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A Distributed Algorithm for Online Convex Optimization with Time-Varying Coupled Inequality Constraints

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Abstract— **This paper considers distributed online optimization with time-varying coupled inequality constraints. The global objective function is composed of local convex cost and regularization functions and the coupled constraint function is the sum of local convex constraint functions. A distributed online primal-dual mirror descent algorithm is proposed to solve this problem, where the local cost, regularization, and constraint functions are held privately and revealed only after each time slot. We first derive regret and constraint violation bounds for the algorithm and show how they depend on the stepsize sequences, the accumulated variation of the comparator sequence, the number of agents, and the network connectivity. As a result, we prove that the algorithm achieves sublinear dynamic regret and constraint violation if the accumulated variation of the optimal sequence also grows sublinearly. We also prove that the algorithm achieves sublinear static regret and constraint violation under mild conditions. In addition, smaller bounds on the static regret are achieved when the objective functions are strongly convex. Finally, numerical simulations are provided to illustrate the effectiveness of the theoretical results.**

I. Introduction

Consider a network of n agents indexed by $i = 1, \dots, n$. For each i, let the local decision set $X_i \subseteq \mathbb{R}^{p_i}$ be a closed convex set with p_i being a positive integer. Let $\{f_{i,t} : X_i \to \mathbb{R}\}$ and $\{g_{i,t} : X_i \to \mathbb{R}^m\}$ be sequences of local convex cost and constraint functions over time slots $t = 1, 2, \dots$, respectively, where m is a positive integer. At each time t, the network's objective is to solve the convex optimization problem $\min_{x_t \in X} \sum_{i=1}^n f_{i,t}(x_{i,t})$ with coupled constraint $\sum_{i=1}^n g_{i,t}(x_{i,t}) \le \mathbf{0}_m$, where the global decision variable $x_t = \mathrm{col}(x_{1,t}, \dots, x_{n,t}) \in X = X_1 \times \cdots \times X_n \subseteq \mathbb{R}^p$ with $p = \sum_{i=1}^n p_i$. We are interested in distributed algorithms to solve this problem, where computations are done by each agent. It is common to influence the structure of the solution using regularization. In this case, each agent i introduces a regularization function $r_{i,t} : X_i \to \mathbb{R}$. Examples of regularization include $r_{i,t}(x_i) = \lambda_i \|x_i\|_1$ and $r_{i,t}(x_i) = \frac{\lambda_i}{2}\|x_i\|$ with $\lambda_i > 0$, i.e., ℓ_1- and ℓ_2-regularization, respectively. Denote $c_t(x_t) = f_t(x_t) + r_t(x_t)$, $f_t(x_t) = \sum_{i=1}^n f_{i,t}(x_{i,t})$, $r_t(x_t) = \sum_{i=1}^n r_{i,t}(x_{i,t})$, and $g_t(x_t) = \sum_{i=1}^n g_{i,t}(x_{i,t})$. This paper is on solving the constrained optimization problem

$$\min_{x_t \in X} \quad c_t(x_t) \quad \text{subject to} \quad g_t(x_t) \le \mathbf{0}_m,\ t = 1, \dots \tag{1}$$

using distributed algorithms. In order to guarantee that problem (1) is feasible, we assume that for any $T \in \mathbb{N}_+$, the set of all feasible sequences $\mathcal{X}_T = \{(x_1, \dots, x_T) :\ x_t \in X,\ g_t(x_t) \le \mathbf{0}_m,\ t = 1, \dots, T\}$ is non-empty. With this standing assumption, an optimal sequence to (1) always exists.

We consider online algorithms. For a distributed online algorithm, at time t, each agent i selects a decision $x_{i,t} \in X_i$. After the selection, the agent receives its cost function $f_{i,t}$ and regularization $r_{i,t}$ together with its constraint function $g_{i,t}$. At the same moment, the agents exchange data with their neighbors over a time-varying directed graph. The performance of an algorithm depends on both the amount of data exchanged between the agents and how they process the data.

For online algorithms, regret and constraint violation are often used as performance metrics. The regret is the accumulation over time of the loss difference between the decision determined by the algorithm and a comparator sequence. Specifically, the efficacy of a decision sequence $\boldsymbol{x}_T = (x_1, \dots, x_T)$ relative to a comparator sequence $\boldsymbol{y}_T = (y_1, \dots, y_T) \in \mathcal{X}_T$ with $y_t = \mathrm{col}(y_{1,t}, \dots, y_{n,t})$ is characterized by the regret

$$\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) = \sum_{t=1}^T c_t(x_t) - \sum_{t=1}^T c_t(y_t). \tag{2}$$

