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# Global consensus in homogeneous networks of discrete-time agents subject to actuator saturation

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***Abstract*— In this paper, we give necessary conditions for achieving global consensus in homogeneous networks of discrete-time linear time-invariant agents with input saturation constraints under fixed undirected topologies. For two special cases, where the agent model is either neutrally stable or a double integrator, these necessary conditions together with a gain condition are also sufficient. We show this by designing a linear protocol based on the combination of state differences between the agent and its neighbors. In particular, for the neutrally stable case, we show that any linear protocol of a particular form which solves the consensus problem for the case without input saturation constraints also solves the global consensus problem for the case with input saturation constraints. For the double integrator case, we show that a subset of linear protocols which solve the consensus problem for the case without saturation constraints also solve the global consensus problem in the presence of input saturation. The results are illustrated by numerical simulations.**

## I. Introduction

In recent years, the consensus problem for multi-agent systems (MAS) has received substantial attention, e.g., [1]–[6]. To design a consensus protocol, each agent has to implement a distributed protocol based on the limited information about itself and its neighboring agents. The design on consensus protocols can be generally divided into two categories depending on whether the agent models are continuous-time or discrete-time. Much attention has been directed to the continuous-time case. The existing works here can be categorized into two directions depending whether the agent models are identical or not. The consensus problem for homogeneous networks (i.e., networks where the agent models are identical) has been considered in [4], [7]–[13], while the consensus problem for heterogenous networks (i.e., networks where the agent models are non-identical) has been considered in [14]–[17]. The studies on the discrete-time case is rather limited, but can be found in [3], [7], [18]–[23].

Most aforementioned papers do not consider the case where the agents are subject to actuator saturation. However, in every physical application, the actuator has bounds on its input, and thus actuator saturation is important to study. The protocol design for achieving consensus for the case with input saturation constraints is a challenging problem, and only few results are available for the continuous-time agent models, e.g., [24]–[27]. In [25], for the single integrator case, the authors showed that any linear protocol based on the relative state information which solves the consensus problem for the case without input saturation constraints under fixed directed network topologies also solves the *global* consensus problem in the presence of input saturation constraints. Meng *et al* [26] proposed a linear protocol based on the relative state information to solve the global consensus problem for a MAS with input saturation constraints under fixed undirected network topologies and time varying topologies. Yang *et al* [27] solved the *semi-global* regulation of output synchronization for heterogeneous networks under fixed directed network topologies.

To the best of the authors' knowledge, all the existing works on the consensus problem for a MAS with input saturation constraints are restricted to the case where the agents models are continuous-time. This motivates us to consider the consensus problem for the case where the agents models are discrete-time. As a first step, in this paper, we assume the network topology is fixed and undirected.

The remainder of the paper is organized as follows. In Section II, some preliminaries and notations are introduced. In Section III, we first formulate the global consensus problem with input saturation constraints, and then give necessary conditions for achieving global consensus under fixed undirected topologies. In Section IV and Section V, we consider the case where the agent model is neutrally stable and a double integrator respectively. Simulation examples are presented in Section VI followed by conclusions.

## II. Preliminaries and notations

In this paper, we assume that the communication topology among the agents is described by a fixed undirected weighted graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, with the set of agents $\mathscr{V} = \{1,\ldots,N\}$, the set of undirected edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if and only if the edge $(i, j) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. We also assume that there is no self-loop, i.e., $a_{ii} = 0$ for $i \in \{1,\ldots,N\}$. The set of neighboring agents of agent $i$ is defined as $\mathscr{N}_i = \{j \in \mathscr{V} | a_{ij} > 0\}$. A path from node $i_1$ to $i_k$ is a sequence of nodes $\{i_1,\ldots,i_k\}$ such that $(i_j, i_{j+1}) \in \mathscr{E}$ for $j = 1,\ldots,k-1$ in the undirected graph. An undirected graph is connected if there exists a path between any pair of distinct nodes.

For an undirected weighted graph $\mathscr{G}$, a matrix $L = [\ell]_{ij} \in \mathbb{R}^{N\times N}$ with $\ell_{ii} = \sum_{j=1}^{N} a_{ij}$ and $\ell_{ij} = -a_{ij}$ for $j \neq i$, is called Laplacian matrix associated with graph $\mathscr{G}$. It is well-known that the Laplacian matrix has the property that all the row sums are zero. If the undirected weighted graph $\mathscr{G}$ is connected, then $L$ has a simple eigenvalue at zero with

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corresponding right eigenvector $\mathbf{1}$ and all other eigenvalues are strictly positive. All the eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N \leq 2\Delta$, where $\Delta = \max_{i\in\{1,\ldots,N\}} \ell_{ii}$.

