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Products of Generalized Stochastic Sarymsakov Matrices

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***Abstract*— In the set of stochastic, indecomposable, aperiodic (SIA) matrices, the class of stochastic Sarymsakov matrices is the *largest* known subset (i) that is closed under matrix multiplication and (ii) the infinitely long left-product of the elements from a compact subset converges to a rank-one matrix. In this paper, we show that a larger subset with these two properties can be derived by generalizing the standard definition for Sarymsakov matrices. The generalization is achieved either by introducing an "SIA index", whose value is one for Sarymsakov matrices, and then looking at those stochastic matrices with larger SIA indices, or by considering matrices that are not even SIA. Besides constructing a larger set, we give sufficient conditions for generalized Sarymsakov matrices so that their products converge to rank-one matrices. The new insight gained through studying generalized Sarymsakov matrices and their products has led to a new understanding of the existing results on consensus algorithms and will be helpful for the design of network coordination algorithms.**

I. Introduction

Over the last decade, there has been considerable interest in consensus problems [1]–[8] that are concerned with a group of agents trying to agree on a specific value of some variable. Similar research interest arose decades ago in statistics [9]. While different aspects of consensus processes, such as convergence rates [10], measurement delays [11], stability [5], [12], and controllability [13], have been investigated, and many variants of consensus problems, such as average consensus [14], asynchronous consensus [11], quantized consensus [15], and constrained consensus [16], have been proposed, some fundamental issues of discrete-time linear consensus processes still remain open.

A discrete-time linear consensus process can typically be modeled by a linear recursion equation of the form

$$x(k+1) = P(k)x(k), \ k \geq 1, \tag{1}$$

where $x(k) = [x_1(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ and each $P(k)$ is an $n \times n$ stochastic matrix. It is well known that reaching a consensus for any initial state in this model is equivalent to the product $P(k) \cdots P(2)P(1)$ converging to a rank-one matrix as k goes to infinity. In this context, one fundamental issue is as follows. Given a set of $n \times n$ stochastic matrices $\mathcal{P}$, what are the conditions on $\mathcal{P}$ such that for any infinite sequence of matrices $P(1), P(2), P(3), \ldots$ from $\mathcal{P}$, the sequence of left-products $P(1), P(2)P(1), P(3)P(2)P(1), \ldots$ converges to a rank-one matrix? We will call $\mathcal{P}$ satisfying this property a consensus set (the formal definition will be given in the next section). The existing literature on characterizing a consensus set can be traced back to at least the work of Wolfowitz [17] in which stochastic, indecomposable, aperiodic (SIA) matrices are introduced. Recently, it has been shown in [18] that deciding whether $\mathcal{P}$ is a consensus set is NP-hard; a combinatorial necessary and sufficient condition for deciding a consensus set has also been provided there. Even in the light of these classical and recent findings, the following fundamental question remains: What is the largest subset of the class of $n \times n$ stochastic matrices whose compact subsets are all consensus sets? In [19], this question is answered under the assumption that each stochastic matrix has positive diagonal entries. For general stochastic matrices, the question has remained open. This paper aims at dealing with this challenging question by checking some well-known classes of SIA matrices.

In the literature, the set of stochastic Sarymsakov matrices, first introduced by Sarymsakov [20], is the *largest* known subset of the class of stochastic matrices whose compact subsets are all consensus sets; in particular, the set is closed under matrix multiplication and the left-product of the elements from its compact subset converges to a rank-one matrix [21]. In this paper, we construct a larger set of stochastic matrices whose compact subsets are all consensus sets. We adopt the natural idea which is to generalize the definition of stochastic Sarymsakov matrices so that the original set of stochastic Sarymsakov matrices are contained.

In the paper, we introduce two ways to generalize the definition and thus study two classes of generalized stochastic Sarymsakov matrices. The first class makes use of the concept of the SIA index (the formal definition will be given in the next section). It is shown that the set of $n \times n$ stochastic matrices with SIA index no larger than k is closed under matrix multiplication only when $k = 1$, which turns out to be the stochastic Sarymsakov class; this result reveals why exploring a consensus set larger than the set of stochastic Sarymsakov matrices is a challenging problem. A set that consists of all stochastic Sarymsakov matrices plus one specific SIA matrix and thus is slightly larger than the stochastic Sarymsakov class is constructed, and we show that it is closed under matrix multiplication. For the other class of generalized Sarymsakov matrices, which contains matrices that are not SIA, sufficient conditions are provided for the convergence of the product of an infinite matrix

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sequence from this class to a rank-one matrix. A special case in which all the generalized Sarymsakov matrices are doubly stochastic is also discussed.