There are two special comparators. One is $\boldsymbol{y}_T = \boldsymbol{x}_T^* = \arg\min_{\boldsymbol{x}_T \in \mathcal{X}_T} \sum_{t=1}^T c_t(x_t)$, i.e., an optimal sequence to (1). In this case $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)$ is called the dynamic regret. Another special comparator is $\boldsymbol{y}_T = \check{\boldsymbol{x}}_T^* = \arg\min_{\boldsymbol{x}_T \in \check{\mathcal{X}}_T} \sum_{t=1}^T c_t(x_t)$, i.e., a static optimal sequence to (1), where $\check{\mathcal{X}}_T$ is the set of feasible static sequences, i.e, $\check{\mathcal{X}}_T = \{(x, \dots, x) :\ x \in X,\ g_t(x) \le \mathbf{0}_m,\ t = 1, \dots, T\} \subseteq \mathcal{X}_T$. In order to guarantee the existence of $\check{\boldsymbol{x}}_T^*$, we assume that $\check{\mathcal{X}}_T$ is non-empty. In this case $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*)$ is called the static regret. It is straightforward to see that $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) \le \mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*),\ \forall \boldsymbol{y}_T \in \mathcal{X}_T$, and that $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \le \mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)$. For a decision sequence $\boldsymbol{x}_T$, the normally used constraint violation measure is $\|[\sum_{t=1}^T g_t(x_t)]_+\|$, i.e., the accumulation of constraint violations. This definition implicitly allows constraint violations at some times to be compensated by strictly feasible decisions at other times.

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This is appropriate for constraints that have a cumulative nature such as energy budgets enforced through average power constraints.

The problem considered in this paper is to develop a distributed online algorithm to solve (1) with guaranteed performance. The performance is measured by the regret and constraint violation. We are normally satisfied with low regret and constraint violation, by which we mean that both $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T)$ and $\|[\sum_{t=1}^{T} g_t(x_t)]_+\|$ grow sublinearly with T, i.e., there exist $\kappa_1,\ \kappa_2 \in (0,1)$ such that $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) = \mathcal{O}(T^{\kappa_1})$ and $\|[\sum_{t=1}^{T} g_t(x_t)]_+\| = \mathcal{O}(T^{\kappa_2})$. This implies that the upper bound of the time averaged difference between the accumulated cost of the decision sequence and the accumulated cost of any comparator sequences tends to zero as T goes to infinity. The same thing holds for the upper bound of the time averaged constraint violation.

A. Literature review

The online optimization problem (1) is related to two bodies of literature: centralized online convex optimization with time-varying inequality constraints ($n = 1$) and distributed online convex optimization with time-varying coupled inequality constraints ($n \geq 2$). Depending on the characteristics of the constraint, there are two important special cases: optimization with static constraints ($g_{i,t} \equiv 0$ for all t and i) and time-invariant constraints ($g_{i,t} = g_i$ for all t and i). Below, we provide an overview of the related works.

Centralized online convex optimization with static set constraints was first studied by Zinkevich [1]. Specifically, Zinkevich [1] developed a projection-based online gradient descent algorithm and achieved $\mathcal{O}(\sqrt{T})$ static regret bound for an arbitrary sequence of convex objective functions with bounded subgradients, which is a tight bound up to constant factors [2]. The regret bound can be reduced under more stringent strong convexity conditions on the objective functions [2]–[5] or by allowing to query the gradient of the objective function multiple times [6]. When the static constrained sets are characterized by inequalities, the conventional projection-based online algorithms are difficult to implement and may be inefficient in practice due to high computational complexity of the projection operation. To overcome these difficulties, some researchers proposed primal-dual algorithms for centralized online convex optimization with time-invariant inequality constraints, e.g., [7]–[10]. The authors of [11] showed that the algorithms proposed in [7], [8] are general enough to handle time-varying inequality constraints. The authors of [12] used the modified saddle-point method to handle time-varying constraints. The papers [13], [14] used a virtual queue, which essentially is a modified Lagrange multiplier, to handle stochastic and time-varying constraints. One common assumption in [12]–[14] is that the time-varying constraint functions satisfy Slater's condition, which is not assumed in this paper.

Distributed online convex optimization has been extensively studied, so here we only list some of the most relevant work. Firstly, the authors of [15]–[19] proposed distributed online algorithms to solve convex optimization problems with static set constraints and achieved sublinear regret. Secondly, the paper [20] extended the adaptive algorithm proposed in [8] to a distributed setting to solve an online convex optimization problem with a static inequality constraint. Finally, the authors of [21], [22] proposed distributed primal-dual algorithms to solve an online convex optimization with static coupled inequality constraints. In the continuous-time setting, the authors of [23] proposed a distributed algorithm to solve consensus-based online optimization with time-varying constraints.

