Balance Conditions in Discrete-Time Consensus Algorithms

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***Abstract*— We study the consensus problem of discrete-time systems under persistent flow and non-reciprocal interactions between agents. An arc describing the interaction strength between two agents is said to be persistent if its weight function has an infinite l_1 norm. We discuss two balance conditions on the interactions between agents which generalize the arc-balance and cut-balance conditions in the literature respectively. The proposed conditions require that such a balance should be satisfied over each time window of a fixed length instead of at each time instant. We prove that in both cases global consensus is reached if and only if the persistent graph, which consists of all the persistent arcs, contains a directed spanning tree. The convergence rates are also provided in terms of the number of node interactions that have taken place.**

I. Introduction

A. Background

In distributed coordination of multi-agent systems, a great deal of attention has been paid to consensus-seeking systems. The study of this type of systems is motivated by opinion forming in social networks [1], [2], flocking behaviors in animal groups [3], [4], data fusion in engineered systems [5] and so on. Ample results on the convergence and convergence rate of the consensus system have been reported. Typical conditions involve the connectivity of the network topology and the interaction strengths between agents for both continuous-time [6]–[12] and discrete-time systems [7], [8], [11], [13]–[17].

In the literature, several types of balance conditions on the interaction weights are considered, among which the cut-balance condition [9], [10] and the arc-balance condition [11] are typical ones. The cut-balance condition requires that at each time instant, if a group of agents in the network influences the remaining ones then it is also influenced by the remaining ones bounded by a constant proportional amount. This type of conditions characterizes a reciprocal interaction relationship among the agents, which covers the symmetric interaction and type-symmetric interaction as special cases [10]. The convergence of the system with the balanced asymmetry property, a stronger notion than the cut-balance condition, is proved under the absolute infinite flow property [18] for deterministic iterations [12].

The arc-balance condition requires that in the persistent graph the weight of each arc is bounded by a proportional amount of the weight of any other arc at each time instant. Under this condition, it was proved that the multi-agent system reaches consensus under the condition that the persistent graph contains a directed spanning tree [11]. This persistent graph property behaves as forms of network Borel-Cantelli lemmas for consensus algorithms over random graphs [19]. If the persistent graph is strongly connected, the arc balance assumption is a special case of the cut-balance condition imposed on the persistent graph, while in the general case, these two conditions do not cover each other.

B. The Algorithm

Consider a network with the node set $\mathcal{V} = \{1, \dots, N\}$, $N \geq 2$. Each node i holds a state $x_i(t) \in \mathbb{R}$. The initial time is $t_0 \geq 0$. The evolution of $x_i(t)$ is given by

$$x_i(t+1) = \sum_{j=1}^{N} a_{ij}(t) x_j(t), \tag{1}$$

where $a_{ij}(t) \geq 0$ stands for the influence of node j on node i at time t and $a_{ii}(t)$ represents the self-confidence of each node. If $a_{ij}(t) > 0$ at time t, then it is considered as the weight of arc (j,i) of the graph $\mathbb{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$.

For the time-varying arc weights $a_{ij}(t)$, we impose the following condition as our standing assumption throughout the paper.

Assumption 1: For all $i, j \in \mathcal{V}$ and $t \geq 0$, (i) $a_{ij}(t) \geq 0$; (ii) $\sum_{j=1}^{N} a_{ij}(t) = 1$; (iii) There exists a constant $0 < \eta < 1$ such that $a_{ii}(t) \geq \eta$.

Denote $x(t) = [x_1(t), \dots, x_N(t)]^T$ and $A(t) = [a_{ij}(t)]_{N \times N}$. We know that $A(t)$ is a stochastic matrix from Assumption 1. System (1) can be rewritten as

$$x(t+1) = A(t)x(t). \tag{2}$$

We continue to introduce the following definition [11].

Definition 1: An arc (j,i) is called a persistent arc if

$$\sum_{t=0}^{\infty} a_{ij}(t) = \infty. \tag{3}$$

The set of all persistent arcs is denoted as $\mathcal{E}_p$ and we call the digraph $\mathbb{G}_p = (\mathcal{V}, \mathcal{E}_p)$ the persistent graph.

The weight function of each arc in the persistent graph has an infinite l_1 norm as can be seen from (3). The notions of

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persistent arcs and persistent graph have also been considered in [9], [10], [12], [20] for studying the consensus problem of discrete-time and continuous-time systems. In [12] the persistent graph $\mathbb{G}_p$ is called an unbounded interactions graph. We will show in the next section that the connectivity of the persistent graph is fundamental for deciding consensus, while those edges whose time-varying interaction weights summing up to a finite number is not critical. The consensus problem considered in this paper is defined as follows.