The rest of the paper is organized as follows. Preliminaries are introduced in Section II. Section III introduces the SIA index and discusses the properties of the set of stochastic matrices with SIA index no larger than $k,\ k \geq 1$. In Section IV, sufficient conditions are provided for the convergence of the product of an infinite sequence of matrices from a class of generalized stochastic Sarymsakov matrices, and the results are applied to the class of doubly stochastic matrices. Section V concludes the paper. Proofs of some assertions in this paper are omitted due to length limit and can be found in an expanded version of the paper [22].

II. Preliminaries

We first introduce some basic definitions. Let n be a positive integer. A square matrix $P = \{p_{ij}\}_{n\times n}$ is said to be *stochastic* if $p_{ij} \geq 0$ for all $i,\ j \in \{1, \ldots, n\} = \mathcal{N}$, and $\sum_{j=1}^{n} p_{ij} = 1$ for all $i = 1, \ldots, n$. Consider a stochastic matrix P. For a set $\mathcal{A} \subseteq \mathcal{N}$, the set of *one-stage consequent indices* [23] of $\mathcal{A}$ is defined by

$$F_P(\mathcal{A}) = \{j :\ p_{ij} > 0 \text{ for some } i \in \mathcal{A}\}$$

and we call F_P the consequent function of P. For a singleton $\{i\}$, we write $F_P(i)$ instead of $F_P(\{i\})$ for simplicity. A matrix P is indecomposable and aperiodic, and thus called an *SIA* matrix, if $\lim_{m\to\infty} P^m = \mathbf{1}c^T$, where $\mathbf{1}$ is the n-dimensional all-one column vector, and $c = [c_1, \ldots, c_n]^T$ is some column vector satisfying $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. P is said to belong to the *Sarymsakov class* or equivalently P is a *Sarymsakov* matrix if for any two disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset, \tag{2}$$

or

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|, \tag{3}$$

where $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. We say that P is a *scrambling* matrix if for any pair of distinct indices $i, j \in \mathcal{N}$, $F_P(i) \cap F_P(j) \neq \emptyset$, which is equivalent to requiring that there always exists an index $k \in \mathcal{N}$ such that both p_{ik} and p_{jk} are positive.

From the definitions, it should be obvious that a scrambling matrix belongs to the Sarymsakov class. It has been proved in [23] that any product of $n-1$ matrices from the Sarymsakov class is scrambling. Since a stochastic scrambling matrix is SIA [24], any stochastic Sarymsakov matrix must be an SIA matrix.

Definition 1: (*Consensus set*) Let $\mathcal{P}$ be a set of $n \times n$ stochastic matrices. $\mathcal{P}$ is a consensus set if for each sequence of matrices $P(1),\ P(2),\ P(3), \ldots$ from $\mathcal{P}$, $P(k)\cdots P(1)$ converges to a rank-one matrix $\mathbf{1}c^T$ as $k \to \infty$, where $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$.

Deciding whether a set is a consensus set or not is critical in establishing the convergence of the state of system (1) to a common value. Necessary and sufficient conditions for $\mathcal{P}$ being a consensus set have been established [17], [18], [24]–[26].

Theorem 1: [26] Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices. The following conditions are equivalent:

1) $\mathcal{P}$ is a consensus set.
2) For each integer $k \geq 1$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ is SIA.
3) There is an integer $\nu \geq 1$ such that for each $k \geq \nu$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ is scrambling.
4) There is an integer $\mu \geq 1$ such that for each $k \geq \mu$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ has a column with only positive elements.
5) There is an integer $\alpha \geq 1$ such that for each $k \geq \alpha$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ belongs to the Sarymsakov class.

For a compact set $\mathcal{P}$ to be a consensus set, it is necessary that every matrix in $\mathcal{P}$ is SIA in view of item (2) in Theorem 1. If a set of SIA matrices is closed under matrix multiplication, then one can easily conclude from item (2) that its compact subsets are all consensus sets. However, it is well known that the product of two SIA matrices may not be SIA. The stochastic Sarymsakov class is the largest known set of stochastic matrices, which is closed under matrix multiplication. Whether there exists a larger class of SIA matrices, which contains the Sarymsakov class as a proper subset and is closed under matrix multiplication, remains unknown. We will explore this by taking a closer look at the definition of the Sarymsakov class and study the properties of classes of generalized Sarymsakov matrices that contain the Sarymsakov class as a subset.

III. SIA index

The key notion in the definition of the Sarymsakov class is the set of one-stage consequent indices. We next introduce the notion of the set of *k-stage consequent indices* and utilize this to define a larger matrix set, which contains the Sarymsakov class.