B. Main contributions

In this paper, we propose a novel distributed online primal-dual mirror descent algorithm to solve the constrained optimization problem (1). The algorithm uses the subgradients of the local cost and constraint functions at the previous decision, but the total number of iterations or any parameters related to the objective or constraint functions are not used. Moreover, in order to influence the structure of the decision, in the primal update the regularization function is not linearized.

We derive regret and constraint violation bounds for the algorithm and show how they depend on the stepsize sequences, the accumulated variation of the comparator sequence, the number of agents, and the network connectivity. Specifically, we prove that the algorithm simultaneously achieves sublinear dynamic regret and constraint violation if the accumulated variation of the optimal sequence grows sublinearly. Moreover, we show that $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$ and $\|[\sum_{t=1}^{T} g_t(x_t)]_+\| = \mathcal{O}(T^{1-\kappa/2})$, where $\kappa \in (0,1)$ is a user-defined trade-off parameter. Compared with [7], [8], [10], [11], [22] which assumed the same assumption on the cost and constraint functions as this paper, the proposed algorithm has the following advantages. The proposed algorithm achieves the same static bound regret as in [8] but generalizes the constraint violation bound. As κ enables the user to trade-off static regret bound for constraint violation bound, we recover the $\mathcal{O}(\sqrt{T})$ static regret bound and $\mathcal{O}(T^{3/4})$ constraint violation bound from [7], [11] as special cases. However, note that the algorithms proposed in [7], [8], [11] are centralized and the constraint functions in [7], [8] are time-invariant. Moreover, in [7], [11] the total number of iterations and in [7], [8], [11] the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to design the stepsizes. We can also see that the proposed algorithm achieves smaller static regret and constraint violation bounds than [22], although time-invariant coupled inequality constraints were considered. Although the algorithm proposed in [10] achieved more strict constraint violation bound than our Algorithm 1, ithe constraint functions there are time-invariant and the algorithm is centralized also. We summarize the detailed results in Table I.

Finally, when the local objective functions are assumed to be strongly convex, we show that the proposed algorithm

TABLE I: Summary of related works on online convex optimization.

Reference	Problem type	Constraint type	Static regret and constraint violation
[7]	Centralized	$g(x) \leq \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{1/2})$, $\Vert[\sum_{t=1}^T g(x_t)]_+\Vert \leq \mathcal{O}(T^{3/4})$
[8]	Centralized	$g(x) \leq \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$, $\Vert[\sum_{t=1}^T g(x_t)]_+\Vert \leq \mathcal{O}(T^{1-\kappa/2}), \kappa \in (0,1)$
[10]	Centralized	$g(x) \leq \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{1/2})$, $\sum_{t=1}^T \Vert[g(x_t)]_+\Vert^2 \leq \mathcal{O}(T^{1/2})$
[11]	Centralized	$g_t(x) \leq 0$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{1/2})$, $\vert[\sum_{t=1}^T g_t(x_t)]_+\vert \leq \mathcal{O}(T^{3/4})$
[22]	Distributed	$g(x) = \sum_{i=1}^n g_i(x_i) \leq \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{1/2+2\kappa})$, $\Vert[\sum_{t=1}^T g(x_t)]_+\Vert \leq \mathcal{O}(T^{1-\kappa/2}), \kappa \in (0,1/4)$
This paper	Distributed	$g_t(x) = \sum_{i=1}^n g_{i,t}(x_i) \leq \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$, $\Vert[\sum_{t=1}^T g_t(x_t)]_+\Vert \leq \mathcal{O}(T^{1-\kappa/2}), \kappa \in (0,1)$

achieves $\mathcal{O}(T^{\kappa})$ static regret bound and $\mathcal{O}(T^{1-\kappa/2})$ cumulative constraint violation bound.

Notations: All inequalities and equalities are understood componentwise. $[n]$ represents the set $\{1,\dots,n\}$ for any $n \in \mathbb{N}_+$. $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced 2-norm for matrices. $\langle x, y\rangle$ represents the standard inner product of two vectors x and y. $\text{col}(z_1,\dots,z_k)$ is the concatenated column vector of vectors $z_i \in \mathbb{R}^{n_i},\ i \in [k]$. $[z]_+$ represents the component-wise projection of a vector $z \in \mathbb{R}^n$ onto $\mathbb{R}^n_+$. For a set $\mathcal{S} \subseteq \mathbb{R}^p$, $\mathcal{P}_{\mathcal{S}}(\cdot)$ is the projection operator. $\nabla f(x)$ denotes the subgradient of function f at point x.