Definition 2: Global consensus is achieved for the considered network if for any initial time $t_0 \geq 0$, and for any initial value $x(t_0)$, there exists $x_* \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = x_*$ for all $i \in \mathcal{V}$.

In addition, we not only derive conditions under which global consensus can be reached, but also characterize the convergence speed in terms of how much interaction among the nodes has happened in the network.

C. Generalized Balance Conditions

A central aim of this paper is to derive conditions under which the convergence to consensus of system (1) can be guaranteed by imposing merely the connectivity of the persistent graph. In this case some balance conditions among the arc weights become essential [10], [11]. We introduce the following two balance conditions.

Assumption 2: (Balance Condition I) There exist an integer $L \geq 1$ and a constant $K \geq 1$ such that for any $(j,i),(l,k) \in \mathcal{E}_p$, we have

$$\sum_{t=s}^{s+L-1} a_{kl}(t) \leq K \sum_{t=s}^{s+L-1} a_{ij}(t) \tag{4}$$

for all $s \geq 0$.

Assumption 3: (Balance Condition II) There exist an integer $L \geq 1$ and a constant $K \geq 1$ such that for any nonempty proper subset S of $\mathcal{V}$, we have

$$\sum_{t=s}^{s+L-1} \sum_{i\notin S, j\in S} a_{ij}(t) \leq K \sum_{t=s}^{s+L-1} \sum_{i\in S, j\notin S} a_{ij}(t) \tag{5}$$

for all $s \geq 0$.

Remark 1: The Balance Condition I is a generalized version of the arc-balance condition introduced in [11] where $L = 1$. The Balance Condition II is a generalized version of the cut-balance condition introduced in [10] where $L = 1$. These conditions require either the balance between the weights of different persistent arcs or the balance between the amount of interactions between one group and its remaining part over each time window of a fixed length. When Assumption 2 or Assumption 3 holds for $L = 1$, (4) or (5) imposes a restriction on such a balance condition which should be satisfied instantaneously. A relatively large L gives more flexibility on the interaction weights and allows possible non-instantaneous reciprocal interactions between agents. □

D. Main results

In this subsection, we first give some basic observations of the state evolution of system (1) and then present the main results.

Let $H(t) \doteq \max_{i\in\mathcal{V}}\{x_i(t)\}$, $h(t) \doteq \min_{i\in\mathcal{V}}\{x_i(t)\}$ be the maximum and minimum state value at time t, respectively. Denote $\Psi\big(t\big) \doteq H(t) - h(t)$ which serves as a metric of consensus. Note that $\Psi\big(t\big)$ measures the maximum difference among the states of the nodes.

Apparently reaching a consensus of system (1) implies that $\lim_{t\to\infty} \Psi\big(t\big) = 0$. In fact the contrary is also true. It is straightforward to see that $H(t)$ is non-increasing, $h(t)$ is non-decreasing and thus $\Psi(t)$ is non-increasing. Therefore, for any initial time $t_0 \geq 0$ and any initial value $x(t_0)$, there exist $H_*, h_* \in \mathbb{R}$ such that $\lim_{t\to\infty} H(t) = H_*$; $\lim_{t\to\infty} h(t) = h_*$. If $\lim_{t\to\infty} \Psi\big(t\big) = 0$, we obtain $H_* = h_*$, which implies that $\lim_{t\to\infty} x_i(t) = H_*$ for all $i \in \mathcal{V}$.

Let $\lceil a \rceil$ represent the smallest integer that is no less than a, and $\lfloor a \rfloor$ represent the largest integer that is no greater than a. We present the following two main results, for the two types of balance conditions, respectively.

Theorem 1: Assume that Assumptions 1 and 2 hold.
(i) Global consensus is achieved for system (1) if and only if the persistent graph $\mathbb{G}_p$ has a directed spanning tree.
(ii) If the persistent graph $\mathbb{G}_p$ has a directed spanning tree, then for any initial time $t_0 \geq 0$, $\epsilon > 0$, and $\varepsilon > 0$, we have

$$\Psi(t) \leq \epsilon\Psi(t_0), \qquad \text{for all } t \geq T_\varepsilon + t^*, \tag{6}$$

where $T_\varepsilon \geq t_0$ such that $\sum_{t=T_\varepsilon}^{\infty} a_{ij}(t) \leq \varepsilon$ for all $(j,i) \in \mathcal{E} \setminus \mathcal{E}_p$,

$$t^* \doteq \inf\Bigg\{ t \geq 1 : \sum_{k=0}^{t-1} \sum_{j=1, j\neq i, (j,i)\in\mathcal{E}_p}^{N} a_{ij}(T_\varepsilon + k) \geq \omega_1 d_0(\delta + 1) \Bigg\}, \tag{7}$$