For a stochastic matrix P and a set $\mathcal{A} \subseteq \mathcal{N}$, let $F_P^k(\mathcal{A})$ be the set of *k-stage consequent indices* of $\mathcal{A}$, which is defined by

$$F_P^1(\mathcal{A}) = F_P(\mathcal{A}) \text{ and } F_P^k(\mathcal{A}) = F_P(F_P^{k-1}(\mathcal{A})),\ k \geq 2.$$

Definition 2: A stochastic matrix P is said to belong to the class $\mathcal{S}$ if for any two disjoint nonempty subsets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an integer $k \geq 1$ such that either

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset, \tag{4}$$

or

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P^k(\mathcal{A}) \cup F_P^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{5}$$

It is easy to see that the Sarymsakov class is a subset of $\mathcal{S}$ since k is 1 in the definition of the Sarymsakov class. An important property of the consequent function F_P given below will be useful.

Lemma 1: [21] Let P and Q be $n \times n$ nonnegative matrices. Then, $F_{PQ}(\mathcal{A}) = F_Q(F_P(\mathcal{A}))$ for all subsets $\mathcal{A} \subseteq \mathcal{N}$.

A direct consequence of Lemma 1 is that $F_{P^k}(\mathcal{A}) = F_P^k(\mathcal{A})$ for any stochastic matrix P, any integer $k \geq 1$ and any subset $\mathcal{A} \subseteq \mathcal{N}$.

The following theorem establishes the relationship between the matrices in $\mathcal{S}$ and SIA matrices.

Theorem 2: [26] A stochastic matrix P is in $\mathcal{S}$ if and only if P is SIA.

From Theorem 4.4 in [27], we know the following result.

Theorem 3: [27] A stochastic matrix P is SIA if and only if for every pair of indices i and j, there exists an integer k, $k \leq n(n-1)/2$, such that $F_P^k(i) \cap F_P^k(j) \neq \emptyset$.

Theorem 3 implies that the index k in (4) and (5) can be bounded by some integer.

Lemma 2: A stochastic matrix P is SIA if and only if for any pair of disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an index k, $k \leq n(n-1)/2$, such that $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset$.

Example 1: Let

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

P is a stochastic matrix. Consider two disjoint nonempty sets $\mathcal{A} = \{2\}, \tilde{\mathcal{A}} = \{3\}$. One knows that $F_P(\mathcal{A}) = \{1\}$ and $F_P(\tilde{\mathcal{A}}) = \{2\}$, implying that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$ and $|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Therefore, P is not a Sarymsakov matrix. However, the fact that $F_P^2(\mathcal{A}) = \{1, 2, 3\}$ and $F_P^2(\tilde{\mathcal{A}}) = \{2\}$ shows that $F_P^2(\mathcal{A}) \cap F_P^2(\tilde{\mathcal{A}}) \neq \emptyset$. This means (4) holds for $k = 2$.

For every other pair of disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, it can be verified that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$. One has that though P is not a Sarymsakov matrix, P is an SIA matrix from Lemma 2. □

From the above example and Lemma 2, one knows that the class of SIA matrices may contain a large number of matrices that do not belong to the Sarymsakov class. Starting from the Sarymsakov class, where $k = 1$ in (4) and (5), we relax the constraint on the value of the index k in (4) and (5), i.e., allowing for $k \leq 2$, $k \leq 3, \ldots,$ and obtain a larger set containing the Sarymsakov class. We formalize the idea below and study whether the derived set is closed under matrix multiplication or not.

Consider a fixed integer n. Denote all the unordered pairs of disjoint nonempty sets of $\mathcal{N}$ as $(\mathcal{A}_1,\ \tilde{\mathcal{A}}_1), \ldots, (\mathcal{A}_m,\ \tilde{\mathcal{A}}_m)$, where m is a finite number.

Definition 3: Let $P \in \mathbb{R}^{n \times n}$ be an SIA matrix. For each pair of disjoint nonempty sets $\mathcal{A}_i,\ \tilde{\mathcal{A}}_i \subseteq \mathcal{N},\ i = 1, \ldots, m$, let s_i be the smallest integer such that either (4) or (5) holds. The *SIA index* s of P is $s = \max\{s_1, s_2, \ldots, s_m\}$.