II. Preliminaries

A. Graph Theory

Interactions between agents in the distributed algorithm are modeled by a time-varying directed graph. Specifically, at time t, agents communicate with each other according to a directed graph $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$, where $\mathcal{V} = [n]$ is the vertex set and $\mathcal{E}_t \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A directed edge $(j,i) \in \mathcal{E}_t$ means that vertex i can receive data broadcasted by vertex j at time t. Let $\mathcal{N}_i^{\text{in}}(\mathcal{G}_t) = \{j \in [n] \mid (j,i) \in \mathcal{E}_t\}$ and $\mathcal{N}_i^{\text{out}}(\mathcal{G}_t) = \{j \in [n] \mid (i,j) \in \mathcal{E}_t\}$ be the sets of in- and out-neighbors, respectively, of vertex i at time t. A directed path is a sequence of consecutive directed edges, and a graph is called strongly connected if there is at least one directed path from any vertex to any other vertex in the graph. The adjacency matrix $W_t \in \mathbb{R}^{n\times n}$ at time t fulfills $[W_t]_{ij} > 0$ if $(j,i) \in \mathcal{E}_t$ or $i = j$, and $[W_t]_{ij} = 0$ otherwise.

B. Bregman Divergences

Each agent $i \in [n]$ uses the Bregman divergence $\mathcal{D}_{\psi_i}(x,y)$ to measure the distance between $x, y \in X_i$. The Bregman divergence is defined as

$$\mathcal{D}_{\psi_i}(x,y) = \psi_i(x) - \psi_i(y) - \langle \nabla\psi_i(y), x-y\rangle, \qquad (3)$$

where $\psi_i : X_i \to \mathbb{R}$ is a differentiable and strongly convex function with convexity parameter $\sigma_i > 0$. Then, for all $x, y \in X_i$, we have $\psi_i(x) \geq \psi_i(y) + \langle\nabla\psi_i(y), x-y\rangle + \frac{\sigma_i}{2}\|x-y\|^2$. Thus, $\mathcal{D}_{\psi_i}(x,y) \geq \frac{\underline{\sigma}}{2}\|x-y\|^2,\ \forall x,y \in X_i,\ \forall i \in [n]$, where $\underline{\sigma} = \min\{\sigma_1,\dots,\sigma_n\}$. Hence, $\mathcal{D}_{\psi_i}(\cdot, y)$ is a strongly convex function with convexity parameter $\underline{\sigma}$ for all $y \in X_i$.

C. Assumptions

The following mild assumption is made on the graph.

Assumption 1. *For any $t \in \mathbb{N}_+$, the graph $\mathcal{G}_t$ satisfies the following conditions:*

1) *There exists a constant $w \in (0,1)$, such that $[W_t]_{ij} \geq w$ if $[W_t]_{ij} > 0$.*
2) *The adjacency matrix W_t is doubly stochastic, i.e., $\sum_{i=1}^n [W_t]_{ij} = \sum_{j=1}^n [W_t]_{ij} = 1,\ \forall i,j \in [n]$.*
3) *There exists an integer $\iota > 0$ such that the graph $(\mathcal{V}, \cup_{l=0,\dots,\iota-1}\mathcal{E}_{t+l})$ is strongly connected.*

We make the following standing assumption on the cost, regularization, and constraint functions.

Assumption 2.
1) *The set X_i is convex and compact for all $i \in [n]$.*
2) *$\{f_{i,t}\}$, $\{r_{i,t}\}$, and $\{g_{i,t}\}$ are convex and uniformly bounded on X_i, i.e., there exists a constant $F > 0$ such that $\|f_{i,t}(x)\| \leq F,\ \|r_{i,t}(x)\| \leq F,\ \|g_{i,t}(x)\| \leq F,\ \forall t \in \mathbb{N}_+,\ \forall i \in [n],\ \forall x \in X_i$.*
3) *The subgradients of $\{f_{i,t}\}$, $\{r_{i,t}\}$, and $\{g_{i,t}\}$ denoted as $\{\nabla f_{i,t}\}$, $\{\nabla r_{i,t}\}$, and $\{\nabla g_{i,t}\}$ exist and they are uniformly bounded on X_i, i.e., there exists a constant $G > 0$ such that $\|\nabla f_{i,t}(x)\| \leq G,\ \|\nabla r_{i,t}(x)\| \leq G,\ \|\nabla g_{i,t}(x)\| \leq G,\ \forall t \in \mathbb{N}_+,\ \forall i \in [n],\ \forall x \in X_i$.*

One mild assumption on the Bregman divergence is stated as follows.

Assumption 3. *For all $i \in [n]$ and $y \in X_i$, $\mathcal{D}_{\psi_i}(\cdot, y) : X_i \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $K > 0$ such that $|\mathcal{D}_{\psi_i}(x_1,y) - \mathcal{D}_{\psi_i}(x_2,y)| \leq K\|x_1 - x_2\|,\ \forall x_1, x_2 \in X_i$.*

This assumption is satisfied when ψ_i is Lipschitz on X_i. From Assumptions 2 and 3 it follows that

$$\mathcal{D}_{\psi_i}(x,y) \leq d(X)K,\ \forall x,y \in X_i,\ \forall i \in [n], \qquad (4)$$

where $d(X)$ is a positive constant such that $\|x - y\| \leq d(X),\ \forall x,y \in X$.

