Distributed Optimization for Second-Order Multi-Agent Systems with Dynamic Event-Triggered Communication

Xinlei Yi, Lisha Yao, Tao Yang, Jemin George, and Karl H. Johansson

Abstract— In this paper, we propose a fully distributed algorithm for second-order continuous-time multi-agent systems to solve the distributed optimization problem. The global objective function is a sum of private cost functions associated with the individual agents and the interaction between agents is described by a weighted undirected graph. We show the exponential convergence of the proposed algorithm if the underlying graph is connected, each private cost function is locally gradient-Lipschitz-continuous, and the global objective function is restricted strongly convex with respect to the global minimizer. Moreover, to reduce the overall need of communication, we then propose a dynamic event-triggered communication mechanism that is free of Zeno behavior. It is shown that the exponential convergence is achieved if the private cost functions are also globally gradient-Lipschitz-continuous. Numerical simulations are provided to illustrate the effectiveness of the theoretical results.

I. INTRODUCTION

Distributed optimization in multi-agent systems is an important class of distributed optimization problems and has received great attention in recent years due to its wide application in wireless networks, sensor networks, smart grids, and multi-robot systems.

From a control point of view, distributed convex optimization in multi-agent systems is the optimal consensus problem, where the global objective function is a sum of private convex cost functions associated with the individual agents and the interaction between agents is described by a graph. Although classical distributed algorithms based on consensus theory and (sub)gradient method are discrete-time [1]–[3], continuous-time algorithms have attracted much attention recently due to the development of cyber-physical systems and the well-developed continuous-time control techniques. For example [4]–[13] propose continuous-time distributed algorithms to solve (constrained or unconstrained) optimal consensus problems and analyze the convergence properties via classic stability analysis.

However, all these existing continuous-time algorithms require continuous information exchange between agents,

X. Yi and K. H. Johansson are with the Department of Automatic Control, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, 100 44, Stockholm, Sweden. {xinleiy, kallej}@kth.se.

L. Yao and T. Yang are with the Department of Electrical Engineering, University of North Texas, Denton, TX 76203 USA. LishaYao@my.unt.edu, Tao.Yang@unt.edu.

J. George is with the U.S. Army Research Laboratory, Adelphi, MD 20783, USA. jemin.george.civ@mail.mil.

which may be impractical in physical applications. The event-triggered communication and control mechanism is introduced partially to tackle this problem [14], [15]. Event-triggered communication and control mechanisms for multi-agent systems have been studied recently [16]–[22]. Key challenges are how to design the control law, determine the event threshold, and avoid Zeno behavior. Zeno behavior means that there are an infinite number of triggers in a finite time interval [23].

There are few works on the optimal consensus problem with event-triggered communication. In [24], the authors design a distributed continuous-time algorithm for first-order multi-agent systems with event-triggered communication. In [25], the authors extend the zero-gradient-sum algorithm proposed in [6] with event-triggered communication. In [26], the authors propose a distributed continuous-time algorithm for second-order multi-agent systems with event-triggered communication. However, these algorithms are not fully distributed since the gain parameters of the algorithms depends on some global parameters, such as the eigenvalues of the graph Laplacian matrix.

In this paper, we consider the distributed optimization problem for second-order multi-agent systems with undirected and connected topologies. In particular, double-integrator dynamics are considered since they are widely applied to mechanical systems. For example, Euler-Lagrangian systems with exact knowledge of nonlinearities can be converted into double integrators and they can be used to describe many mechanical systems, such as autonomous vehicles, see [27], [28]. Moreover, the considered distributed optimization problem has many applications, such as the targeted agreement problem for a group of Lagrangian systems [29]. A fully distributed continuous-time algorithm is first proposed to solve the problem. One related existing work is [8], which also proposes a continuous-time distributed algorithm for second-order multi-agent systems. However, in [8], the parameters of the algorithm depend on some global information and the speed information of each agent has to be exchanged between neighbors, and only asymptotic convergence is established for the case where private cost functions are strongly convex and globally gradient-Lipschitz-continuous. In contrast, in this paper, no global information is needed to be known in advance and each agent does not need its neighbors' speed information. For the case where private cost functions are convex, we show the asymptotic convergence. Furthermore, we establish the exponential convergence for the case where each private cost function is locally gradient-Lipschitz-continuous and the