From Lemma 2, we know that for an SIA matrix P, its SIA index s is upper bounded by $n(n-1)/2$. Assume that the largest value of the SIA indices of all the $n \times n$ SIA matrices is l, which depends on the order n. We define several subsets of the class of SIA matrices. For $1 \leq k \leq l$, let

$$\mathcal{V}_k = \{P \in \mathbb{R}^{n \times n} | P \text{ is SIA and its SIA index is } k\} \tag{7}$$

and

$$\mathcal{S}_k = \cup_{r=1}^k \mathcal{V}_r. \tag{8}$$

Obviously $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_l$ and $\mathcal{S}_1 = \mathcal{V}_1$ is the class of stochastic Sarymsakov matrices. One can easily check that when $n = 2$, all SIA matrices are scrambling matrices and hence belong to the Sarymsakov class. When $n \geq 3$, the set $\mathcal{V}_{n-1}$ is nonempty. To see this, consider an $n \times n$ stochastic matrix

$$P = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For an index $i \in \mathcal{N},\ i \neq n$, it is easy to check that $F_P^{n-1}(i) = \mathcal{N}$. Hence, for any two nonempty disjoint sets $\mathcal{A}, \tilde{\mathcal{A}} \in \mathcal{N}$, it must be true that $F_P^{n-1}(\mathcal{A}) \cap F_P^{n-1}(\tilde{\mathcal{A}}) \neq \emptyset$, proving that P is an SIA matrix. Consider the specific pair of sets $\mathcal{A} = \{n\}, \tilde{\mathcal{A}} = \{n-1\}$. One has that $F_P^{n-2}(n) = \{2\}$, $F_P^{n-2}(n-1) = \{1\}$, and $F_P^{n-1}(n) \cap F_P^{n-1}(n-1) \neq \emptyset$, implying that $P \in \mathcal{V}_{n-1}$. From this example, we know that a lower bound for l is $n - 1$.

In the next three subsections, we first discuss the properties of $\mathcal{S}_i,\ i = 1, \ldots, l$, then construct a set, closed under matrix multiplication, consisting of a specific SIA matrix and all stochastic Sarymsakov matrices, and finally discuss the class of pattern-symmetric matrices.

A. Properties of $\mathcal{S}_i$

The following novel result reveals the properties of the sets $\mathcal{S}_i,\ 1 \leq i \leq l$.

Theorem 4: Suppose that $n \geq 3$. Among the sets $\mathcal{S}_1,\ \mathcal{S}_2, \ldots,\ \mathcal{S}_l$, the set $\mathcal{S}_1$ is the only set that is closed under matrix multiplication.

Note that a compact subset $\mathcal{P}$ of $\mathcal{S}_1$ is a consensus set. However, if $\mathcal{P}$ is a compact set consisting of matrices in $\mathcal{V}_i,\ i \geq 2$, Theorem 4 shows that there is no guarantee that $\mathcal{P}$ is still a consensus set.

The proof of Theorem 4 relies on the following key lemma, based on which the conclusion of Theorem 4 immediately follows. Before stating the lemma, we define a matrix Q in terms of a matrix $P \in \mathcal{V}_i,\ i \geq 2$.

For a given matrix $P \in \mathcal{V}_i,\ i \geq 2$, from the definition of the Sarymsakov class, one has that there exist two disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset$ and

$$|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \leq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{9}$$

Define a matrix $Q = (q_{ij})_{n \times n}$ as follows

$$q_{ij} = \begin{cases} \frac{1}{|\mathcal{A}|}, & i \in F_P(\mathcal{A}),\ j \in \mathcal{A}, \\ 0, & i \in F_P(\mathcal{A}),\ j \in \bar{\mathcal{A}}, \\ \frac{1}{|\tilde{\mathcal{A}}|}, & i \in F_P(\tilde{\mathcal{A}}),\ j \in \tilde{\mathcal{A}}, \\ 0, & i \in F_P(\tilde{\mathcal{A}}),\ j \in \bar{\tilde{\mathcal{A}}}, \\ \frac{1}{n}, & \text{otherwise}, \end{cases} \tag{10}$$

where $\bar{\mathcal{A}}$ denotes the complement of $\mathcal{A}$ with respect to $\mathcal{N}$.

Lemma 3: Suppose that $n \geq 3$. For any $i = 2, \ldots, l$, given a stochastic matrix $P \in \mathcal{V}_i$, then the matrix Q given in (10) belongs to the set $\mathcal{S}_2$ and PQ, QP are not SIA. In addition, $Q \in \mathcal{V}_1$ if (9) holds with the equality sign; $Q \in \mathcal{V}_2$ if the inequality (9) is strict.

