 + \sum_{i=1}^{n}\|\dot{q}_i(t)\|^2 \to 0$, as $t \to \infty$. Therefore, $\lim_{t \to \infty}\|q_i(t)\|_{X_i} = 0$, and $\lim_{t \to \infty}\dot{q}_i(t) = 0$, for all $i \in \mathcal{V}$. $\qquad \square$

Next, we define $x_i = q_i$, $x_{n+i} = q_i + \dot{q}_i$, for all $i \in \mathcal{V}$. After some manipulations, (10) can be written as

$$\dot{x}_i = -(x_i - x_{n+i}), \tag{12a}$$

$$\dot{x}_{n+i} = -\sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(x_{n+i} - x_{n+j}) + \delta_i, \tag{12b}$$

where $i \in \mathcal{V}$, and $\delta_i = (1-k)\dot{q}_i + \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(\dot{q}_i - \dot{q}_j) - (q_i - P_{\mathcal{X}_i}(q_i))$, for all $i \in \mathcal{V}$. Note that Lemma 3 implies that $\lim_{t \to \infty} \delta_i(t) = 0$, for all $i \in \mathcal{V}$. We next present the main result of this section.

Theorem 2. *Suppose that Assumptions 3.1 and 3.2 hold and choose k large enough. The multi-agent system (5) with (9) achieves set target aggregation in the sense of Definition 1 if the communication topology $\mathcal{G}_{\sigma(t)}$ is uniformly jointly connected.*

Proof: Consider system (12) as a new multi-agent system with node set $\overline{\mathcal{V}} = \{1, 2, \ldots, 2n\}$. We associate this system with a new graph $\overline{\mathcal{G}}_{\sigma(t)} = (\overline{\mathcal{V}}, \overline{\mathcal{E}}_{\sigma(t)})$, the corresponding neighbor set $\overline{\mathcal{N}}_i(\sigma(t))$ and the adjacency matrix $\overline{A}_{\sigma(t)}$. Note that the connections and weights for agents $\{n+1, n+2, \ldots, 2n\}$ are defined by $\mathcal{E}_{\sigma(t)}$ and $A_{\sigma(t)}$. In addition, there exists persistent arc $a_{i(i+n)}(t) = 1$, for all $i = 1, 2, \ldots, n$ and all $t \geq t_0$. It is not hard to see that $\overline{\mathcal{G}}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected with the uniform constant T, where $\overline{\mathcal{G}}_{\sigma(t)}$ is said to be uniformly jointly quasi-strongly connected if there exists a node $i \in \overline{\mathcal{V}}$ and a constant $T > 0$ such that there exists a directed path from i to any other node for $\overline{\mathcal{G}}([t, t+T))$, $\forall t \geq t_0$.

Then, based on Proposition 4.10 of [33] and the fact that $\lim_{t \to \infty} \delta_i(t) = 0$, for all $i \in \mathcal{V}$, we know that $\lim_{t \to \infty}(q_i(t) - q_j(t)) = 0$, $\forall i, j \in \mathcal{V}$. Following (8) in the proof of Theorem 1, we know that $\lim_{t \to \infty} \|q_i(t)\|_{\mathcal{X}_0} = 0$, $\forall i \in \mathcal{V}$. This shows that set target aggregation is achieved in the sense of Definition 1.
$\square$

VI. SIMULATION RESULTS

In this section, we use numerical simulations to validate the effectiveness of the theoretical results obtained in this paper. We assume that there are four agents ($n = 4$) in the group. The system dynamics of the followers are given by the following two-link manipulators [20], [28],

$$\begin{bmatrix} M_{11,i} & M_{12,i} \\ M_{21,i} & M_{22,i} \end{bmatrix} \begin{bmatrix} \ddot{q}_{ix} \\ \ddot{q}_{iy} \end{bmatrix} + \begin{bmatrix} C_{11,i} & C_{12,i} \\ C_{21,i} & C_{22,i} \end{bmatrix} \begin{bmatrix} \dot{q}_{ix} \\ \dot{q}_{iy} \end{bmatrix}$$
$$= \begin{bmatrix} \tau_{ix} \\ \tau_{iy} \end{bmatrix}, i = 1, 2, 3, 4,$$