global objective function is restricted strongly convex with respect to the global minimizer. Note that not all private cost functions need to be so or strongly convex, which is a less restricted condition compared with that in [8]. To reduce the overall need of communication, inspired by the distributed dynamic event-triggered control mechanism for multi-agent systems proposed in [22], we then extend our algorithm with dynamic event-triggered communication. The proposed dynamic event-triggered communication mechanism is also fully distributed since no global information, such as the Laplacian matrix, is required. We show that the proposed dynamic event-triggered communication mechanism is free of Zeno behavior by a contradiction argument. Moreover, we also show that the extended algorithm with the proposed event-triggered communication mechanism exponentially converges to the global minimizer when each private cost function is globally gradient-Lipschitz-continuous and the global objective function is restricted strongly convex.

The rest of this paper is organized as follows. Section II introduces the preliminaries. The main results are stated in Sections III and IV. Simulations are given in Section V. Finally, the paper is concluded in Section VI.

Notations: $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices. $\mathbf{1}_n$ denotes the column vector with each component being 1 and dimension n. I_n is the n-dimensional identity matrix. Given a vector $[x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, $\mathrm{diag}([x_1, \ldots, x_n])$ is a diagonal matrix with the i-th diagonal element being x_i. The notation $A \otimes B$ denotes the Kronecker product of matrices A and B. $\rho(\cdot)$ stands for the spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues. Given two symmetric matrices M, N, $M \geq N$ means that $M - N$ is positive semi-definite.

II. PRELIMINARIES

In this section, we present some definitions from algebraic graph theory [30] and the problem formulation.

A. Algebraic Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ denote a weighted undirected graph with the set of vertices (nodes) $\mathcal{V} = \{1, \ldots, n\}$, the set of links (edges) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $A = A^\top = (a_{ij})$ with nonnegative elements a_{ij}. A link of $\mathcal{G}$ is denoted by $(i, j) \in \mathcal{E}$ if $a_{ij} > 0$, i.e., if vertices i and j can communicate with each other. It is assumed that $a_{ii} = 0$ for all $i \in \mathcal{V}$. Let $\mathcal{N}_i = \{j \in \mathcal{V} \mid a_{ij} > 0\}$ and $\deg_i = \sum_{j=1}^{n} a_{ij}$ denotes the neighbor index set and weighted degree of vertex i, respectively. The degree matrix of graph $\mathcal{G}$ is $\mathrm{Deg} = \mathrm{diag}([\deg_1, \cdots, \deg_n])$. The Laplacian matrix is $L = (L_{ij}) = \mathrm{Deg} - A$. A path of length k between vertices i and j is a subgraph with distinct vertices $i_0 = i, \ldots, i_k = j \in \mathcal{V}$ and edges $(i_j, i_{j+1}) \in \mathcal{E}$, $j = 0, \ldots, k - 1$. An undirected graph is connected if there exists at least one path between any two vertices.

B. Problem Formulation

Consider a network of n agents and the underlying interaction between agents is described by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. Each agent is described by a double integrator

$$\ddot{x}_i(t) = u_i(t), \ i \in \mathcal{V}, \ t \geq 0, \tag{1}$$

where $x_i \in \mathbb{R}^p$ is the state and $u_i \in \mathbb{R}^p$ is the control input of agent i. Each agent i is also associated with a private convex cost function $f_i(x_i): \ \mathbb{R}^p \mapsto \mathbb{R}$.

The goal of the distributed optimization problem is to design an algorithm, i.e., design the control input u_i for every agent, so that all agents find an optimizer x^* that minimizes the sum of the f_i's collaboratively in a distributed manner, i.e.,

$$x^* \in \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} f_i(x). \tag{2}$$

The existence of the global minimizer x^* is guaranteed by the following assumption.

Assumption 1. *(Convex) For each $i \in \mathcal{V}$, the function f_i is continuously differentiable and convex.*

Moreover, if the following assumption also holds, then the global minimizer x^* is unique.