Remark 1: Note that whether a stochastic matrix is SIA or not only depends on the positions of its nonzero elements but not their magnitudes. One can derive other matrices based on Q in (10) such that PQ is not SIA by varying the magnitudes of the positive elements of Q as long as each row sum equals to 1 and the positive elements are kept positive.

There has been research work on defining another class of matrices that is a subset of the SIA matrices and larger than the stochastic scrambling matrices. We establish its relationship with the stochastic Sarymsakov class in view of Lemma 3.

Definition 4: [24] $P \in \mathcal{G}$, if P is SIA and for any SIA matrix Q, QP is SIA.

Proposition 1: For $n \geq 3$, $\mathcal{G}$ is a proper subset of the class of stochastic Sarymsakov matrices $\mathcal{S}_1$.

B. A set closed under matrix multiplication

In this subsection, we construct a subset of $\mathcal{S}$, which is closed under matrix multiplication. This subset consists of the set $\mathcal{S}_1$ and one specific matrix in $\mathcal{V}_2$.

Let R be a matrix in $\mathcal{V}_2$ and satisfies that for any disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) \neq \emptyset, \tag{11}$$

or

$$F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| \geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{12}$$

Such a matrix exists. An example is

$$R = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}. \tag{13}$$

To verify that R satisfies this condition, we only have to consider the pair of sets $\mathcal{A} = \{2\},\ \tilde{\mathcal{A}} = \{3\}$ since for other pairs of $\mathcal{A},\ \tilde{\mathcal{A}}$, $F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) \neq \emptyset$. It is clear that $|F_R(2) \cup F_R(3)| = |\{1, 2\}| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$ and $F_R^2(2) \cap F_R^2(3) = \{1\}$.

Theorem 5: Suppose that R is a matrix in $\mathcal{V}_2$ and satisfies that for any disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, (11) or (12) holds. Then, the set $\mathcal{S}_1' = \mathcal{S}_1 \cup \{R\}$ is closed under matrix multiplication and a compact subset of $\mathcal{S}_1'$ is a consensus set.

Proof: Let P be a matrix in $\mathcal{S}_1$. We first show that $RP, PR \in \mathcal{S}_1$. Given two disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, assume that $F_{RP}(\mathcal{A}) \cap F_{RP}(\tilde{\mathcal{A}}) = \emptyset$. Since $F_{RP}(\mathcal{A}) = F_P(F_R(\mathcal{A}))$ and $F_{RP}(\tilde{\mathcal{A}}) = F_P(F_R(\tilde{\mathcal{A}}))$ based on Lemma 1, one has that $F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) = \emptyset$. In view of the fact that P is a Sarymsakov matrix, one has

$$\begin{aligned} |F_{RP}(\mathcal{A}) \cup F_{RP}(\tilde{\mathcal{A}})| &= |F_P(F_R(\mathcal{A})) \cup F_P(F_R(\tilde{\mathcal{A}}))| \\ &> |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| \\ &\geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \end{aligned}$$

It follows that RP is a Sarymsakov matrix. Consider the matrix PR. Suppose that $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$ are two disjoint nonempty sets satisfying that $F_{PR}(\mathcal{A}) \cap F_{PR}(\tilde{\mathcal{A}}) = \emptyset$. One similarly derives that

$$\begin{aligned} |F_{PR}(\mathcal{A}) \cup F_{PR}(\tilde{\mathcal{A}})| &= |F_R(F_P(\mathcal{A})) \cup F_R(F_P(\tilde{\mathcal{A}}))| \\ &\geq |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \\ &> |\mathcal{A} \cup \tilde{\mathcal{A}}|. \end{aligned}$$

Therefore, PR is a Sarymsakov matrix.

We next show that $R^2 \in \mathcal{S}_1$. Since $R \in \mathcal{V}_2$, for any disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an integer $k \leq 2$ such that either

$$F_R^k(\mathcal{A}) \cap F_R^k(\tilde{\mathcal{A}}) \neq \emptyset \tag{14}$$

or

$$F_R^k(\mathcal{A}) \cap F_R^k(\tilde{\mathcal{A}}) = \emptyset, \text{ and } |F_R^k(\mathcal{A}) \cup F_R^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{15}$$

When (14) holds, it follows from Lemma 1 that $F_{R^2}(\mathcal{A}) \cap F_{R^2}(\tilde{\mathcal{A}}) \neq \emptyset$. When (15) holds, suppose that $F_{R^2}(\mathcal{A}) \cap F_{R^2}(\tilde{\mathcal{A}}) = \emptyset$. If (15) holds for $k = 1$, then from the assumption on R, we have

$$|F_{R^2}(\mathcal{A}) \cup F_{R^2}(\tilde{\mathcal{A}})| \geq |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|;$$

if (15) holds for $k = 2$, then we immediately have that $|F_{R^2}(\mathcal{A}) \cup F_{R^2}(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Hence, $R^2 \in \mathcal{S}_1$.