where $M_{11,i} = \theta_{1i} + 2\theta_{2i} \cos q_{iy}$, $M_{12,i} = M_{21,i} = \theta_{3i} + \theta_{2i} \cos q_{iy}$, $M_{22,i} = \theta_{3i}$, $C_{11,i} = -\theta_{2i} \sin q_{iy} \dot{q}_{iy}$, $C_{12,i} = -\theta_{2i} \sin q_{iy}(\dot{q}_{ix} + \dot{q}_{iy})$, $C_{21,i} = \theta_{2i} \sin q_{iy} \dot{q}_{ix}$, $C_{22,i} = 0$. We choose $\theta_{1i} = 1.301$, $\theta_{2i} = 0.256$, $\theta_{3i} = 0.096$, $i = 1, 2, 3, 4$. The available local sets of all the agents are rectangles given by $X_1 = [-0.5, 1] \times [-0.5, 1]$, $X_2 = [0.5, -1] \times [0.5, -1]$, $X_3 = [-0.5, 1] \times [-0.5, 1]$, and $X_4 = [0.5, -1] \times [0.5, -1]$. The initial states of the agents are given by $q_1(0) = [-2, 2]^T$,
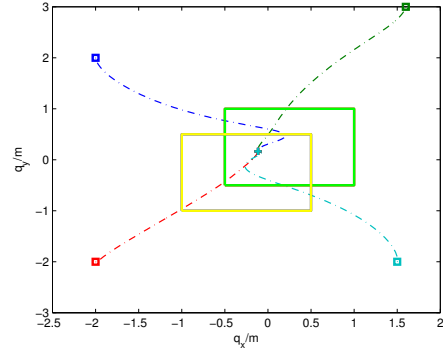

Fig. 1. The trajectories of the generalized coordinates of the agents using (6) for (5) under a fixed communication topology. The squares and the cross denote, respectively, the initial and final generalized coordinates of the agents. The big rectangles denote the available local sets for each agent.

$q_2(0) = [1.6, 3]^T$, $q_3(0) = [-2, -2]^T$, $q_4(0) = [1.5, -2]^T$, $\dot{q}_1(0) = [-0.1, 0.1]^T$, $\dot{q}_2(0) = [0.2, -0.2]^T$, $\dot{q}_3(0) = [0.7, -0.7]^T$, and $\dot{q}_4(0) = [0.4, -0.4]^T$.

For the case of the fixed communication topology, the control parameter is chosen by $k = 1$. The weighted adjacency matrix $\mathcal{A}$ of the generalized coordinates associated with $\mathcal{G}$ is chosen to be $\mathcal{A} = \begin{bmatrix} 0 & 2 & 0 & 4 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 0 \end{bmatrix}$. Using (6) for (5), the trajectories of the generalized coordinates of the agents are shown in Fig. 1. We see that set target aggregation is achieved.

The control parameter is chosen by $k = 5$. The weighted adjacency matrix $\mathcal{A}(t)$ of the generalized coordinates associated with $\mathcal{G}(t)$ switches between $\mathcal{A}_1$ and $\mathcal{A}_2$ at time instants $t_\varepsilon = 5\varepsilon$, $\varepsilon = 0, 1, \ldots$, where $\mathcal{A}_1 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and

$\mathcal{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$. Using (9) for (5), the trajectories of the generalized coordinates of the agents are shown in Fig. 2. We see that set target aggregation is achieved even when the communication topology is switching.

VII. CONCLUSIONS

In this paper, we studied set target aggregation of multiple mechanical systems. Each system is modelled by the Lagrangian dynamics. The objective is to achieve generalized coordinate derivative convergence and generalized coordinate targeted agreement for all the agents. By exchanging information with local neighbors, we proposed set target aggregation algorithms and showed that all the agents converge into the intersection of all the local target sets over both fixed and switching communication topologies. Simulation are given to validate the theoretical results.
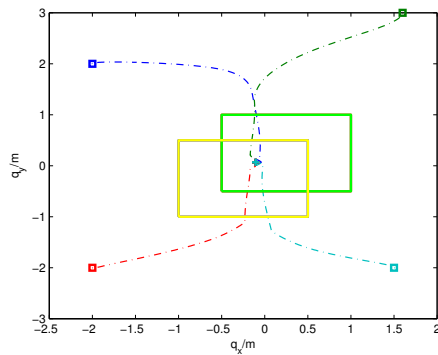
Fig. 2. The trajectories of the generalized coordinates of the agents using (9) for (5) under switching communication topologies. The squares and the cross denote, respectively, the initial and final generalized coordinates of the agents. The big rectangles denote the available local sets for each agent.

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