$\delta > L(N-1)(1-\eta)$ is a constant, d_0 is the diameter of $\mathbb{G}_p$, $\omega_1 \doteq \left\lceil \frac{\log \epsilon^{-1}}{\log\left(1-\frac{1}{2}\mathbf{Q}^{2d_0}\mathbf{R}^{d_0}\right)^{-1}} \right\rceil$ with $\mathbf{R} \doteq K^{-1}\left[\frac{\delta}{N-1} - L(1-\eta)\right]$, $\mathbf{Q} \doteq e^{-\frac{(N-1)(K(1-\eta+\delta)+L(1-\eta)+\varepsilon)\ln\eta}{\eta-1}}$.

Theorem 2: Assume that Assumptions 1 and 3 hold.
(i) Global consensus is achieved for system (1) if and only if the persistent graph $\mathbb{G}_p$ has a directed spanning tree.
(ii) If the persistent graph $\mathbb{G}_p$ has a directed spanning tree, then for any initial time $t_0 \geq 0$ and $\epsilon > 0$, we have

$$\Psi(t) \leq \epsilon\Psi(t_0), \qquad \text{for all } t \geq k^* L + t_0, \tag{8}$$

where

$$k^* \doteq \inf\Bigg\{ t \geq 1 : \min_{|S(0)|=\cdots=|S(t-1)|} W \sum_{k=0}^{t-1} \sum_{\substack{i\notin S(k+1)\\ j\in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL+u+t_0) \geq \omega_2 \left\lfloor \frac{N}{2} \right\rfloor (\eta^L + 1) \Bigg\}, \tag{9}$$

with $W = \frac{\eta^L}{(N-1)L}$, $\omega_2 = \left\lceil \frac{\log \epsilon^{-1}}{\log\left(1 - K_*^{-\lfloor \frac{N}{2} \rfloor}/(8N^2)^{\lfloor \frac{N}{2} \rfloor}\right)^{-1}} \right\rceil$, $K_* = \max\left\{\frac{(N-1)K}{\eta^{L-1}}, \frac{N-1}{\eta^L}\right\}$, $S(k),\ k \geq 0$, being nonempty proper subsets of $\mathcal{V}$ with the same cardinality, and $|S(k)|$ being the cardinality of $S(k)$.

For both cases, the conclusions (ii) establish the convergence rates of system (1) to consensus in terms of the interactions between agents having taken place. In the following two sections, we prove these two theorems. Finally we conclude this paper with a few remarks and future directions.

II. Proof of Theorem 1

In this section, we first establish two key technical lemmas, and then present the proof of Theorem 1.

A. Key Lemmas

First we present two lemmas. The first one is a pure algebraic inequality and the second lemma follows from a similar analysis in [11], and thus we omit their detailed proofs.

Lemma 1: Let $b_k, k = 1, \ldots, m$ be a sequence of real numbers of length m satisfying $b_k \in [\eta, 1]$, $m \geq 0$, where $0 < \eta < 1$ is a given constant. Then we have $\prod_{k=1}^{m} b_k \geq e^{-\frac{\zeta \ln \eta}{\eta - 1}}$ if $\sum_{k=1}^{m}(1 - b_k) \leq \zeta$.

Lemma 2: For system (1), suppose Assumption 1 holds and $x_i(s) \leq \mu h(s) + (1-\mu)H(s)$ for some $s \geq t_0$ and $0 \leq \mu < 1$. Then we have

$$x_i(s+\tau) \leq \mu \prod_{k=0}^{T-1} a_{ii}(s+k) \cdot h(s) + \left(1 - \mu \prod_{k=0}^{T-1} a_{ii}(s+k)\right) \cdot H(s) \tag{10}$$

for all $\tau \leq T$ and $T = 0, 1, \ldots$.

B. Proof of Theorem 1 (i)

(Sufficiency) We introduce $\mathbf{A}_i(t) = \sum_{j=1, j\neq i, (j,i)\in\mathcal{E}_p}^{N} a_{ij}(t)$ for each node $i \in \mathcal{V}$ and $t \geq 0$. According to the definition of the persistent graph, for any initial time t_0 and any $\varepsilon > 0$, there exists an integer $T_\varepsilon \geq t_0$ such that $\sum_{t=T_\varepsilon}^{\infty} a_{ij}(t) \leq \varepsilon$ for all $(j,i) \in \mathcal{E} \setminus \mathcal{E}_p$.

We divide the rest of the proof into three steps.