Recall that the product of matrices in $\mathcal{S}_1$ still lies in $\mathcal{S}_1$. It is clear that P_2P_1 is a Sarymsakov matrix for $P_1, P_2 \in \mathcal{S}_1'$. By induction $P_k \cdots P_2P_1 \in \mathcal{S}_1$ for $P_i \in \mathcal{S}_1',\ i = 1, \ldots, k$, and any integer $k \geq 2$, implying that $\mathcal{S}_1'$ is closed under matrix multiplication. Then, it immediately follows from Theorem 1 (5) that a compact subset of $\mathcal{S}_1'$ is a consensus set. □

For a set consisting of the Sarymsakov class and two or more matrices which belong to $\mathcal{V}_2$ and satisfy that for any disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, (11) or (12) holds, whether it is closed under matrix multiplication depends on those specific matrices in $\mathcal{V}_2$.

C. Pattern-symmetric matrices

In this subsection, we discuss the SIA index of a class of $n \times n$ stochastic matrices, each element P of which satisfies the following pattern-symmetric condition

$$p_{ij} > 0 \Leftrightarrow p_{ji} > 0, \text{for } i \neq j. \tag{16}$$

System (1) with bidirectional interactions between agents induces a system matrix satisfying (16), which arises often in the literature.

We present the following lemma regarding the property of a matrix satisfying (16).

Proposition 2: Suppose that P satisfies (16) and is an SIA matrix. Then,

1) $P \in \mathcal{S}_2$;
2) if P is symmetric, then $P \in \mathcal{S}_1$.

For an SIA and nonsymmetric matrix P satisfying (16), P is not necessarily a Sarymsakov matrix. An example of such a P is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

P is not a Sarymsakov matrix, but $P \in \mathcal{S}_2$.

With the knowledge of the SIA index, the condition for a consensus set of stochastic symmetric matrices in the literature can be derived directly. It has been established in Example 7 in [18] that a compact set $\mathcal{P}$ of stochastic symmetric matrices is a consensus set if and only if P is an SIA matrix for every $P \in \mathcal{P}$. The necessity part holds for any consensus set. As we know from Proposition 2, a stochastic symmetric matrix P is SIA if and only if P is a Sarymsakov matrix. The sufficient part becomes clear as the Sarymsakov class is closed under matrix multiplication.

The above claim for stochastic symmetric matrices cannot be extended to stochastic matrices that satisfy (16). The reason is that a stochastic matrix satisfying (16), is not necessarily a Sarymsakov matrix. Hence, in view of Theorem 4, the product of two such matrices may not be SIA anymore. An example to illustrate this is a set $\mathcal{P}$ consisting of two matrices

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

P_1, P_2 both satisfy (16). However, the matrix product $(P_1P_2)^k$ does not converge to a rank-one matrix as $k \to \infty$.

IV. A CLASS OF GENERALIZED SARYMSAKOV MATRICES

We have seen in Theorem 5 that the Sarymsakov class plus one specific SIA matrix can lead to a closed set under matrix multiplication that contains $\mathcal{S}_1$. The property (12) of the matrix R turns out to be critical and we next consider a class of matrices containing all such matrices.

Definition 5: A stochastic matrix P is said to belong to a set $\mathcal{W}$ if for any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either (11) or (12) with R replaced by P holds.

The definition of the set $\mathcal{W}$ relaxes that of the Sarymsakov class by allowing the inequality in (3) not to be strict. It is obvious that $\mathcal{S}_1$ is a subset of $\mathcal{W}$. In addition, $\mathcal{W}$ is a set that is closed under matrix multiplication. To see this, we show $PQ \in \mathcal{W}$ for $P, Q \in \mathcal{W}$. For any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, suppose that $F_{PQ}(\mathcal{A}) \cap F_{PQ}(\tilde{\mathcal{A}}) = \emptyset$. It follows from (12) that

$$\begin{aligned} |F_{PQ}(\mathcal{A}) \cup F_{PQ}(\tilde{\mathcal{A}})| &= |F_Q(F_P(\mathcal{A})) \cup F_Q(F_P(\tilde{\mathcal{A}}))| \\ &\geq |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \\ &\geq |\mathcal{A} \cup \tilde{\mathcal{A}}|, \end{aligned}$$

which implies that $PQ \in \mathcal{W}$.