Given a matrix $A$, $A^{\mathrm{T}}$ denotes its transpose and $\|A\|$ denotes its induced norm. A symmetric matrix $A$ is positive (negative) definite if and only if all its eigenvalues are positive (negative), and is positive (negative) semi-definite if and only if all its eigenvalues are non-negative (non-positive). We denote by $A \otimes B$ the Kronecker product between matrices $A$ and $B$. For two column vectors $a$ and $b$ of the same dimensions, $a < (\leq) b$ means that each entry of $a-b$ is negative (non-positive), while $a > (\geq) b$ means that each entry of $a-b$ is positive (non-negative). $I_N$ denotes an identity matrix of dimension $N \times N$. $\mathbf{1}_N$ denotes the column vector with each entry being 1. For column vectors $x_1,\ldots,x_N$, the stacking column vector of $x_1,\ldots,x_N$ is denoted by $[x_1;\ldots;x_N]$.

## III. Problem Formulation

We consider a MAS of $N$ identical discrete-time agents

$$x_i(k+1) = Ax_i(k) + B\sigma(u_i(k)), \quad i \in \mathscr{V}, \tag{1}$$

where $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^m$,

$$\sigma(u_i(k)) = [\sigma_1(u_{i,1}(k)); \sigma_1(u_{i,2}(k)); \ldots; \sigma_1(u_{i,m}(k))],$$

and each $\sigma_1(u)$ is the standard saturation function

$$\sigma_1(u) = \begin{cases} 1 & \text{if } u > 1, \\ u & \text{if } |u| \leq 1, \\ -1 & \text{if } u < -1. \end{cases}$$

The only information available for agent $i$ comes from the network. In particular, agent $i$ receives a linear combination of its own state relative to that of neighboring agents, i.e., agent $i$ has access to the quantity

$$\zeta_i(k) = \sum_{j\in\mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)).$$

Our goal is to design distributed protocols $u_i(k)$ by using $\zeta_i(k)$ to solve the *global* consensus problem, that is, for any initial conditions $x_i(0)$, $i \in \mathscr{V}$, $\lim_{k\to\infty}(x_i(k) - x_j(k)) = 0$ for all $i, j \in \mathscr{V}$.

Each agent is subject to the input saturation constraints. These nonlinearities make the protocol design for achieving global consensus difficult since we have to guarantee that consensus is achieved for all initial conditions.

*Assumption 1:* The agent model (1) is asymptotically null controllable with bounded controls (ANCBC), i.e., the pair $(A,B)$ is stabilizable and all the eigenvalues of the matrix $A$ are within or on the unit circle.

Based on the result in [28], we have the following result.

*Proposition 1:* Global consensus for a MAS of $N$ agents (1) via distributed protocols $u_i(k) = f_i(\zeta_i(k), k)$ is possible only if Assumption 1 is satisfied.

From the saturation literature [28], [29], in general we need to design nonlinear protocols to solve the global consensus problem. In this paper, we shall concentrate on a linear protocol

$$u_i = K\zeta = K \sum_{j\in\mathscr{N}_i} a_{ij}(x_i - x_j), \quad i \in \mathscr{V}, \tag{2}$$

as such a protocol may suffice in some cases. [1]

Given a fixed undirected graph and Assumption 1, it follows from [23][Theorem 3.1] that a MAS achieves consensus under the protocol (2) if and only if the following assumption is satisfied.

*Assumption 2:* The graph $\mathscr{G}$ is connected.

This together with Proposition 1 yields the following result:

*Proposition 2:* Assumptions 1 and 2 are necessary for a MAS of $N$ agents (1) to achieve global consensus under the protocol (2).

There is limited knowledge regarding which linear systems with input saturation allow for global stabilization via linear state feedback control laws. It is known that for some special discrete-time cases, that is, open-loop neutrally stable system[2] [30], and double integrator [31], there exist linear state feedback control laws which globally asymptotically stabilize the linear system in the presence of input saturation. Hence, in the following sections, we consider the global consensus problem under a distributed linear protocol (2) for such special cases. We show that Assumptions 1 and 2 are also sufficient for achieving global consensus for such special cases by designing the matrix $K$ for the linear protocol (2) to solve the global consensus problem.

## IV. Neutrally stable case

In this section, we consider the case where the agent model (1) is open-loop neutrally stable. Under Assumption 1, there exists a non-singular state transformation $T^{-1}$, such that

$$A = T^{-1}\begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix} T, \quad B = T^{-1}\begin{bmatrix} B_c \\ B_s \end{bmatrix},$$

where $A_c^{\mathrm{T}}A_c = I$, $A_s$ is Schur stable (i.e., all its eigenvalues are within the unit circle), and the pair $(A_c, B_c)$ is controllable.

As shown in [23], the asymptotically stable modes can be ignored since we can set the corresponding gain matrix to zero. Thus, without loss of generality, we make the following assumption in this section.

*Assumption 3:* $A^{\mathrm{T}}A = I_n$ and the pair $(A,B)$ is controllable.

Under Assumption 3, controllability of the pair $(A,B)$ is equivalent to stabilizability of the pair $(A,B)$.