Assumption 2. *(Restricted strongly convex, see [31]) The global objective function $\sum_{i=1}^{n} f_i(x)$ is restricted strongly convex with respect to its global minimizer x^* with convexity parameter $m_f > 0$, i.e., for all $x \in \mathbb{R}^p$,*

$$\sum_{i=1}^{n} (\nabla f_i(x) - \nabla f_i(x^*))^\top (x - x^*) \geq m_f \|x - x^*\|^2.$$

Remark 1. *Assumption 2 is weaker than the assumption that the global object function is strongly convex, thus it is also weaker than the assumption that each private convex cost function is strongly convex.*

In addition, same as the existing literature, we assume that each private cost function has a locally (globally) Lipschitz continuous gradient.

Assumption 3. *(Locally gradient-Lipschitz-continuous) For each $i \in \mathcal{V}$, for any compact set $D \subseteq \mathbb{R}^p$, there exists a constant $M_i(D) > 0$, such that $\|\nabla f_i(a) - \nabla f_i(b)\| \leq M_i(D)\|a - b\|$, $\forall a, b \in D$.*

Assumption 4. *(Globally gradient-Lipschitz-continuous) For each $i \in \mathcal{V}$, there exists a constant $\overline{M}_i > 0$, such that $\|\nabla f_i(a) - \nabla f_i(b)\| \leq \overline{M}_i\|a - b\|$, $\forall a, b \in \mathbb{R}^p$.*

III. DISTRIBUTED CONTINUOUS-TIME ALGORITHMS

In this section, we propose a distributed continuous-time algorithm to solve the optimization problem stated in (2) and analyze its convergence.

For each agent $i \in \mathcal{V}$, we first design the following algorithm,

$$\dot{v}_i(t) = \beta \sum_{j=1}^{n} L_{ij} x_j(t), \ \sum_{i=1}^{n} v_i(0) = 0, \tag{3a}$$

$$u_i(t) = -\gamma\dot{x}_i(t) - \alpha\beta\sum_{j=1}^{n} L_{ij}x_j(t) - \theta v_i(t)$$
$$-\alpha\nabla f_i(x_i(t)), \ t \geq 0, \tag{3b}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\theta > 0$ are gain parameters.

Remark 2. *In the design of right-hand side in* (3b), $-\gamma\dot{x}_i(t)$ *is to ensure the convergence of* (3), $-\alpha\beta\sum_{j=1}^{n} L_{ij}x_j(t)$ *is to ensure the consensus among agents,* $-\alpha\nabla f_i(x_i(t))$ *is to optimize each agent's private cost function, and* $-\theta v_i(t)$ *together with* $\sum_{i=1}^{n} v_i(0) = 0$ *and* (3a) *are to maintain the equilibrium point at the optimal point. Moreover, by setting* $v_i(0) = 0$, $\forall i \in \mathcal{V}$, *the coordination between agents to let* $\sum_{i=1}^{n} v_i(0) = 0$ *can be avoided.*

Denote $y_i(t) = \dot{x}_i(t)$. Then we can rewrite (1) and (3) as

$$\dot{x}_i(t) = y_i(t), \ \forall x_i(0), \ t \geq 0, \tag{4a}$$
$$\dot{y}_i(t) = -\gamma y_i(t) - \alpha\beta\sum_{j=1}^{n} L_{ij}x_j(t) - \theta v_i(t)$$
$$-\alpha\nabla f_i(x_i(t)), \ \forall y_i(0), \tag{4b}$$
$$\dot{v}_i(t) = \beta\sum_{j=1}^{n} L_{ij}x_j(t), \ \sum_{i=1}^{n} v_i(0) = 0. \tag{4c}$$

Remark 3. *If there is only one agent, the algorithm* (4) *becomes the heavy ball with friction system* [32]:

$$\ddot{x} + \gamma\dot{x} + \alpha\nabla f(x) = 0.$$

Denote $\boldsymbol{x} = [x_1^\top, \cdots, x_n^\top]^\top$, $\boldsymbol{y} = [y_1^\top, \cdots, y_n^\top]^\top$, $\boldsymbol{v} = [v_1^\top, \cdots, v_n^\top]^\top$, and $f(\boldsymbol{x}) = \sum_{i=1}^{n} f_i(x_i)$. Then, we can rewrite (4) in the following compact form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{y}(t), \ \forall x(0), \ t \geq 0, \tag{5a}$$
$$\dot{\boldsymbol{y}}(t) = -\gamma\boldsymbol{y}(t) - \alpha\beta(L \otimes I_p)\boldsymbol{x}(t) - \theta\boldsymbol{v}(t)$$
$$-\alpha\nabla f(\boldsymbol{x}(t)), \ \forall \boldsymbol{y}(0), \tag{5b}$$
$$\dot{\boldsymbol{v}}(t) = \beta(L \otimes I_p)\boldsymbol{x}(t), \ \sum_{i=1}^{n} v_i(0) = 0, \tag{5c}$$

The following result establishes sufficient conditions on the private cost function f_i; the gain parameters α, γ, θ; and the underlying graph to guarantee the (exponential) convergence of (4).