Step 1. Take $T_0 = T_\varepsilon$ and $\delta > L(N-1)(1-\eta)$, where η is the constant in Assumption 1 and L is the integer in Assumption 2. Let i_0 be a root of the persistent graph $\mathbb{G}_p$ and $(i_0, i_1) \in \mathcal{E}_p$. Such an i_1 exists since $\mathbb{G}_p$ contains a directed spanning tree. Define

$$t_1 \doteq \inf\left\{t \geq 1 : \sum_{k=0}^{t-1} \mathbf{A}_{i_1}(T_0+k) \geq \delta\right\}.$$

Let s be the integer satisfying that $(s-1)L \leq t_1 < sL$. With Assumption 1, we have that $\sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0+k) \leq 1 - \eta + \delta$.

Since $a_{i_1 i_1}(T_0+k) = 1 - \sum_{j=1, j\neq i_1}^{N} a_{i_1 j}(T_0+k)$ and based on Assumption 1, we have

(i) $a_{i_1 i_1}(T_0+k) \in [\eta, 1]$ for all $k = 0, \ldots, t_1 - 1$;

(ii)
$$\begin{aligned}\sum_{k=0}^{t_1-1}\left(1 - a_{i_1 i_1}(T_0+k)\right) &= \sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0+k) \\ &\quad + \sum_{k=0}^{t_1-1} \sum_{j=1, j\neq i_1, (j,i_1)\notin\mathcal{E}_p}^{N} a_{i_1 j}(T_0+k) \\ &\leq 1 - \eta + \delta + \varepsilon(N-1).\end{aligned}$$

Therefore, we conclude from Lemma 1 that

$$\prod_{k=0}^{t_1-1} a_{i_1 i_1}(T_0+k) \geq e^{-\frac{(1-\eta+\delta+\varepsilon(N-1))\ln\eta}{\eta-1}} \doteq \mathbf{S}. \tag{11}$$

It is clear from the definition of $\mathbf{A}_i(t)$ and the fact $(s-1)L \leq t_1 < sL$ that

$$\sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0+k) \leq \sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0+k) \leq 1 - \eta + \delta,$$

for all $(i_r, i_1) \in \mathcal{E}_p$. From Assumption 2 and $t_1 < sL$, one has that for any $(j,i) \in \mathcal{E}_p$,

$$\begin{aligned}&\sum_{k=0}^{t_1-1} a_{ij}(T_0+k) \\ &\leq K \sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0+k) + \sum_{(s-1)L}^{sL-1} a_{ij}(T_0+k) \\ &\leq K(1-\eta+\delta) + L(1-\eta).\end{aligned}$$

For any $i \neq i_1$, it is true that

$$\begin{aligned}&\sum_{k=0}^{t_1-1}\left(1 - a_{ii}(T_0+k)\right) \\ &= \sum_{k=0}^{t_1-1} \mathbf{A}_i^*(T_0+k) + \sum_{k=0}^{t_1-1} \sum_{j=1, j\neq i, (j,i)\notin\mathcal{E}_p}^{N} a_{ij}(t) \\ &\leq (N-1)(K(1-\eta+\delta) + L(1-\eta) + \varepsilon).\end{aligned}$$

Thus in view of Lemma 1, we have that

$$\prod_{k=0}^{t_1-1} a_{ii}(T_0+k) \geq e^{-\frac{(N-1)(K(1-\eta+\delta)+L(1-\eta)+\varepsilon)\ln\eta}{\eta-1}} = \mathbf{Q} \tag{12}$$

for $i \neq i_1$. Note that $\mathbf{Q} < \mathbf{S}$.

Assume that $x_{i_0}(T_0) \leq \frac{1}{2}h(T_0) + \frac{1}{2}H(T_0)$. In this step, we establish an upper bound for $x_{i_0}(T_0+\tau), \tau = 0, \ldots, t_1$.

Based on Lemma 2, we obtain

$$x_{i_0}(T_0+\tau) \leq \frac{1}{2}\prod_{k=0}^{t_1-1} a_{i_0 i_0}(T_0+k) \cdot h(T_0) + \left(1 - \frac{1}{2}\prod_{k=0}^{t_1-1} a_{i_0 i_0}(T_0+k)\right) \cdot H(T_0) \tag{13}$$

for all $\tau = 0, \ldots, t_1$. Then (12) and (13) further imply

$$x_{i_0}(T_0+\tau) \leq \frac{\mathbf{Q}}{2}h(T_0) + \left(1 - \frac{\mathbf{Q}}{2}\right)H(T_0). \tag{14}$$

for all $\tau = 0, \dots, t_1$.