Consider the following protocol

$$u_i = -\varepsilon B^{\mathrm{T}}A \sum_{j\in\mathscr{N}_i} a_{ij}(x_i - x_j), \quad i \in \mathscr{V}. \tag{3}$$

Note that the above protocol (3) is of the form (2) with $K = -\varepsilon B^{\mathrm{T}}A$, where $\varepsilon$ is a designed parameter to be specified. The following lemma shows that the protocol (3) with properly chosen $\varepsilon$ solves the consensus problem for a MAS without input saturation.

[1] To simplify the notation, sometimes $x$ or $u$ without explicitly indicating the time instant will refer to $x(k)$ or $u(k)$ respectively.

[2] A discrete-time system is said to be open-loop neutrally stable if, all its open-loop poles are within or on the unit circle with those on the unit circle being simple.

*Lemma 1:* Consider a MAS of $N$ identical agents (1) in the absence of actuator saturation. Assume that Assumptions 2 and 3 are satisfied. Then any linear protocol (3) with $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$ solves the consensus problem.

*Proof:* It is well known [9], [22] that the consensus problem for a network of $N$ identical agents is equivalent to the simultaneous stabilization problem of $N-1$ systems. Hence, it can be verified that consensus is achieved via (3) if all the matrices $A - \varepsilon\lambda_i BB^{\mathrm{T}}A$, where $\lambda_i$, $i \in \{2,\dots,N\}$ are the nonzero eigenvalues of the Laplacian matrix, are Schur stable. It then follows from [32] that all these matrices are Schur stable if $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$. ■

The following theorem shows that any protocol of the form (3) with $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$ also solves the global consensus problem for a MAS with input saturation constraints.

*Theorem 1:* Consider a MAS of $N$ identical agents (1). Assume that Assumptions 1, 2 and 3 are satisfied. Then any protocol (3) with $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$ solves the global consensus problem.

*Proof:* Define $x(k) = [x_1(k);\dots;x_N(k)]$ and $u(k) = [u_1(k);\dots,u_N(k)]$. With these quantities, we obtain the following dynamics

$$x(k+1) = (I_N \otimes A)x(k) + (I_N \otimes B)\sigma(u(k)), \tag{4a}$$
$$u(k) = -\varepsilon(L \otimes B^{\mathrm{T}}A)x(k). \tag{4b}$$

Motivated by the Lyapunov candidate in [24], we consider the following Lyapunov candidate

$$V(k) = \frac{1}{2}x^{\mathrm{T}}(k)(L \otimes I_n)x(k).$$

Define a manifold where all the agents' states are identical as $M := \{x \in \mathbb{R}^{Nn} | x_1 = x_2 = \dots = x_N\}$. Note that $V(k) \geq 0$ and $V(k) = 0$ if and only if $x \in M$.

From the dynamics of (4), we obtain

$$\begin{aligned}\Delta V &= \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}A)x + \frac{1}{2}x^{\mathrm{T}}(L \otimes A^{\mathrm{T}}B)\sigma(u) \\ &\quad + \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}B)\sigma(u) \\ &= -\frac{1}{\varepsilon}\sigma^{\mathrm{T}}(u)u + \frac{1}{2}\sigma^{\mathrm{T}}(u)(L \otimes B^{\mathrm{T}}B)\sigma(u) \\ &\leq -\sigma^{\mathrm{T}}(u)(\frac{1}{\varepsilon}I_{Nm} - \frac{1}{2}L \otimes B^{\mathrm{T}}B)\sigma(u),\end{aligned}$$

where we have used that $L = L^{\mathrm{T}}$ for undirected graph and that $z^{\mathrm{T}}\sigma(z) \geq \sigma^{\mathrm{T}}(z)\sigma(z)$ for any column vector $z$.

Since $\varepsilon \in (0, \frac{2}{\lambda_N \|B^{\mathrm{T}}B\|})$, $\Delta V \leq 0$ and $\Delta V = 0$ if and only if $(L \otimes B^{\mathrm{T}}A)x = 0$. We shall show that $(L \otimes B^{\mathrm{T}}A)x = 0$ if and only if $x \in M$, which in turn implies that $\Delta V = 0$ if and only if $x \in M$. We first note that if $x \in M$, then $(L \otimes B^{\mathrm{T}}A)x = 0$ since the graph is connected. We then need to show that $(L \otimes B^{\mathrm{T}}A)x = 0$ implies that $x \in M$. Note that $(L \otimes B^{\mathrm{T}}A)x = 0$ implies that $(\tilde{L} \otimes B^{\mathrm{T}}A)q = 0$, where the relative state $q = [q_2;\dots;q_N]$, $q_i = x_i - x_1$ for $i \in \{2,\dots,N\}$, and

$$\tilde{L} = \begin{bmatrix} \ell_{2,2} - \ell_{1,2} & \dots & \ell_{2,N} - \ell_{1,N} \\ \vdots & \ddots & \vdots \\ \ell_{N,2} - \ell_{1,2} & \dots & \ell_{N,N} - \ell_{1,N} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}. \tag{5}$$

Since the graph is connected, from [22, Lemma 1], we know that the eigenvalues of $\tilde{L}$ are nonzero eigenvalues of the matrix $L$, which are positive. Thus, the matrix $\tilde{L}$ is non-singular, i.e., $\operatorname{rank}(\tilde{L}) = N-1$.