Theorem 1. *Suppose that Assumption 1 holds, and that the underlying undirected graph $\mathcal{G}$ is connected. If every agent $i \in \mathcal{V}$ runs the distributed algorithm with continuous-time communication given in* (4) *and* $\theta < \alpha\gamma$, *then every individual solution $x_i(t)$ asymptotically converges to one global minimizer. Moreover, if Assumptions 2 and 3 are also satisfied, then every individual solution $x_i(t)$ exponentially converges to the unique global minimizer x^* with a rate no less than* $\frac{\varepsilon_3}{2\varepsilon_4}$, *where*

$$\varepsilon_1 = \min\{\gamma(1-\varepsilon_0), \ \alpha\gamma\varepsilon_0 m_1\} > 0, \tag{6}$$
$$\varepsilon_2 = \max\left\{\frac{\gamma}{\alpha} + \frac{\gamma^2}{\theta} + \frac{\theta}{\alpha^2}, \ \frac{\alpha^2(M(D))^2}{\theta}\right\} > 0, \tag{7}$$

$$\varepsilon_3 = \min\left\{\varepsilon_1, \ \frac{\varepsilon\theta}{2}\right\} > 0, \tag{8}$$
$$\varepsilon_4 = \max\left\{1 + \frac{\varepsilon\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon}{\alpha},\right.$$
$$\left(1 + \frac{\varepsilon\varepsilon_2}{\varepsilon_1}\right)(\gamma^2\varepsilon_0 + \alpha\beta\rho(L) + \frac{\alpha M(D)}{2}) + \frac{\varepsilon M(D)}{2},$$
$$\left.\left(1 + \frac{\varepsilon\varepsilon_2}{\varepsilon_1}\right)\frac{\theta\gamma\varepsilon_0}{\beta\rho_2(L)} + \varepsilon\alpha\right\} > 1, \tag{9}$$

where $\varepsilon > 0$ *and* $\varepsilon_0 \in (\frac{\theta}{\alpha\gamma}, 1)$ *are design parameters and can be freely chosen in the given intervals,* $m_1 = \min\left\{\frac{m_f}{2}, \frac{\rho_2(L)m_f^2\alpha\gamma\varepsilon_0}{2(\alpha\gamma\varepsilon_0 - \theta)(m_f^2 + 16M^2(D))}\right\} > 0$ *and* $M(D) = \max_{i \in \mathcal{V}}\{M_i(D)\} > 0$ *are constants, and* $D \subseteq \mathbb{R}^p$ *is a compact convex set and its definition is given in the proof.*

Proof. Due to the space limitations, the proof is omitted here, but can be found in [33]. The proof is based the Lyapunov stability analysis. A novel Lyapunov function is constructed, which is different from that in the existing literature. $\square$

Remark 4. *The algorithm* (4) *is fully distributed in the sense that it does not require any global parameters to design the gain parameters α, β, γ, and θ. On the other hand, the algorithms proposed in* [8], [11] *do not have such a property.*

Remark 5. *We could also construct an alternative algorithm:*

$$\dot{x}_i(t) = y_i(t), \ \forall x_i(0), \ t \geq 0, \tag{10a}$$
$$\dot{y}_i(t) = -\gamma y_i(t) - \alpha\beta\sum_{j=1}^{n} L_{ij}x_j(t)$$
$$-\theta\sum_{j=1}^{n} L_{ij}v_j(t) - \alpha\nabla f_i(x_i(t)), \ \forall y_i(0), \tag{10b}$$
$$\dot{v}_i(t) = \beta\sum_{j=1}^{n} L_{ij}x_i(t), \ \forall v_i(0). \tag{10c}$$

Similar results as shown in Theorem 1 could be given and proven. We omit the details due to space limitations.