Step 2. In this step, we establish a bound for $x_{i_1}(T_0 + t_1)$. Since $\sum_{k=0}^{t_1-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta$, there must exist a node i_r such that $(i_r, i_1) \in \mathcal{E}_p$ and $\sum_{k=0}^{t_1-1} a_{i_1 i_r}(T_0 + k) \geq \frac{\delta}{N-1}$. Combining with $t_1 < sL$ and $\sum_{(s-1)L}^{sL} a_{i_1 i_r}(T_0+k) \leq L(1-\eta)$, simple calculation shows that

$$\sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) \geq \frac{\delta}{N-1} - L(1-\eta).$$

From Assumption 2, for any arc $(i, j) \in \mathcal{E}_p$, one has that

$$\begin{aligned}\sum_{k=0}^{t_1-1} a_{ij}(T_0 + k) &\geq K^{-1} \sum_{k=0}^{(s-1)L-1} a_{i_1 i_r}(T_0 + k) \\ &\geq K^{-1}\left[\frac{\delta}{N-1} - L(1-\eta)\right] = \mathbf{R}. \quad (15)\end{aligned}$$

The above inequality also holds for the arc (i_0, i_1) since $(i_0, i_1) \in \mathcal{E}_p$.

First according to (14), we have

$$\begin{aligned}x_{i_1}(T_0 + 1) &= \sum_{j=1}^{N} a_{i_1 j}(T_0) x_j(T_0) \\ &\leq a_{i_1 i_0}(T_0) x_{i_0}(T_0) + \big(1 - a_{i_1 i_0}(T_0)\big) H(T_0) \\ &= \frac{\mathbf{Q}}{2} a_{i_1 i_0}(T_0) h(T_0) + \Big(1 - \frac{\mathbf{Q}}{2} a_{i_1 i_0}(T_0)\Big) H(T_0).\end{aligned}$$

Then for $T_0 + 2$, we have

$$\begin{aligned}x_{i_1}(T_0 + 2) &= \sum_{j=1}^{N} a_{i_1 j}(T_0+1) x_j(T_0+1) \\ &\leq a_{i_1 i_0}(T_0+1) x_{i_0}(T_0+1) + a_{i_1 i_1}(T_0+1) x_{i_1}(T_0+1) \\ &\quad + \big(1 - a_{i_1 i_0}(T_0+1) - a_{i_1 i_1}(T_0+1)\big) H(T_0+1) \\ &= \frac{\mathbf{Q}}{2}\Big[a_{i_1 i_0}(T_0+1) + a_{i_1 i_1}(T_0+1) a_{i_1 i_0}(T_0)\Big] h(T_0) \\ &\quad + \Big[1 - \frac{\mathbf{Q}}{2}\Big[a_{i_1 i_0}(T_0+1) + a_{i_1 i_0}(T_0+1) a_{i_1 i_0}(T_0)\Big]\Big] H(T_0).\end{aligned}$$

By induction it is straightforward to find that

$$x_{i_1}(T_0 + t_1) \leq \frac{1}{2}\mathbf{SQR} h(T_0) + \Big(1 - \frac{1}{2}\mathbf{SQR}\Big) H(T_0). \quad (16)$$

Step 3. Let $\mathcal{V}_0 = \{i_0\}$ and $\mathcal{V}_1 = \{i : (i_0, i) \in \mathcal{E}_p\}$. It is obvious from (16) and in view of (12) that for any $i \in \mathcal{V}_1$,

$$x_i(T_0 + t_1) \leq \frac{1}{2}\mathbf{Q}^2\mathbf{R} h(T_0) + \Big(1 - \frac{1}{2}\mathbf{Q}^2\mathbf{R}\Big) H(T_0).$$

Let $\mathcal{V}_2$ be a subset of $\mathcal{V} \backslash (\mathcal{V}_0 \cup \mathcal{V}_1)$ and consist of all the nodes each of which has a neighbor in $\mathcal{V}_0 \cup \mathcal{V}_1$ in $\mathbb{G}_p$. We continue to define

$$t_2 \doteq \inf\Big\{t \geq t_1 + 1 : \sum_{k=t_1}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta\Big\}.$$

Following similar calculations in the above two steps, for any $i_2 \in \mathcal{V}_2$, we have

$$x_{i_2}(T_0 + t_2) \leq \frac{1}{2}\mathbf{Q}^4\mathbf{R}^2 h(T_0) + \Big(1 - \frac{1}{2}\mathbf{Q}^4\mathbf{R}^2\Big) H(T_0). \quad (17)$$