Since $A^{\mathrm{T}} = A^{-1}$ which can obtained by $A^{\mathrm{T}}A = I_n$, we see that $(\tilde{L} \otimes B^{\mathrm{T}}A)q = 0$ implies that $q^{\mathrm{T}}(\tilde{L} \otimes A^{-1}B) = 0$. We then note that $q(k+1) = (I_{N-1} \otimes A)q(k)$ since $u(k) = -\varepsilon(L \otimes B^{\mathrm{T}}A)x = 0$. Therefore

$$(\tilde{L} \otimes B^{\mathrm{T}}A)q(k+1) = (\tilde{L} \otimes B^{\mathrm{T}}A)(I_{N-1} \otimes A)q = (\tilde{L} \otimes B^{\mathrm{T}}A^2)q,$$

which is equivalent to $q^{\mathrm{T}}(\tilde{L} \otimes A^{-2}B) = 0$. By iteration, we obtain $q^{\mathrm{T}}(\tilde{L} \otimes A^{-r}B) = 0$ for $r = 3,4,\dots,n+1$. Hence,

$$q^{\mathrm{T}}\left(\tilde{L} \otimes A^{-(n+1)}\begin{bmatrix} A^nB & \dots & AB & B \end{bmatrix}\right) = 0. \tag{6}$$

Since the pair $(A,B)$ is controllable, we know that

$$\operatorname{rank}\left(\begin{bmatrix} A^nB & \dots & AB & B \end{bmatrix}\right) = n.$$

Since the matrix $A$ is non-singular, it is easy to see that

$$\operatorname{rank}\left(A^{-(n+1)}\begin{bmatrix} A^nB & \dots & AB & B \end{bmatrix}\right) = n.$$

Finally, using the property of Kronecker product, we obtain

$$\begin{aligned}&\operatorname{rank}\left(\tilde{L} \otimes A^{-(n+1)}\begin{bmatrix} A^nB & \dots & AB & B \end{bmatrix}\right) \\ &= \operatorname{rank}(\tilde{L})\operatorname{rank}(A^{-(n+1)}\begin{bmatrix} A^nB & \dots & AB & B \end{bmatrix}) = (N-1)n.\end{aligned}$$

Therefore, the only solution of (6) is $q = 0$, which is equivalent to $x_1 = \dots = x_N$, i.e., $x \in M$. Hence, we have shown that $\Delta V \leq 0$ and $\Delta V = 0$ if and only if $x \in M$.

Since $\Delta V \leq 0$, we conclude that $V(k)$ is non-increasing. Thus, $\lim_{k\to\infty} V(k) = V_*$ for some $V_* \geq 0$. This implies that $\Delta V(k) \to 0$ as $k \to \infty$ and hence $x(k) \to M$ as $k \to \infty$ as shown above. Hence, global consensus is achieved. ■

## V. Double Integrator

In this section, we consider the case where the agent model (1) is a double integrator, that is, we make the following assumption in this section.

*Assumption 4:* In (1), $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Let us first recall the following result which gives conditions on the feedback gain parameters for achieving the consensus without saturation constraints.

*Lemma 2:* [33] Consider a MAS of $N$ identical agents described by

$$\begin{bmatrix} x_i(k+1) \\ v_i(k+1) \end{bmatrix} = A\begin{bmatrix} x_i(k) \\ v_i(k) \end{bmatrix} + Bu_i, \quad i \in \{1,\dots,N\}. \tag{7}$$

Assume that Assumptions 2 and 4 is satisfied. Then the protocol of the form (2) with $K = -[\alpha, \beta]$, that is,

$$u_i(k) = -\alpha\sum_{j\in\mathscr{N}_i} a_{ij}(x_i(k) - x_j(k)) - \beta\sum_{j\in\mathscr{N}_i} a_{ij}(v_i(k) - v_j(k)), \tag{8}$$

solves the consensus problem if and only if

$$0 < \alpha < \beta < \frac{\alpha}{2} + \frac{2}{\lambda_N}. \tag{9}$$

The following theorem shows that a subset of the protocols (8) which solve the consensus problem for a MAS without input saturation constraints, that is (9) is satisfied, also solves the global consensus problem for a MAS with input saturation constraints.

*Theorem 2:* Consider a MAS of $N$ identical agents (1). Assume that Assumptions 2 and 4 are satisfied. Then the linear protocol (8) with

$$0<\sqrt{3}\alpha<\beta<\frac{3}{2\lambda_N}, \tag{10}$$

solves the global consensus problem.