Different from the requirement that $\sum_{i=1}^{n} v_i(0) = 0$ *in the algorithm* (5), $v_i(0)$ *can be arbitrarily chosen in the algorithm* (10). *In other words, the algorithm* (10) *is robust to the initial condition $v_i(0)$. However, the algorithm* (10) *requires additional communication of v_j in* (10b), *compared to the algorithm* (5).

IV. EVENT-TRIGGERED COMMUNICATION

To implement the distributed algorithm (4), every agent $i \in \mathcal{V}$ has to know the continuous-time state $x_j(t)$, $\forall j \in \mathcal{N}_i$. In other words, continuous communication between agents is needed. However, distributed networks are normally resources-constrained and communication is energy-consuming. To avoid continuous communication, inspired by the idea of event-triggered control for multi-agent systems [16], we consider event-triggered communication. More specifically, we extend the algorithm (4) with event-triggered communication mechanism as:

$$\dot{x}_i(t) = y_i(t), \ \forall x_i(0), \ t \geq 0, \tag{11a}$$

$$\dot{y}_i(t) = -\gamma y_i(t) - \alpha\beta\sum_{j=1}^{n} L_{ij}x_j(t_{k_j(t)}^j) - \theta v_i(t)$$
$$- \alpha\nabla f_i(x_i(t)), \ \forall y_i(0), \quad (11b)$$

$$\dot{v}_i(t) = \beta\sum_{j=1}^{n} L_{ij}x_j(t_{k_j(t)}^j),$$
$$t \in [t_k^i, t_{k+1}^i), \ k = 1, 2, \ldots, \ \sum_{i=1}^{n} v_i(0) = 0, \quad (11c)$$

where the increasing sequence $\{t_k^i\}_{k=1}^{\infty}, \ \forall i \in \mathcal{V}$ to be determined later is the triggering times and $t_{k_j(t)}^j = \max\{t_k^j : \ t_k^j \leq t\}$. We assume $t_1^j = 0, \ \forall j \in \mathcal{V}$. For simplicity, let $\hat{x}_j(t) = x_j(t_{k_j(t)}^j)$ and $e_j^x(t) = \hat{x}_j(t) - x_j(t)$.

Denote $\hat{\boldsymbol{x}} = [\hat{x}_1^\top, \cdots, \hat{x}_n^\top]^\top$ and $\boldsymbol{e}^x = [(e_1^x)^\top, \cdots, (e_n^x)^\top]^\top$. Then, we can rewrite (11) in the following compact form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{y}(t), \ \forall\boldsymbol{x}(0), \ t \geq 0, \quad (12a)$$
$$\dot{\boldsymbol{y}}(t) = -\gamma\boldsymbol{y}(t) - \alpha\beta(L \otimes I_p)\hat{\boldsymbol{x}}(t) - \theta\boldsymbol{v}(t)$$
$$- \alpha\nabla f(\boldsymbol{x}(t)), \ \forall\boldsymbol{y}(0), \quad (12b)$$
$$\dot{\boldsymbol{v}}(t) = \beta(L \otimes I_p)\hat{\boldsymbol{x}}(t), \ \sum_{i=1}^{n} v_i(0) = 0. \quad (12c)$$

In the following theorem, we propose a dynamic event-triggered law to determine the triggering times such that the solution of the distributed optimization problem can still be reached exponentially.

Theorem 2. *Suppose that Assumptions 1, 2, and 4 hold, and that the underlying undirected graph $\mathcal{G}$ is connected. Suppose that each agent $i \in \mathcal{V}$ runs the distributed algorithm with event-triggered communication given in (11) and $\theta < \alpha\gamma$. Given the first triggering time $t_1^i = 0$, every agent $i \in \mathcal{V}$ determines the triggering times $\{t_k^i\}_{k=2}^{\infty}$ by the following dynamic event-triggered law:*

$$t_{k+1}^i = \min\Big\{t : \ \kappa_i\Big(\|e_i^x(t)\|^2 - \frac{(\alpha\gamma\varepsilon_0 - \theta)\beta\sigma_i}{4\varphi_i}\hat{q}_i(t)\Big)$$
$$\geq \chi_i(t), \ t \geq t_k^i\Big\}, \ k = 1, 2, \ldots \quad (13)$$