Compared with the definition of $\mathcal{S}_1$, the subtle difference in the inequality in (12) drastically changes the property of $\mathcal{W}$. A matrix in $\mathcal{W}$ is not necessarily SIA. For example, permutation matrices belong to the class $\mathcal{W}$ since for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$,

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|. \quad (17)$$

One may expect that the set $\mathcal{W} \cap \mathcal{S}$ is closed under matrix multiplication. However, the claim is false and an example to show this is the following two SIA matrices

$$P_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

where

$$P_1P_2 = \begin{bmatrix} + & + & + \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is not SIA anymore.

Instead of looking at whether a subset of $\mathcal{W}$ is a consensus set which concerns the convergence of the matrix product formed by an arbitrary matrix sequence from the subset, we explore the sufficient condition for the convergence of the matrix product of the elements from $\mathcal{W}$ and its application to doubly stochastic matrices.

A. Sufficient conditions for consensus

Theorem 6: Let $\mathcal{P}$ be a compact subset of $\mathcal{W}$ and let $P(1), P(2), P(3), \ldots$ be a sequence of matrices from $\mathcal{P}$. Suppose that $j_1, j_2, \ldots$ is an infinite increasing sequence of the indices such that $P(j_1), P(j_2), \ldots$ are Sarymsakov matrices and $\cup_{r=1}^{\infty}\{P(j_r)\}$ is a compact set. Then, $P(k) \cdots P(1)$ converges to a rank-one matrix as $k \to \infty$ if there exists an integer T such that $j_{r+1} - j_r \leq T$ for all $r \geq 1$.

Remark 2: In Theorem 6, if $T_r = j_{r+1} - j_r, \; r \geq 1$ is not uniformly upper bounded, we may not be able to draw the conclusion. The reason is that the compactness condition of Theorem 1 may not be satisfied so that Theorem 1 does not apply.

When the set $\mathcal{P}$ is a finite set, we have the following corollary.

Corollary 1: Let $\mathcal{P}$ be a finite subset of $\mathcal{W}$ and let $P(1), P(2), P(3), \ldots$ be a sequence of matrices from $\mathcal{P}$. Suppose that $j_1, j_2, \ldots$ is an infinite increasing sequence of the indices such that $P(j_1), P(j_2), \ldots$ are Sarymsakov matrices. Then, $P(k) \cdots P(1)$ converges to a rank-one matrix as $k \to \infty$ if there exists an integer T such that $j_{r+1} - j_r \leq T$ for all $r \geq 1$.

B. Applications to doubly stochastic matrices

In fact, the set of matrices $\mathcal{W}$ contains all doubly stochastic matrices. We can establish the following property of doubly stochastic matrices using the Birkhoff–von Neumann theorem [28].

Lemma 4: Let P be a doubly stochastic matrix. For any nonempty set $\mathcal{A} \subseteq \mathcal{N}$, $|F_P(\mathcal{A})| \geq |\mathcal{A}|$.

Proof: From the Birkhoff–von Neumann theorem [28], P is doubly stochastic if and only if P is a convex combination of permutation matrices, i.e., $P = \sum_{i=1}^{n!} \alpha_i P_i$ where $\sum_{i=1}^{n!} \alpha_i = 1$, $a_i \geq 0$ for $i = 1, \ldots, n!$ and P_i are permutation matrices. For any permutation matrix P_i, it is obvious that $|F_{P_i}(\mathcal{A})| = |\mathcal{A}|$ for any set $\mathcal{A} \subseteq \mathcal{N}$. In view of the Birkhoff–von Neumann theorem, it holds that

$$F_P(\mathcal{A}) = \cup_{\alpha_i \neq 0} F_{P_i}(\mathcal{A}).$$

It then immediately follows that $|F_P(\mathcal{A})| \geq |\mathcal{A}|$. □

From the above lemma, it is easy to see that for a doubly stochastic matrix P, either (11) or (12) holds. Hence, doubly stochastic matrices belong to the set $\mathcal{W}$. The following lemma reveals when a doubly stochastic matrix is a Sarymsakov matrix.

Lemma 5: Let P be a doubly stochastic matrix. P is a Sarymsakov matrix if and only if for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, $|F_P(\mathcal{A})| > |\mathcal{A}|$.

Lemma 5 provides a condition to decide whether a doubly stochastic matrix belongs to $\mathcal{S}_1$ or not. We have the following corollary based on Theorem 6.