*Proof:* Define $x(k) = [x_1(k);\ldots;x_N(k)]$, $v(k) = [v_1(k);\ldots;v_N(k)]$, $u(k) = [u_1(k);\ldots;u_N(k)]$, $y_i(k) = [x_i(k);v_i(k)]$, and $y = [y_1(k);\ldots;y_N(k)]$. With these quantities, we obtain the following dynamics:

$$y(k+1) = (I_N\otimes A)y(k)+(I_N\otimes B)\sigma(u(k)), \tag{11a}$$
$$u(k) = \left(L\otimes\begin{bmatrix}-\alpha & -\beta\end{bmatrix}\right)y(k). \tag{11b}$$

We also obtain the following dynamics

$$x(k+1) = x(k)+v(k), \tag{12a}$$
$$v(k+1) = v(k)+\sigma(u(k)). \tag{12b}$$

Note that $u(k)$ can be written in terms of $x(k)$ and $v(k)$ as

$$u(k) = -\alpha Lx(k)-\beta Lv(k). \tag{13}$$

Hence, we get that

$$u(k+1) = u(k)-\alpha Lv(k)-\beta L\sigma(u(k)). \tag{14}$$

Motivated by the Lyapunov candidate for a discrete-time double integrator with saturated linear state feedback control laws in [31], we consider the following Lyapunov candidate

$$V = -\sigma^{\mathrm{T}}(u)\sigma(u)+2\sigma^{\mathrm{T}}(u)u+2\alpha\sigma^{\mathrm{T}}(u)Lv+\alpha v^{\mathrm{T}}Lv.$$

Similar as the proof of Theorem 1, we define a manifold where all the agents' states are identical as

$$M := \{y\in\mathbb{R}^{2N}|x_1=x_2=\ldots=x_N,\, v_1=v_2=\ldots v_N\}.$$

Note that $V=0$ if $y\in M$. We will show that $V=0$ only if $y\in M$. Since $\sigma^{\mathrm{T}}(z)z\geq\sigma^{\mathrm{T}}(z)\sigma(z)$ for any column vector $z$, where the equality holds if and only if $-\mathbf{1}\leq z\leq\mathbf{1}$, we obtain

$$V \geq \sigma^{\mathrm{T}}(u)\sigma(u)+2\alpha\sigma^{\mathrm{T}}(u)Lv+\alpha v^{\mathrm{T}}Lv \tag{15}$$
$$= \begin{bmatrix}\sigma(u)\\ Lv\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}1 & \alpha\\ \alpha & \frac{2}{3}\alpha\beta\end{bmatrix}\begin{bmatrix}\sigma(u)\\ Lv\end{bmatrix}+v^{\mathrm{T}}(\alpha L-\frac{2}{3}\alpha\beta L^{\mathrm{T}}L)v, \tag{16}$$

where the equality of (15) holds if and only if $-\mathbf{1}\leq u\leq\mathbf{1}$. Since $\beta>\sqrt{3}\alpha>\frac{3}{2}\alpha>0$, the first term of (16) is non-negative, and equal to zero if and only if $\sigma(u)=0$ and $Lv=0$. Note that from (13), we see that $u=-\alpha Lx-\beta Lv=-\alpha Lx$, therefore, $Lx=0$ since $\alpha\neq 0$. Thus, the first term equal to zero if and only if $y\in M$. We then show that the second term is also non-negative. Since $L=L^{\mathrm{T}}$, we see that the eigenvalues of the matrix $\alpha L-\frac{2}{3}\alpha\beta L^{\mathrm{T}}L$ are $\alpha\lambda_i(1-\frac{2}{3}\beta\lambda_i)$, where $\lambda_i$, $i\in\{1,\ldots,N\}$ are the eigenvalues of the Laplacian matrix $L$. Since $\beta\lambda_N<\frac{3}{2}$, the second term is non-negative and equal to zero if and only if $Lv=0$. Therefore, $V\geq 0$ and $V=0$ if and only if $y\in M$.

Next, we show that $\Delta V(k)=V(k+1)-V(k)\leq 0$. Let $t=\sigma(u(k+1))$. Note that $-\mathbf{1}\leq t\leq\mathbf{1}$ by the definition of the saturation function. With some algebra, we get

$$\begin{aligned}V(k+1) = &-t^{\mathrm{T}}t+2t^{\mathrm{T}}u+2(\alpha-\beta)t^{\mathrm{T}}L\sigma(u)+\alpha v^{\mathrm{T}}Lv\\ &+2\alpha v^{\mathrm{T}}L\sigma(u)+\alpha\sigma^{\mathrm{T}}(u)L\sigma(u).\end{aligned}$$