$$\hat{q}_i(t) = -\frac{1}{2}\sum_{j \in \mathcal{N}_i} L_{ij}\|\hat{x}_j(t) - \hat{x}_i(t)\|^2 \geq 0, \quad (14)$$

$$\dot{\chi}_i(t) = -\delta_i\Big(\|e_i^x(t)\|^2 - \frac{(\alpha\gamma\varepsilon_0 - \theta)\beta\sigma_i}{4\varphi_i}\hat{q}_i(t)\Big)$$
$$- \phi_i\chi_i(t), \ \forall\chi_i(0) > 0, \quad (15)$$

where $\sigma_i \in [0, 1), \ \phi_i > 0, \ \delta_i \in [0, 1],$ and $\kappa_i > \frac{1-\delta_i}{\phi_i}$ are design parameters and can be freely chosen in the given interval; and

$$m_2 = \min\Big\{\frac{m_f}{2}, \ \frac{4\rho_2(L)m_f^2\alpha}{(\alpha\gamma\varepsilon_0 - \theta)\beta(m_f^2 + 16\overline{M}^2)}\Big\}, \quad (16)$$

$$\varepsilon_5 = \min\Big\{\frac{\gamma(1-\varepsilon_0)}{2}, \ m_2\alpha\Big\} > 0, \quad (17)$$

$$\varepsilon_6 = \max\Big\{\frac{\gamma}{\alpha} + \frac{\gamma^2}{\theta} + \frac{\theta}{\alpha^2}, \ \frac{\alpha^2\overline{M}^2}{\theta}\Big\} > 0, \quad (18)$$

$$\varepsilon_7 = 1 + \frac{\varepsilon\varepsilon_6}{\varepsilon_5} > 1, \quad (19)$$

$$\varepsilon_8 = \frac{\varepsilon}{4\varepsilon_7} > 0, \quad (20)$$

$$\varphi_i = \frac{(\alpha\gamma\varepsilon_0 - \theta)\beta}{4}L_{ii} + (\alpha\gamma\varepsilon_0 - \theta)\beta L_{ii} + \frac{\gamma^2\theta\varepsilon_0^2}{4\varepsilon_8}$$
$$+ \frac{\alpha^2\beta^2}{\gamma(1-\varepsilon_0)}\Big(L_{ii} - \sum_{j=1, j\neq i}^{n} L_{jj}L_{ij}\Big) \quad (21)$$

with $\overline{M} = \max_{i \in \mathcal{V}}\{M_i\}$ is a constant, $\varepsilon > 0$ and $\varepsilon_0 \in (\frac{\theta}{\alpha\gamma}, 1)$ are design parameters, then (i) there is no Zeno behavior, and (ii) every individual solution $x_i(t)$ exponentially converges to the unique global minimizer x^* with a rate no less than $\frac{\varepsilon_9}{2\varepsilon_{10}}$, where

$$\varepsilon_9 = \min\Big\{\varepsilon_5, \ \frac{\varepsilon\theta}{4}, \ k_d\Big\} > 0, \quad (22)$$

$$\varepsilon_{10} = \max\Big\{\varepsilon_7 + \frac{\varepsilon}{\alpha}, \ \varepsilon_7\Big(\gamma^2\varepsilon_0 + \alpha\beta\rho(L) + \frac{\alpha\overline{M}}{2}\Big) + \frac{\varepsilon\overline{M}}{2},$$
$$\varepsilon_7\frac{\theta\gamma\varepsilon_0}{\beta\rho_2(L)} + \frac{\varepsilon\alpha}{\rho_2(L)}\Big\} > 1, \quad (23)$$

with $k_d = \min_{i \in \mathcal{V}}\Big\{\phi_i - \frac{1-\delta_i}{\kappa_i}\Big\} > 0.$

Proof. Due to the space limitations, the proof is omitted here, but can be found in [33]. $\square$

Remark 6. *The proposed dynamic event-triggered communication has several nice features: i) the exchange of $x_i(t)$ only occurs at the discrete time points $\{t_k^i, \ i \in \mathcal{V}\}_{k=1}^{\infty}$, ii) it is free of Zeno behavior, and iii) the implementation does not require any global information such as the Laplacian matrix. One potential drawback of the proposed dynamic event-triggered law is that when determining φ_i the global parameters $\rho_2(L), \ m_f,$ and $\overline{M}$ are needed. One solution to overcome this drawback is let $\sigma_i = \delta_i = 0, \ i \in \mathcal{V}$, since in this case we do not need to know φ_i.*