III. Distributed Online Primal-Dual Mirror Descent Algorithm

In this section, we propose a distributed online primal-dual mirror descent algorithm for solving convex optimization problem (1).

The augmented Lagrangian function associated with problem (1) at each time t is

$$\mathcal{A}_t(x_t, u_t) = f_t(x_t) + r_t(x_t) + u_t^\top g_t(x_t) - \frac{\beta_{t+1}}{2}\|u_t\|^2, \tag{5}$$

where $\{u_t \in \mathbb{R}_+^m\}$ is the dual variable or Lagrange multiplier vector sequence and $\{\beta_t > 0\}$ is the regularization sequence. Inspired by composite objective mirror descent algorithm [24], a centralized online primal-dual mirror decent algorithm to solve problem (1) is

$$\begin{aligned} x_{t+1} =& \arg\min_{x\in X}\{\alpha_{t+1}(\langle x, \nabla f_t(x_t) + (\nabla g_t(x_t))^\top u_t\rangle \\ &+ r_t(x_t)) + \mathcal{D}_\psi(x, x_t)\}, \end{aligned} \tag{6a}$$

$$u_{t+1} =[u_t + \gamma_{t+1}(g_t(x_t) - \beta_{t+1}u_t)]_+, \tag{6b}$$

where $\{\alpha_t > 0\}$ and $\{\gamma_t > 0\}$ are the stepsize sequences used in the primal and dual updates, respectively, and ψ is a strongly convex function to define the Bregman divergence. When r_t is a constant mapping, then the centralized online algorithm (6) is Algorithm 1 proposed in [11]. The potential drawback of that algorithm is that the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to choose the stepsize sequences. In order to avoid using these upper bounds, inspired by the algorithm proposed in [14], we slightly modify the dual update equation (6b) as

$$\begin{aligned} u_{t+1} = [&u_t + \gamma_{t+1}(g_t(x_t) + \nabla g_t(x_t)(x_{t+1} - x_t) \\ &- \beta_{t+1}u_t)]_+, \end{aligned} \tag{7}$$

Then we modify the centralized online primal-dual mirror decent algorithm (6a) and (7) to a distributed manner, which is given in pseudo-code as in Algorithm 1.

Remark 1. *At time t, each agent i needs to know the regularization function at the previous time $t-1$, i.e., $r_{i,t-1}(\cdot)$. This is in many situations a mild assumption since regularization functions are normally predefined to influence the structure of the decision. Furthermore, $g_{i,t-1}(x_{i,t-1})$, $\nabla f_{i,t-1}(x_{i,t-1})$, and $\nabla g_{i,t-1}(x_{i,t-1})$ rather than the full knowledge of $f_{i,t-1}(\cdot)$ and $g_{i,t-1}(\cdot)$ are needed, similar to the assumption on most online algorithms in the literature, cf., [7], [8], [10], [11], [22]. Note that the total number of iterations or any parameters related to the objective or constraint functions, such as upper bounds of the objective and constraint functions or their subgradients, are not used in the algorithm. Also note that no local information related to the primal is exchanged between the agents, but only the local dual variables.*

Algorithm 1 Distributed Online Primal-Dual Mirror Descent

Input: non-increasing sequences $\{\alpha_t > 0\}$, $\{\beta_t > 0\}$, and $\{\gamma_t > 0\}$; differentiable and strongly convex functions $\{\psi_i,\ i \in [n]\}$.
Initialize: $x_{i,0} \in X_i$, $f_{i,0}(\cdot) = r_{i,0}(\cdot) \equiv 0$, $g_{i,0}(\cdot) \equiv \mathbf{0}_m$, and $q_{i,0} = \mathbf{0}_m,\ \forall i \in [n]$.
for $t = 1, \dots, T$ **do**
  for $i = 1, \dots, n$ **do**
    Observe $\nabla f_{i,t-1}(x_{i,t-1})$, $\nabla g_{i,t-1}(x_{i,t-1})$, $g_{i,t-1}(x_{i,t-1})$, and $r_{i,t-1}(\cdot)$;
    Receive $[W_{t-1}]_{ij}q_{j,t-1},\ j \in \mathcal{N}_i^{\text{in}}(\mathcal{G}_{t-1})$;
    Update