Thus,

$$\begin{aligned}\Delta V = &-t^{\mathrm{T}}t+2t^{\mathrm{T}}u+2(\alpha-\beta)t^{\mathrm{T}}L\sigma(u)\\ &+\sigma^{\mathrm{T}}(u)(\alpha L+I)\sigma(u)-2\sigma^{\mathrm{T}}(u)u.\end{aligned}$$

Without loss of generality, we assume that $u_i>1$ for $i\in\{1,\ldots,N_1\}:=S_p$, $|u_i|\leq 1$ for $i\in\{N_1+1,\ldots,N_2\}:=S_m$, and $u_i<-1$ for $i\in\{N_2+1,\ldots,N\}:=S_q$, since if this is not the case, we can always relabel the nodes to achieve this. Note that the sets $S_p$, $S_m$, and $S_q$ may be empty. We then partition $t=[t_p;t_m;t_q]$, $u=[u_p;u_m;u_q]$, where $t_p,u_p\in\mathbb{R}^{N_1}$, $t_m,u_m\in\mathbb{R}^{N_2-N_1}$, and $t_q,u_q\in\mathbb{R}^{N-N_2}$ are defined accordingly. Finally, we partition the Laplacian matrix $L$ as $L=\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\\ L_{pm}^{\mathrm{T}} & L_{mm} & L_{mq}\\ L_{pq}^{\mathrm{T}} & L_{mq}^{\mathrm{T}} & L_{qq}\end{bmatrix}$, where $L_{pp}$, $L_{pm}$, $L_{pq}$, $L_{mm}$, $L_{mq}$ and $L_{qq}$ are real matrices of appropriate dimensions.

With some algebra, we obtain

$$\begin{aligned}\Delta V = &-t_p^{\mathrm{T}}t_p-t_m^{\mathrm{T}}t_m-t_q^{\mathrm{T}}t_q+2t_p^{\mathrm{T}}u_p+2t_m^{\mathrm{T}}u_m+2t_q^{\mathrm{T}}u_q\\ &+2(\alpha-\beta)\begin{bmatrix}t_p^{\mathrm{T}} & t_m^{\mathrm{T}} & t_q^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\\ L_{pm}^{\mathrm{T}} & L_{mm} & L_{mq}\\ L_{pq}^{\mathrm{T}} & L_{mq}^{\mathrm{T}} & L_{qq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}\\ &+\alpha\begin{bmatrix}\mathbf{1}_p^{\mathrm{T}} & u_m^{\mathrm{T}} & -\mathbf{1}_q^{\mathrm{T}}\end{bmatrix}L\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p^{\mathrm{T}}\mathbf{1}_p+u_m^{\mathrm{T}}u_m+\mathbf{1}_q^{\mathrm{T}}\mathbf{1}_q\\ &-2\begin{bmatrix}\mathbf{1}_p^{\mathrm{T}} & u_m^{\mathrm{T}} & -\mathbf{1}_q^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}u_p\\ u_m\\ u_q\end{bmatrix}\\ = &\,2(t_p-\mathbf{1}_p)^{\mathrm{T}}(u_p-\mathbf{1}_p)+2t_p^{\mathrm{T}}\mathbf{1}_p-2\mathbf{1}_p^{\mathrm{T}}\mathbf{1}_p\\ &+2(t_q+\mathbf{1}_q)^{\mathrm{T}}(u_q+\mathbf{1}_q)-2t_q^{\mathrm{T}}\mathbf{1}_q-2\mathbf{1}_q^{\mathrm{T}}\mathbf{1}_q\\ &-t_p^{\mathrm{T}}t_p+2t_p^{\mathrm{T}}\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]-2t_p^{\mathrm{T}}\mathbf{1}_p\\ &-t_m^{\mathrm{T}}t_m+2t_m^{\mathrm{T}}\left[(\alpha-\beta)\begin{bmatrix}L_{pm}^{\mathrm{T}} & L_{mm} & L_{mq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+u_m\right]\\ &-t_q^{\mathrm{T}}t_q+2t_q^{\mathrm{T}}\left[(\alpha-\beta)\begin{bmatrix}L_{pq}^{\mathrm{T}} & L_{mq}^{\mathrm{T}} & L_{qq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}-\mathbf{1}_q\right]+2t_q^{\mathrm{T}}\mathbf{1}_q\\ &+\alpha\begin{bmatrix}\mathbf{1}_p^{\mathrm{T}} & u_m & -\mathbf{1}_q^{\mathrm{T}}\end{bmatrix}L\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p^{\mathrm{T}}\mathbf{1}_p+\mathbf{1}_q^{\mathrm{T}}\mathbf{1}_q-u_m^{\mathrm{T}}u_m.\end{aligned}$$