Remark 7. *If we let $\delta_1 = \cdots = \delta_n = 0$ and $\phi_1 = \cdots = \phi_n \in (0, \frac{\varepsilon_9}{\varepsilon_{10}}]$ in (15), where ε_9 and ε_{10} is defined in (22) and (23), respectively, then similar to the proof of Theorem 3.2 in [17], for each agent $i \in \mathcal{V}$, we can find a positive constant τ_i, such that $t_{k+1}^i - t_k^i \geq \tau_i, \ k = 1, 2, \ldots$. Since the proof is similar, we omit the detailed analysis here.*

V. SIMULATIONS

In this section, we illustrate and validate the proposed algorithm through numerical examples and compare the results with other existing algorithms. Consider a simple network of $n = 3$ agents with the Laplacian matrix

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

We first consider the case that the private cost functions f_i and the global objective function $\sum_{i=1}^{n} f_i(x)$ are just convex. We choose $f_i(x) = \frac{1}{2}(x - a_i)^\top A_i(x - a_i)$,

$$A_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1.5 & -0.5 \\ -1 & -0.5 & 1.5 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$
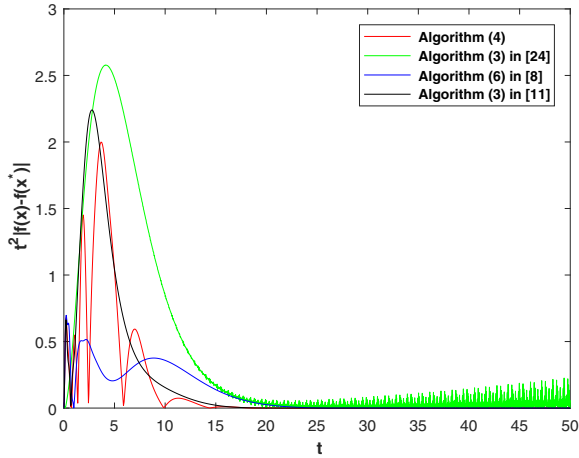
Fig. 1: Simulation results for non-strongly convex private cost and global objective functions.

Fig. 2: Simulation results for non-strongly convex private cost functions but strongly convex global objective function.

$$A_3 = \begin{bmatrix} 2.5 & 0 & -2.5 \\ 0 & 10 & -10 \\ -2.5 & -10 & 12.5 \end{bmatrix}, a_1 = \begin{bmatrix} 0.6132 \\ -0.5278 \\ 1.2416 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} -0.1576 \\ -1.3736 \\ 0.8708 \end{bmatrix}, a_3 = \begin{bmatrix} -1.5685 \\ -1.8443 \\ 0.2884 \end{bmatrix}.$$

Fig. 1 shows the comparison between the distributed algorithm (4) with $\alpha = \beta = 2$, $\gamma = 6$, $\theta = 5$; algorithm (3) in [24] with $\alpha = \beta = 2$; algorithm (6) in [8] with $\alpha = \beta = 2$, $k = 6$; and algorithm (3) in [11] with $k = 6$. It can be seen that distributed gradient descent algorithm (algorithm (3) in [24]) cannot achieve a $O(\frac{1}{t^2})$ convergence when the global objective and all the private cost functions are just convex.

We then consider the case that the private cost functions f_i are just convex but $\sum_{i=1}^{n} f_i(x)$ is strongly convex. We choose $f_i(x) = \|x - b_i\|^4$ with $x \in \mathbb{R}^3$,

$$b_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 2.5 \\ 2 \\ 3 \end{bmatrix}, \ b_3 = \begin{bmatrix} -3.5 \\ -2.7 \\ -1 \end{bmatrix}.$$

Fig. 2 shows the comparison between the distributed algorithm (4) with $\alpha = \beta = 2$, $\gamma = 6$, $\theta = 5$; algorithm (3) in [24] with $\alpha = \beta = 2$; algorithm (6) in [8] with $\alpha = \beta = 2$, $k = 6$; and algorithm (3) in [11] with $k = 6$. We can see that the proposed algorithm (4) achieves a faster convergence in this simulation.