$$\tilde{q}_{i,t} =\sum_{j=1}^n [W_{t-1}]_{ij}q_{j,t-1}, \tag{8}$$

$$\begin{aligned} a_{i,t} =&\nabla f_{i,t-1}(x_{i,t-1}) \\ &+ (\nabla g_{i,t-1}(x_{i,t-1}))^\top \tilde{q}_{i,t}, \end{aligned} \tag{9}$$

$$\begin{aligned} x_{i,t} =& \arg\min_{x\in X_i}\{\alpha_t\langle x, a_{i,t}\rangle + \alpha_t r_{i,t-1}(x) \\ &+ \mathcal{D}_{\psi_i}(x, x_{i,t-1})\}, \end{aligned} \tag{10}$$

$$\begin{aligned} b_{i,t} =&\nabla g_{i,t-1}(x_{i,t-1})(x_{i,t} - x_{i,t-1}) \\ &+ g_{i,t-1}(x_{i,t-1}), \end{aligned} \tag{11}$$

$$q_{i,t} =[\tilde{q}_{i,t} + \gamma_t(b_{i,t} - \beta_t\tilde{q}_{i,t})]_+; \tag{12}$$

    Broadcast $q_{i,t}$ to $\mathcal{N}_i^{\text{out}}(\mathcal{G}_t)$.
  end for
end for
Output: $\boldsymbol{x}_T$.

IV. Regret and Cumulative Constraint Violation Bounds

This section presents the main results on regret and constraint violation bounds for Algorithm 1. For space purposes, all proofs are omitted here, but can be found in [25].

A. Dynamic Regret and Constraint Violation Bounds

Theorem 1. *Suppose Assumptions 1–3 hold. For any $T \in \mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1 with*

$$\alpha_t = \frac{1}{t^c},\ \beta_t = \frac{1}{t^\kappa},\ \gamma_t = \frac{1}{t^{1-\kappa}},\ \forall t \in \mathbb{N}_+,$$

where $\kappa \in (0,1)$ and $c \in (0,1)$ are constants. Then,

$$\begin{aligned} &\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*) \le C_1 T^{\max\{1-c,c,\kappa\}} + 2KT^cV(\boldsymbol{x}_T^*), \\ &\|[\sum_{t=1}^T g_t(x_t)]_+\| \le \sqrt{C_2}T^{\max\{1-c/2,1-\kappa/2\}}, \end{aligned}$$

where $C_1 = \frac{C_{1,1}}{\kappa} + \frac{C_{1,2}}{1-c} + 2nd(X)K$, $C_2 = C_{2,1}(2nF + C_1)$, $C_{1,1} = \frac{3n^2\tau B_1 F}{1-\lambda} + \frac{n(B_1)^2}{2}$, $C_{1,2} = \frac{4nG^2}{\sigma}$, $\tau = (1 - w/2n^2)^{-2} > 1$, $B_1 = 2F + Gd(X)$, $\lambda = (1 - w/2n^2)^{1/\iota}$ and $C_{2,1} = 2n(\frac{2G^2}{(1-c)\sigma} + \frac{1}{1-\kappa} + 2)$ are constants independent of T; and $V(\boldsymbol{x}_T^*) = \sum_{t=1}^{T-1}\sum_{i=1}^n \|x_{i,t+1}^* - x_{i,t}^*\|$ is the accumulated variation of the sequence $\boldsymbol{x}_T^*$.

Remark 2. *Note that the dependence on the stepsize sequences, the accumulated variation of the comparator sequence, the number of agents, and the network connectivity is characterized in the regret and constraint violation bounds above. Sublinear dynamic regret and constraint violation is thus achieved if $V(\boldsymbol{x}_T^*)$ grows sublinearly. If, in this case, there exists a constant $\nu \in [0, 1)$, such that $V(\boldsymbol{x}_T^*) = \mathcal{O}(T^\nu)$, then setting $c \in (0, 1 - \nu)$ in Theorem 1 gives $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*) = \mathbf{o}(T)$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathbf{o}(T)$.*

B. Static Regret and Constraint Violation Bounds

Replacing $\boldsymbol{x}_T^*$ by the static sequence $\check{\boldsymbol{x}}_T^*$ in Theorem 1 gives the following results.