Note that

$$\begin{aligned}&-t_p^{\mathrm{T}}t_p+2t_p^{\mathrm{T}}\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\\ &=-\left\{t_p-\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\}^{\mathrm{T}}\\ &\quad\times\left\{t_p-\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]\right\}\\ &\quad+\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right]^{\mathrm{T}}\\ &\quad\times\left[(\alpha-\beta)\begin{bmatrix}L_{pp} & L_{pm} & L_{pq}\end{bmatrix}\begin{bmatrix}\mathbf{1}_p\\ u_m\\ -\mathbf{1}_q\end{bmatrix}+\mathbf{1}_p\right].\end{aligned}$$

Similar completion of squares for

$$-t_m^{\mathrm{T}} t_m + 2t_m^{\mathrm{T}} \left[ (\alpha-\beta) \left[ L_{pm}^{\mathrm{T}} \; L_{mm} \; L_{mq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + u_m \right],$$

and

$$-t_q^{\mathrm{T}} t_q + 2t_q^{\mathrm{T}} \left[ (\alpha-\beta) \left[ L_{pq}^{\mathrm{T}} \; L_{mq}^{\mathrm{T}} \; L_{qq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} - \mathbf{1}_q \right],$$

yields

$$\begin{aligned} \Delta V = {} & 2(t_p - \mathbf{1}_p)^{\mathrm{T}} (u_p - \mathbf{1}_p) + 2(t_q + \mathbf{1}_p)^{\mathrm{T}} (u_q + \mathbf{1}_p) && (17) \\ & - \left\{ t_p - \left[ (\alpha-\beta) \left[ L_{pp} \; L_{pm} \; L_{pq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + \mathbf{1}_p \right] \right\}^{\mathrm{T}} \\ & \times \left\{ t_p - \left[ (\alpha-\beta) \left[ L_{pp} \; L_{pm} \; L_{pq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + \mathbf{1}_p \right] \right\} && (18) \\ & - \left\{ t_m - \left[ (\alpha-\beta) \left[ L_{pm}^{\mathrm{T}} \; L_{mm} \; L_{mq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + u_m \right] \right\}^{\mathrm{T}} \\ & \times \left\{ t_m - \left[ (\alpha-\beta) \left[ L_{pm}^{\mathrm{T}} \; L_{mm} \; L_{mq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} + u_m \right] \right\} && (19) \\ & - \left\{ t_q - \left[ (\alpha-\beta) \left[ L_{pq}^{\mathrm{T}} \; L_{mq}^{\mathrm{T}} \; L_{qq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} - \mathbf{1}_q \right] \right\}^{\mathrm{T}} \\ & \times \left\{ t_q - \left[ (\alpha-\beta) \left[ L_{pq}^{\mathrm{T}} \; L_{mq}^{\mathrm{T}} \; L_{qq} \right] \begin{bmatrix} \mathbf{1}_p \\ u_m \\ -\mathbf{1}_q \end{bmatrix} - \mathbf{1}_q \right] \right\} && (20) \\ & + s^{\mathrm{T}} \tilde{M} s, && (21) \end{aligned}$$

where $s = [\mathbf{1}_p; u_m; -\mathbf{1}_q]$ and $\tilde{M} = (\alpha-\beta)^2 L^2 + (3\alpha - 2\beta)L$ since $L = L^{\mathrm{T}}$. Note that the two terms in (17) are negative since $t_p - \mathbf{1}_p < 0$, $u_p - \mathbf{1}_p > 0$, $t_q + \mathbf{1}_q > 0$, $u_q + \mathbf{1}_q < 0$, and that the terms in (18), (19), (20) are all non-positive. Therefore, in order to show that $\Delta V \le 0$, it is sufficient to show that the term (21) is also non-positive, i.e., to show that the matrix $\tilde{M}$ is negative semidefinite. It is also easy to see that the eigenvalues of the matrix $\tilde{M}$ are $(\alpha-\beta)^2 \lambda_i^2 + (3\alpha - 2\beta)\lambda_i$, $i \in \{1, \ldots, N\}$. Hence, $\tilde{M}$ has one simple eigenvalue at zero with the corresponding right eigenvector $\mathbf{1}$, while all other eigenvalues are $(\alpha-\beta)^2 \lambda_i^2 + (3\alpha - 2\beta)\lambda_i$, $i \in \{2, \ldots, N\}$. We shall show that all these eigenvalues are negative. Since $\lambda_i > 0$ and $\lambda_i \le \lambda_N$, it is sufficient to show that $\lambda_N < \frac{2\beta - 3\alpha}{(\alpha-\beta)^2}$. We note that $\lambda_N < \frac{3}{2\beta}$ from (10). Thus, it is sufficient to show that $\frac{3}{2\beta} < \frac{2\beta - 3\alpha}{(\alpha-\beta)^2}$. With some algebra, we see that this is equivalent to show that $\beta > \sqrt{3}\alpha$, which is true given (10).