Next, we consider the case where all private cost functions f_i are strongly convex. In particular, $f_i(x) = \frac{1}{2}x^\top C_i x + a_i^\top x$ with

$$C_1 = \begin{bmatrix} 4.7471 & 1.2843 & 0.5836 \\ 1.2843 & 5.0861 & -2.4209 \\ 0.5836 & -2.4209 & 2.2270 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1.3528 & 0.5141 & -2.1684 \\ 0.5141 & 1.2333 & -0.5857 \\ -2.1684 & -0.5857 & 4.0361 \end{bmatrix},$$
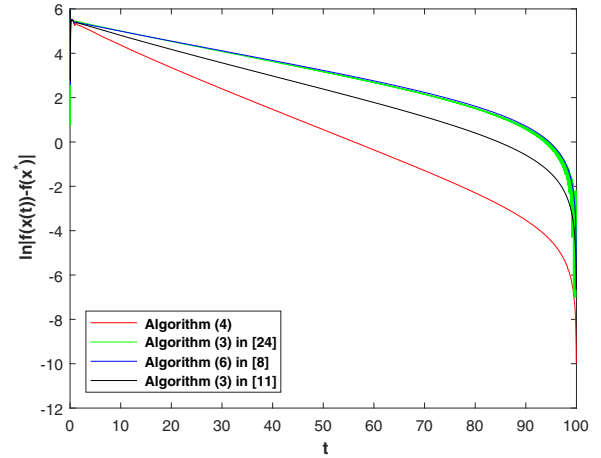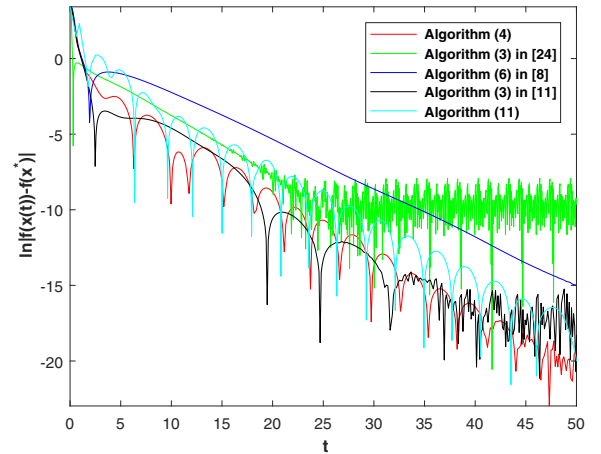

Fig. 3: Simulation results for strongly convex private cost functions.

$$C_3 = \begin{bmatrix} 1.0223 & 1.2630 & -0.4907 \\ 1.2630 & 2.1391 & -0.1378 \\ -0.4907 & -0.1378 & 0.7207 \end{bmatrix}.$$

Fig. 3 shows the comparison between the distributed algorithm (4) with $\alpha = \beta = 2$, $\gamma = 6$, $\theta = 3.5$; algorithm (3) in [24] with $\alpha = \beta = 2$; algorithm (6) in [8] with $\alpha = \beta = 2$, $k = 6$; algorithm (3) in [11] with $k = 6$; and the distributed event-triggered algorithm (11) with dynamic event-triggered communication determined by (13). In our simulation, the sample length is 0.01. During time interval $[0, 50]$, agents 1–3 triggered 1199, 139, and 664 times, respectively, under our dynamic event-triggered communication mechanism. Therefore, our dynamic event-triggered communication mechanism is very efficient and avoids about 85% sampling in this simulation.

VI. CONCLUSION

In this paper, we considered the distributed optimization problem for second-order continuous-time multi-agent sys-

tems. We first proposed a fully distributed continuous-time algorithm that does not require to know any global information in advance. We established the asymptotic convergence when the private cost functions are convex and exponential convergence when each private cost function is locally gradient-Lipschitz-continuous and the global objective function is restricted strongly convex with respect to the global minimizer. To avoid continuous communication, we then extended the continuous-time algorithm with dynamic event-triggered communication. We again showed that the global minimizer can be reached exponentially when each private cost function is globally gradient-Lipschitz-continuous and the global objective function is restricted strongly convex. Furthermore, the dynamic event-triggered communication was shown to be free of Zeno behavior. Future research directions include quantifying the convergence speed when the private cost functions are just convex.

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