Continuing this process, $\mathcal{V}_3, \dots, \mathcal{V}_{d_0}$ can be defined similarly with d_0 being the diameter of $\mathbb{G}_p$ and a time sequence $t_1, \dots, t_{d_0}$ can be defined as

$$t_r \doteq \inf\Big\{t \geq t_{r-1} + 1 : \sum_{k=t_{r-1}}^{t-1} \mathbf{A}_{i_1}(T_0 + k) \geq \delta\Big\} \quad (18)$$

for $r = 1, \dots, d_0$, with $t_0 = 0$. It is easy to see that the root i_0 can be selected such that $\cup_{i=0}^{d_0} \mathcal{V}_i = \mathcal{V}$. The bound for $x_i(T_0 + t_{d_0})$ can be established as

$$x_i(T_0 + t_{d_0}) \leq \frac{1}{2}\mathbf{Q}^{2d_0}\mathbf{R}^{d_0} h(T_0) + \Big(1 - \frac{1}{2}\mathbf{Q}^{2d_0}\mathbf{R}^{d_0}\Big) H(T_0), \quad (19)$$

for all $i = 1, \dots, N$. A bound for $\Psi(T_0 + t_{d_0})$ is thus derived

$$\Psi(T_0 + t_{d_0}) \leq \Big(1 - \frac{1}{2}\mathbf{Q}^{2d_0}\mathbf{R}^{d_0}\Big)\Psi(T_0).$$

When $x_{i_0}(T_0) > \frac{1}{2}h(T_0) + \frac{1}{2}H(T_0)$, one can establish a lower bound for $x_i(T_0 + t_{d_0})$ by a symmetric argument and derive the same inequality for $\Psi(T_0 + t_{d_0})$ as above.

Repeating the above estimate, one can find an infinite increasing time sequence $t_1, \dots, t_{d_0}, t_{d_0+1}, \dots, t_{2d_0}, \dots$, defined by (18) and we have

$$\Psi(T_0 + t_{rd_0}) \leq \Big(1 - \frac{1}{2}\mathbf{Q}^{2d_0}\mathbf{R}^{d_0}\Big)^r \Psi(T_0), \quad (20)$$

for $r = 1, 2, \dots$. It implies that the sequence $\Psi(T_0 + t_{rd_0})$, $r = 1, 2, \dots$, converges to 0 as r goes to infinity. Therefore, $\Psi(t)$ converges to 0 as t goes to infinity as well.

(Necessity) The proof of the necessity part is similar to that of Theorem 3.1 in [11] and is thus omitted here.

C. Proof of Theorem 1 (ii)

Note that from the definition of t_r in (18) and the definition of $\mathbf{A}_{i_1}$, one knows that for any $r \geq 1$, $\sum_{k=t_{r-1}}^{t_r-1} \mathbf{A}_{i_1}(T_\varepsilon + k) \leq 1 + \delta$. It follows that $\sum_{k=0}^{t_{\omega_1}-1} \mathbf{A}_{i_1}(T_\varepsilon + k) \leq \omega_1 d_0(1+\delta)$. By the definition of t^* in (7), $t^* \geq t_{\omega_1} d_0$. For $t \geq T_\varepsilon + t^*$, applying (20) we have

$$\Psi(t) \leq \Psi(T_\varepsilon + t^*) \leq \Psi(T_\varepsilon + t_{\omega_1} d_0) \leq \epsilon\Psi(T_\varepsilon).$$

III. Proof of Theorem 2

In this section, we establish the convergence statement in Theorem 2 (i) and the contraction rate of $\Psi(t)$ claimed in Theorem 2 (ii).

A. Proof of Theorem 2 (i)

Consider system (1) with the initial time t_0. Let $y(t) = x(tL + t_0)$ and $B(t) = A((t+1)L - 1 + t_0) \cdots A(tL + 1 + t_0) A(tL + t_0)$. Then the dynamics of y-system are given by

$$y(t+1) = B(t) y(t), \ t \geq 0. \quad (21)$$

Letting $\Phi(t) \doteq \max_{i\in\mathcal{V}} y_i(t) - \min_{i\in\mathcal{V}} y_i(t)$, one has that $\Phi(t) = \Psi(tL + t_0)$. One can conclude that $\lim_{t\to\infty} \Psi(t) = 0$ if and only if $\lim_{t\to\infty} \Phi(t) = 0$ since $\Psi(t)$ is a nonincreasing function of t. Hence we establish the global consensus of system (1) by studying the property of the y-system (21).

We first establish that the system matrix $B(t)$ in (21) satisfies the cut-balance condition [10] under Assumption 3, whose proof is omitted due to space limitation.

Lemma 3: If Assumptions 1 and 3 hold, then each matrix $B(t),\ t \geq 0$, has positive diagonals lower bounded by η^L and satisfies the cut-balance condition

$$\sum_{i \notin S, j \in S} b_{ij}(t) \leq M_* \sum_{i \in S, j \notin S} b_{ij}(t) \tag{22}$$

for any nonempty proper subset S of $\mathcal{V}$ with $M_* = (N-1)K\eta^{-L+1}$. Let $\mathbb{G}'_p = (\mathcal{V}, \mathcal{E}'_p)$ be a directed graph where $(j,i) \in \mathcal{E}'_p$ if and only if $\sum_{t=0}^{\infty} b_{ij}(t) = \infty$. The persistent graph $\mathbb{G}_p$ contains a directed spanning tree if and only if $\mathbb{G}'_p$ contains a directed spanning tree.