Hence, we have shown that $\Delta V \le 0$. We then show that $\Delta V = 0$ if and only if $y \in M$. To show this, we first note that $\Delta V < 0$ if the first two terms (17) are not empty since they are negative. Therefore, $\Delta V = 0$ only if these terms are empty. This is the case when $|u_i| \le 1$ for all the agents $i \in \{1, \ldots, N\}$, i.e., when the sets $S_p$ and $S_q$ are empty. In this case, we have

$$\begin{aligned} & \Delta V = -t^{\mathrm{T}} t + 2t^{\mathrm{T}} u + 2(\alpha-\beta) t^{\mathrm{T}} L u + u^{\mathrm{T}} (\alpha L - I) u \\ & = -\{ t - [(\alpha-\beta)L + I_N] u \}^{\mathrm{T}} \{ t - [(\alpha-\beta)L + I_N] u \} + u^{\mathrm{T}} \tilde{M} u. \end{aligned}$$

Note that the first term is non-positive and it is equal to zero if and only if $t = [(\alpha-\beta)L + I_N]u$.

Recall that $\tilde{M}$ has exactly one zero eigenvalue with the corresponding right eigenvector $\mathbf{1}$, while all other eigenvalues are negative, thus the second term $u^{\mathrm{T}} \tilde{M} u$ is also non-positive and it is equal to zero if and only if $Lu = 0$.

Hence, we conclude that $\Delta V = 0$ if and only if $t = [(\alpha-\beta)L + I]u$ and $Lu = 0$. Since $Lu = 0$, we obtain that $t = u$. On the other hand, from (14), we get that

$$t = \sigma(u(k+1)) = u(k+1) = u - \alpha L v - \beta L \sigma(u) = u - \alpha L v.$$

Thus, we see that $Lv = 0$ since $\alpha \neq 0$. Thus $v_1 = \ldots = v_N$ since the graph is connected. From (13), we then get that $u = -\alpha L x - \beta L v = -\alpha L x$. This together with the fact that $Lu = 0$ implies that $\tilde{L} q = 0$, where the relative state $q = [q_2; \ldots; q_N]$, $q_i = x_i - x_1$ for $i \in \{2, \ldots, N\}$, and $\tilde{L}$ is given by (5). Recall that the matrix $\tilde{L}$ is non-singular from the proof of Theorem 1. We then see that $q = 0$, that is, $x_1 = \ldots = x_N$. Therefore $\Delta V = 0$ if and only if $y \in M$.

Hence, we have shown that $\Delta V \le 0$ and $\Delta V = 0$ if and only if $y \in M$. It then follows from a similar analysis as in the end of the proof of Theorem 1, that $y(k) \to M$ as $k \to \infty$. Hence, global consensus is achieved. ■

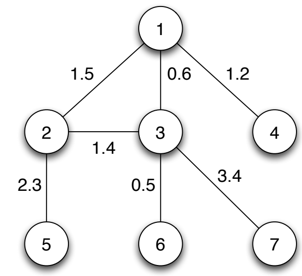


Fig. 1. Network with seven agents

## VI. Illustrative Example

In this section, we illustrate our results on global consensus with input saturation constraints for a network with $N = 7$ double integrators, whose topology is given in Fig. 1. Choose $\alpha = 0.07$ and $\beta = 0.15$ such that the condition (10) is satisfied. The simulation results shown in Fig. 2 confirm the results of Theorem 2.

## VII. Conclusions and Future Work

This paper considered the global consensus problem for a MAS of discrete-time identical linear agents, where the agent dynamics are either neutrally stable or a double integrator, with input saturation constraints under fixed undirected network topologies. Extensions to directed topologies and time-varying topologies are currently under investigation. Another interesting topic is to consider the heterogeneous network.

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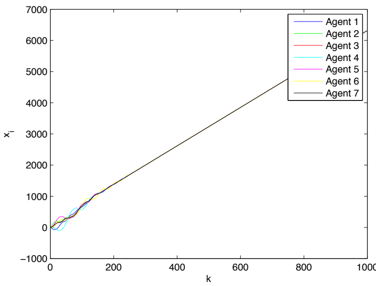
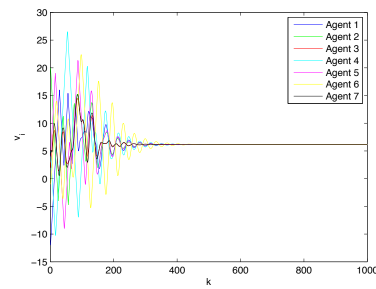
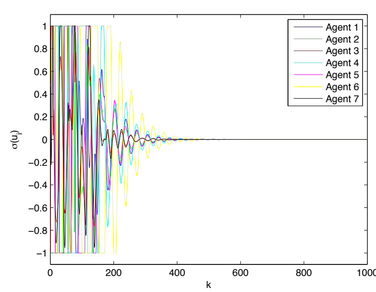



(a) The evolution of $x_i$




(b) The evolution of $v_i$




(c) Saturated input $\sigma(u_i)$

Fig. 2. Simulation results with input saturation constraint

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