Corollary 2: Let $\mathcal{P}$ be a compact set of doubly stochastic matrices and let $P(1),\ P(2),\ P(3),\ldots$ be a sequence of matrices from $\mathcal{P}$. Suppose that $j_1, j_2, \ldots$ is an infinite increasing sequence of the indices such that $P(j_1), P(j_2), \ldots$ are Sarymsakov matrices and $\cup_{r=1}^{\infty}\{P(j_r)\}$ is a compact set. Then, $P(k)\cdots P(1)$ converges to a rank-one matrix as $k \to \infty$ if there exists an integer T such that $j_{r+1} - j_r \leq T$ for all $r \geq 1$.

For doubly stochastic matrices with positive diagonals, a sharp statement can be made.

Proposition 3: Let P be a doubly stochastic matrix with positive diagonals. If P is SIA, then $P \in \mathcal{S}_1$.

For doubly stochastic matrices satisfying condition (16), we have a similar result.

Proposition 4: Let P be a doubly stochastic matrix satisfying condition (16). If P is SIA, then $P \in \mathcal{S}_1$.

V. Conclusion

In this paper, we have discussed products of generalized stochastic Sarymsakov matrices. With the notion of SIA index, we have shown that the set of all SIA matrices with SIA index no larger than k is closed under matrix multiplication only when $k = 1$. Sufficient conditions for the convergence of the matrix product of an infinite matrix sequence to a rank-one matrix have been provided with the help of the Sarymsakov matrices. The results obtained underscore the critical role of the stochastic Sarymsakov class in the set of SIA matrices and in constructing a convergent matrix sequence to consensus. Construction of a larger set than the one constructed in the paper which is closed under matrix multiplication is a subject for future research.

References

[1] J. N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation*. PhD thesis, Department of Electrical Engineering and Computer Science, MIT, 1984.

[2] J. Tsitsiklis, D. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986.

[3] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):985–1001, 2003.

[4] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[5] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[6] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[7] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47(2):575–600, 2008.

[8] B. Touri and A. Nedić. Product of random stochastic matrices. *IEEE Transactions on Automatic Control*, 59(2):437–448, 2014.

[9] M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[10] A. Olshevsky and J. T. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48(1):33–55, 2009.

[11] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: Convergence rates, measurement delays and asynchronous events. *SIAM Journal on Control and Optimization*, 47(2):601–623, 2008.

[12] J. Liu, A. S. Morse, A. Nedić, and T. Başar. Internal stability of linear consensus processes. In *Proc. of the 53th IEEE Conference on Decision and Control*, pages 922–927, 2014.

[13] M. Egerstedt, S. Martini, M. Cao, M. K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus networks? *Control Systems Magazine*, 32(4):66–73, 2012.

[14] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione. Gossip algorithms for distributed signal processing. *Proceedings of the IEEE*, 98(11):1847–1864, 2010.

[15] A. Kashyap, T. Basar, and R. Srikant. Quantized consensus. *Automatica*, 43(7):1192–1203, 2007.

[16] A. Nedić, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–937, 2010.

[17] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 14:733–737, 1963.

[18] V. D. Blondel and A. Olshevsky. How to decide consensus: A combinatorial necessary and sufficient condition and a proof that consensus is decidable but NP-hard. *SIAM Journal on Control and Optimization*, 52(5):2707–2726, 2014.

[19] J. Liu, A. S. Morse, B. D. O. Anderson, and C. Yu. Contractions for consensus processes. In *Proc. of the 50th IEEE Conference on Decision and Control*, pages 1974–1979, 2011.

[20] T. A. Sarymsakov. Inhomogeneous Markov chains (in russian). *Teor. Verojatnost. i Primen.*, 6:194–201, 1961.

[21] D. J. Hartfiel. *Nonhomogeneous Matrix Products*. World Scientific, Singapore, 2002.

[22] W. Xia, J. Liu, M. Cao, K. H. Johansson, and T. Başar. Products of generalized stochastic Sarymsakov matrices. arXiv:1509.03900 [math.OC], 2015.

[23] E. Seneta. Coefficients of ergodicity: Structure and applications. *Advances in Applied Probability*, 11(3):576–590, 1979.

[24] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer-Verlag, 2006. 2nd edition.

[25] J. M. Anthonisse and H. Tijms. Exponential convergence of products of stochastic matrices. *Journal of Mathematical Analysis and Applications*, 59(2):360–364, 1977.

[26] W. Xia and M. Cao. Sarymsakov matrices and asynchronous implementation of distributed coordination algorithms. *IEEE Transcations on Automatic Control*, 59(8):2228–2233, 2014.

[27] A. Paz. *Introduction to Probabilistic Automata*. Academic Press, 1971.

[28] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, U.K., 1985.