Corollary 1. *Under the same conditions as stated in Theorem 1, it holds that*

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \le C_1 T^{\max\{1-\kappa,\kappa\}},$$
$$\|[\sum_{t=1}^T g_t(x_t)]_+\| \le \sqrt{C_2} T^{1-\kappa/2}.$$

Remark 3. *From Corollary 1, we know that Algorithm 1 achieves the same static bound regret as in [8] but generalizes the constraint violation bound. As discussed in [8], $\kappa \in (0, 1)$ is a user-defined trade-off parameter which enables the user to trade-off static regret bound for constraint violation bound depending on the application. Corollary 1 recovers the $\mathcal{O}(\sqrt{T})$ static regret bound and $\mathcal{O}(T^{3/4})$ constraint violation bound from [7], [11] when $\kappa = 0.5$. Moreover, the result extends the $\mathcal{O}(T^{2/3})$ bound for both static regret and constraint violation achieved in [7] for linear constraint functions. However, the algorithms proposed in [7], [8], [11] are centralized and the constraint functions considered in [7], [8] are time-invariant. Moreover, in [7], [11] the total number of iterations and in [7], [8], [11] the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to choose the stepsize sequences. Furthermore, Corollary 1 achieves smaller static regret and constraint violation bounds than [22], although [22] considered time-invariant coupled inequality constraints. However, [22] did not require the time-varying directed graph to be balanced. Although the algorithm proposed in [10] achieved more strict constraint violation bound than our Algorithm 1, it is time-invariant constraint functions that were considered and the algorithm is centralized also.*

The static regret bound in Corollary 1 can be reduced, if a generalized strong convexity of the local objective functions $f_{i,t}+r_{i,t}$ is assumed. We put the generalized strong convexity assumption on the local cost functions $f_{i,t}$, so $r_{i,t}$ can be simply convex, such as an ℓ_1-regularization.

Assumption 4. *For any $i \in [n]$ and $t \in \mathbb{N}_+$, $\{f_{i,t}\}$ are μ_i-strongly convex over X_i with respect to ψ_i with $\mu_i > 0$, i.e., for all $x, y \in \mathcal{X}_i$ and $t \in \mathbb{N}_+$,*

$$f_{t,i}(x) \ge f_{t,i}(y) + \langle x - y, \nabla f_{t,i}(y)\rangle + \mu_i \mathcal{D}_{\psi_i}(x, y). \quad (13)$$

Theorem 2. *Suppose Assumptions 1–4 hold. For any $T \in \mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1 with*

$$\alpha_t = \frac{1}{t^{\max\{1-\kappa,\kappa\}}},\ \beta_t = \frac{1}{t^\kappa},\ \gamma_t = \frac{1}{t^{1-\kappa}},\ \forall t \in \mathbb{N}_+,$$

where $\kappa \in (0, 1)$. Then,

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \le \max\{C_1, C_4\} T^\kappa,$$
$$\|[\sum_{t=1}^T g_t(x_t)]_+\| \le \sqrt{C_2} T^{1-\kappa/2},$$

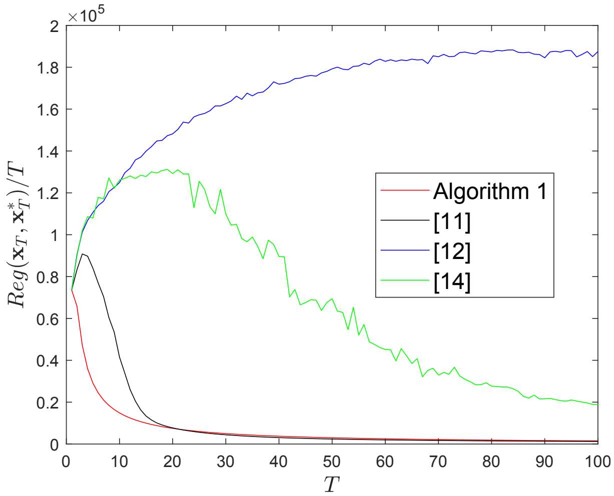
where $C_4 = \frac{n(B_1)^2}{2\kappa} + \frac{B_1 C_{1,1}}{\kappa} + \frac{C_{1,2}}{\kappa} + 2nd(X)K(B_4)^{1-\kappa}$, $B_4 = (\underline{\mu})^{-1/\kappa} + 1$, and $\underline{\mu} = \min\{\mu_1, \dots, \mu_n\}$ are constants independent of T.