Proof of Theorem 2 (i): Lemma 3 shows that the y-system (21) satisfies the assumptions of Theorem 5 in [20]. One concludes that $\mathbb{G}'_p$ defined in Lemma 3 contains a directed spanning tree if and only if global consensus of system (21) is reached. Combining with Lemma 3, the conclusion of Theorem 2 (i) immediately follows.

B. Proof of Theorem 2 (ii)

In this subsection, we provide a contraction rate of $\Phi(t)$ and hence a corresponding contraction rate of $\Psi(t)$ can be obtained. Note that system (21) satisfies the cut-balanced condition (22). Instead of considering the cut-balance condition, we consider the following balanced asymmetric condition.

Assumption 4: (Balanced Asymmetry) [12] There exists a constant $M \geq 1$ such that for any two nonempty proper subsets S_1, S_2 of $\mathcal{V}$ with the same cardinality, the matrices $B(t),\ t \geq 0$, satisfy that

$$\sum_{i \notin S_1, j \in S_2} b_{ij}(t) \leq M \sum_{i \in S_1, j \notin S_2} b_{ij}(t). \tag{23}$$

Remark 2: As pointed out in Remark 1 in [12], the balanced asymmetry condition is stronger than the cut-balance condition (22). But since $B(t)$ in (21) has positive diagonal elements lower bounded by a positive constant η^L and satisfies (22), then it satisfies the balanced asymmetry condition with $M = \max\{M_*, \frac{N-1}{\eta^L}\}$. □

The following notion of absolute infinite flow property [12], [21] is needed which has a close relationship with the connectivity of persistent graphs.

Definition 3: The sequence of matrices $B(t),\ t \geq 0$ is said to have the absolute infinite flow property if the following holds

$$\sum_{t=0}^{\infty} \Big(\sum_{\substack{i \notin S(t+1) \\ j \in S(t)}} b_{ij}(t) + \sum_{\substack{i \in S(t+1) \\ j \notin S(t)}} b_{ij}(t) \Big) = \infty \tag{24}$$

for every sequence $S(t),\ t \geq 0$, of nonempty proper subsets of $\mathcal{V}$ with the same cardinality.

Since the persistent graph $\mathcal{G}_p$ contains a directed spanning tree, the persistent graph $\mathbb{G}'_p$ contains a directed spanning tree as well by Lemma 3. Then we can show that the matrix sequence $B(t),\ t \geq 0$, has the absolute infinite flow property. In addition, $B(t),\ t \geq 0$, satisfies the balanced asymmetry condition by Remark 2. We can define an infinite time sequence $t_0, t_1, t_2, \ldots$ based on the infinite flow property. Let $t_0^0 = t_0$ and define a finite time sequence $t_p^0, t_p^1, \ldots, t_p^{\lfloor \frac{N}{2} \rfloor},\ p \geq 0$. t_p^{q+1} is defined by

$$t_p^{q+1} \doteq \inf \Bigg\{ t \geq t_p^q + 1 : \min_{|S(t_p^q)| = \cdots = |S(t-1)|} \sum_{k=t_p^q}^{t-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} b_{ij}(k) \geq 1 \Bigg\}. \tag{25}$$

We derive an infinite time sequence $t_0, t_1, t_2, \ldots$.

Proposition 1: If Assumption 1 and 3 hold and the persistent graph $\mathcal{G}_p$ contains a directed spanning tree, then for system (21),

$$\Phi(t_{p+1}) \leq \Big(1 - M^{-\lfloor \frac{N}{2} \rfloor}/(8N^2)^{\lfloor \frac{N}{2} \rfloor}\Big)\Phi(t_p), \tag{26}$$

where $M = \max\{M_*, \frac{N-1}{\eta^L}\}$ with M_* given in Lemma 3.