V. NUMERICAL SIMULATIONS

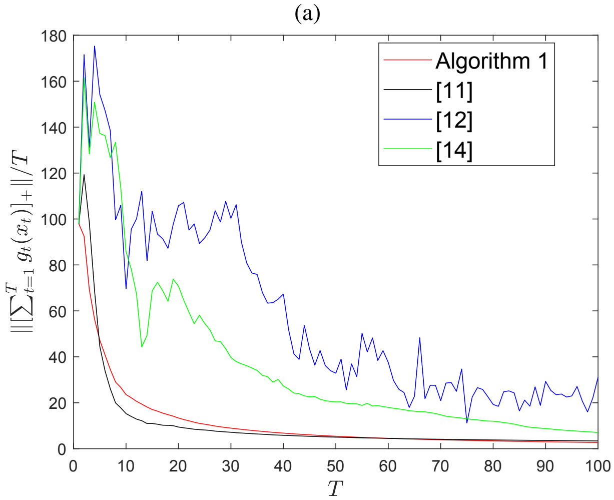
Consider online convex optimization with local cost functions $f_{i,t}(x_i) = \zeta_{i,1}\langle \pi_{i,t}, x_i\rangle + \zeta_{i,2}\|x_i - y_{i,t}\|^2$, where $\zeta_{i,1}$ and $\zeta_{i,2}$ are nonnegative constants, and $\pi_{i,t}, y_{i,t} \in \mathbb{R}^{p_i}$ are time-varying and unknown at time t; local regularization functions $r_{i,t}(x_i) = \lambda_{i,1}\|x_i\|_1 + \lambda_{i,2}\|x_i\|^2$, where $\lambda_{i,1}$ and $\lambda_{i,2}$ are nonnegative constants; and local constraint functions $g_{i,t}(x_i) = D_{i,t}x_i - d_{i,t}$, where $D_{i,t} \in \mathbb{R}^{m\times p_i}$ and $d_{i,t} \in \mathbb{R}^m$ are time-varying and unknown at time t. The above problem formulation arises often in network resource allocation, smart grid control, estimation in sensor networks, and so on.

In the simulations, for each agent $i \in [n]$, the strongly convex function $\psi_i(x) = \sigma\|x\|^2$ is used to define the Bregman divergence $\mathcal{D}_{\psi_i}$. Thus, $\mathcal{D}_{\psi_i}(x, y) = \sigma\|x-y\|^2, \forall i \in [n]$. The stepsize sequences proposed in Theorem 2 are used. At each time t, an undirected graph is used as the communication graph. Specifically, connections between vertices are random and the probability of two vertices being connected is ρ. Moreover, in order to guarantee Assumption 1 holds, edges $(i, i + 1)$, $i \in [n - 1]$ are added and $[W_t]_{ij} = \frac{1}{n}$ if $(j, i) \in \mathcal{E}_t$ and $[W_t]_{ii} = 1 - \sum_{j\in\mathcal{N}_i^{\mathrm{in}}(\mathcal{G}_t)}[W_t]_{ij}$. We assume $n = 50$, $m = 5$, $\sigma = 10$, $p_i = 6$, $X_i = [0, 5]^{p_i}$, $\zeta_{i,1} = \lambda_{i,1} = 1$, $\zeta_{i,2} = \lambda_{i,2} = 30$, $i \in [n]$, and $\rho = 0.2$. Each component of $\pi_{i,t}$ is drawn from the discrete uniform distribution in $[0, 10]$ and each component of $D_{i,t}$ is drawn from the discrete uniform distribution in $[-5, 5]$. We let $y_{i,t} = [2(\zeta_{i,2} + \lambda_{i,2})x_{i,t}^0 + \zeta_{i,1}\pi_{i,t} + \lambda_{i,1}\mathbf{1}_{p_i}]/(2\zeta_{i,2})$, where $x_{i,t+1}^0 = A_{i,t}x_{i,t}^0$ with $A_{i,t}$ being a doubly stochastic matrix and $x_{i,1}^0$ being a vector that is uniformly drawn from X_i. In order to guarantee the constraints are feasible, we let $d_{i,t} = D_{i,t}x_{i,t}^0$.

We compare Algorithm 1 with the centralized online algorithms in [11], [12], [14]. Here, Algorithm 1 in [11] with $\alpha = 10$, $\delta = 1$, and $\mu = 1/\sqrt{T}$, Algorithm 1 in [12] with $\alpha = \mu = T^{-1/3}$, and the virtual queue algorithm in [14] with $V = \sqrt{T}$ and $\alpha = V^2$ are used. Figs. 1 (a) and (b) show the evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|\sum_{t=1}^T [g_t(x_t)]_+\|/T$, respectively, for these algorithms. From these two figures, we can see that in this example Algorithm 1 achieves smaller dynamic regret and constraint violation than the algorithms in [12], [14] and almost the same values as the algorithm in [11].



(a)



(b)

Fig. 1: Comparison of different algorithms: (a) Evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$; (b) Evolutions of $\|[\sum_{t=1}^{T} g_t(x_t)]_+\|/T$.

VI. CONCLUSION

In this paper, we considered an online convex optimization problem with time-varying coupled inequality constraints. We proposed a distributed online primal-dual mirror descent algorithm to solve this problem. We derived regret and constraint violation bounds for the algorithm and showed how they depend on the stepsize sequences, the accumulated variation of the comparator sequence, the number of agents, and the network connectivity. As a result, we proved that the algorithm achieves sublinear regret and constraint violation for both arbitrary and strongly convex objective functions. We showed that the algorithm and results in this paper can be cast as extensions of existing algorithms and results. Future research directions include extending the algorithm with bandit feedback.

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