To show Proposition 1, we need to introduce an equivalent order-preserving system. For $t \geq 0$, let σ_t be a permutation of $\mathcal{V}$ such that for $i < j$, either $y_{\sigma_t(i)}(t) < y_{\sigma_t(j)}(t)$ or $y_{\sigma_t(i)}(t) = y_{\sigma_t(j)}(t)$ and $\sigma_t(i) < \sigma_t(j)$ holds. Define $z_i(t) \doteq y_{\sigma_t(i)}(t),\ t \geq 0$. From the definition of the permutation σ_t, one knows that for all $t \geq 0$, if $i < j$, then $z_i(t) \leq z_j(t)$. Hence $z(t) = [z_1(t), \ldots, z_N(t)]^T$ is a sorted state vector. It is easy to see that $\Phi(t) = \max_{i \in \mathcal{V}} y_i(t) - \min_{i \in \mathcal{V}} y_i(t) = z_N(t) - z_1(t)$. Define $c_{ij}(t) \doteq b_{\sigma_{t+1}(i), \sigma_t(j)}(t)$. Obviously $\sum_{j=1}^N c_{ij}(t) = 1$ for all $i \in \mathcal{V}$, $t \geq 0$. One can easily show that

$$z_i(t+1) = \sum_{j=1}^N c_{ij}(t) z_j(t), \tag{27}$$

In addition, one can show that $C(t) = [c_{ij}(t)]_{N \times N},\ t \geq 0$, have the balanced asymmetry property with the same constant M in (23) since $B(t), t \geq 0$, satisfy Assumption 4.

With these notations, one can prove Proposition 1 by checking $z_N(t) - z_1(t)$ using similar ideas to the proof of Proposition 2 in [9]. The detailed proof is omitted due to space limitation.

Now we are in a position to prove Theorem 2 (ii).

Proof of Theorem 2 (ii): For system (1) and any given initial time $t_0 \geq 0$, let $k_0^0 = k_0 = 0$ and define a finite time sequence $k_p^0, k_p^1, \ldots, k_p^{\lfloor \frac{N}{2} \rfloor},\ p \geq 0$. k_p^{q+1} is defined by

$$k_p^{q+1} \doteq \inf \Big\{ t \geq k_p^q + 1 : \min_{|S(k)| = \cdots = |S(t-1)|} W \sum_{k=k_p^q}^{t-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL + u + t_0) \geq 1 \Big\}, \tag{28}$$

where $W = \frac{\eta^L}{(N-1)L}$ is a constant. Let $k_{p+1} = k_p^{\lfloor \frac{N}{2} \rfloor}$ and $k_{p+1}^0 = k_{p+1}$. We derive an infinite time sequence $k_0, k_1, k_2, \ldots$. Under Assumptions 1 and 3, it can be shown that when the persistent graph $\mathbb{G}_p$ contains a directed spanning tree, the time sequence $k_0, k_1, k_2, \ldots$ is well-defined.

We first show that if the persistent graph $\mathbb{G}_p$ contains a directed spanning tree, then

$$\Psi(k_{p+1}L+t_0) \leq \left(1 - {K_*}^{-\lfloor \frac{N}{2} \rfloor}/(8N^2)^{\lfloor \frac{N}{2} \rfloor}\right)\Psi(k_pL+t_0), \quad (29)$$

where $K_* = \max\{\frac{(N-1)K}{\eta^{L-1}}, \frac{N-1}{\eta^L}\}$.

Consider system (21) derived based on system (1). Some calculations can verify that k_p^{q+1} defined in (28) satisfies $\sum_{k=k_p^q}^{k_p^{q+1}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} b_{ij}(k) \geq 1$. Noting that $\Phi(t) = \Psi(tL+t_0)$ and $M_* = (N-1)K\eta^{-L+1}$, applying (26) in Proposition 1 immediately gives (29).

Next we prove (8). It can be shown that for any sequence $S(k)$, $k \geq 0$, of nonempty proper subsets of $\mathcal{V}$ with the same cardinality, it always holds that

$$W \sum_{k=0}^{k_{\omega_2}-1} \sum_{\substack{i \notin S(k+1) \\ j \in S(k)}} \sum_{u=0}^{L-1} a_{ij}(kL+u+t_0) \leq \omega_2 \left\lfloor \frac{N}{2} \right\rfloor (\eta^L + 1).$$

By the definition of (9), $k^* \geq k_{\omega_2}$. Applying (29), one has that if $t \geq k^*L + t_0$, then

$$\Psi(t) \leq \Psi(k^*L+t_0) \leq \Psi(k_{\omega_2}L+t_0) \leq \epsilon\Psi(t_0).$$

This proves the desired contraction rate. ■

IV. CONCLUSIONS

In this paper, we have generalized the cut-balance and arc-balance conditions in the literature so as to allow for non-instantaneous reciprocal interactions between agents. The assumption on the existence of a lower bound on the nonzero weights a_{ij} of the arcs has been relaxed. It has been shown that global consensus is reached if and only if the persistent graph contains a directed spanning tree. The estimate of the convergence rate of the discrete-time system has been given which is not established for the cut-balance case in [20]. Future work may consider multi-agent systems consisting of agents interacting with each other through attractive and repulsive couplings [22]–[28].

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