Reaching a Quantum Consensus: Master Equations That Generate Symmetrization and Synchronization

Guodong Shi, *Member, IEEE*, Daoyi Dong, *Senior Member, IEEE*, Ian R. Petersen, *Fellow, IEEE*, and Karl Henrik Johansson, *Fellow, IEEE*

***Abstract*—In this paper, we propose and study a master-equation based approach to drive a quantum network with n qubits to a consensus (symmetric) state introduced by Mazzarella *et al.* The state evolution of the quantum network is described by a Lindblad master equation with the Lindblad terms generated by continuous-time swapping operators, which also introduce an underlying interaction graph. We establish a graphical method that bridges the proposed quantum consensus scheme and classical consensus dynamics by studying an induced graph (with 2^{2n} nodes) of the quantum interaction graph (with n qubits). A fundamental connection is then shown that quantum consensus over the quantum graph is equivalent to componentwise classical consensus over the induced graph, which allows various existing works on classical consensus to be applicable to the quantum setting. Some basic scaling and structural properties of the quantum induced graph are established via combinatorial analysis. Necessary and sufficient conditions for exponential and asymptotic quantum consensus are obtained, respectively, for switching quantum interaction graphs. As a quantum analogue of classical synchronization of coupled oscillators, quantum synchronization conditions are also presented, in which the reduced states of all qubits tend to a common trajectory.**

***Index Terms*—Consensus seeking, quantum networks, qubits synchronization.**

I. Introduction

In the past decades, distributed control and optimization methods have witnessed a wide range of applications in network systems such as multi-vehicle systems, wireless communication networks, smart grids, and social networks [2]–[6]. A networked system consists of a number of interconnected nodes, often denoted agents, each of which represents an individual functioning unit ranging from a robot, a power generator, to a member of a society. Recent development in quantum physics and quantum information science suggests the possibility of modeling and analyzing quantum systems as networks of quantum nodes [7]–[10]. In these networks, each quantum node (agent) represents a photon, an electron, an atom, or a finite dimensional quantum system. Nodes in a quantum network are described by quantum mechanics and the interactions between different agents involve non-classical correlations. These unique quantum characteristics make the development of distributed solutions in quantum networks more difficult than classical network systems [7]. It is interesting to understand how synchronization and consensus in quantum networks relate to traditional networks, and if the wealth of graph-theoretic tools recently developed for traditional networks are also applicable to quantum networks.

One of the primary objectives in distributed control and coordination is to drive a network to a consensus, where all agents hold the same state, by local interactions [11]–[13]. Several efforts have been made to investigate the consensus problem in the quantum domain under discrete-time settings. Sepulchre *et al.* [14] generalized consensus algorithms to non-commutative spaces and presented convergence results for quantum stochastic maps. They showed how the Birkhoff theorem can be used to analyze the asymptotic convergence of a quantum system to a fully mixed state. Mazzarella *et al.* [15] made a systematic study regarding consensus-seeking in quantum networks, and pointed out that consensus in a quantum network has close connection to distributed quantum computation, quantum communication and quantum random walk. Four classes of consensus quantum states based on invariance and symmetry properties were introduced, and a quantum generalization to the gossip iteration algorithm (e.g., [16]) was proposed for reaching a symmetric state (consensus) over a quantum network. Such a quantum gossip iteration algorithm is realized through discrete-time quantum swapping operations between two subsystems in a quantum network and can make the quantum network converge to symmetric states while preserving the expected values of permutation-invariant global observables. The class of quantum gossip algorithms can be further extended to so-called symmetrization problems in a group-theoretic framework and be applied to consensus on probability distributions and quantum dynamical decoupling [17].

Quantum systems with external inputs are modeled using master equations that define continuous-time quantum state evolution [18]–[20]. One of the simplest cases is when a Markovian approximation can be applied under the assumption of a short environmental correlation time permitting the neglect of memory effects [23], where a so-called Lindblad equation

Manuscript received May 26, 2014; revised October 26, 2014, February 28, 2015, and April 4, 2015; accepted May 4, 2015. Date of publication May 15, 2015; date of current version January 26, 2016. A preliminary version of this paper was presented at the 11th World Congress on Intelligent Control and Automation. This work was supported in part by the Australian Research Council under projects DP130101658 and FL110100020, the Knut and Alice Wallenberg Foundation, and the Swedish Research Council. Recommended by Associate Editor C. Altafini.

G. Shi is with the Research School of Engineering, The Australian National University, Canberra, ACT 0200, Australia (e-mail: guodong.shi@anu.edu.au).

D. Dong and I. R. Petersen are with the School of Engineering and Information Technology, University of New South Wales, Canberra, ACT 2600, Australia (e-mail: daoyidong@gmail.com; i.r.petersen@gmail.com).

K. H. Johansson is with the ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 100 44, Sweden (e-mail: kallej@kth.se).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2015.2434034

0018-9286 © 2015 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.

can be employed to describe the quantum state evolution. In this paper, we show that a Lindblad master equation [23], [24] can be obtained with the Lindblad terms generated by swapping operators among the qubits, for the dynamical evolution of the quantum network. The swapping operations also introduce an underlying interaction graph for the quantum network, which indeed leads to a distributed structure for the master equation. In this way, a continuous-time generalization of the work of [15], [17] is introduced, under legitimate quantum state evolution.[1]

The contributions of the current paper are highlighted as follows.

- A graphical approach is established bridging the proposed quantum consensus scheme and classical consensus dynamics by introducing an induced graph (with 2^{2n} nodes) of the quantum interaction graph (with n qubits). A fundamental connection is shown that quantum consensus evolution over the n-qubit network is equivalent to a number of independent classical consensus processes, running in parallel over the connected components of the induced graph. Several fundamental scaling and structural properties are obtained for the induced graph. The number of components is characterized; tight bounds of component sizes and node degrees are explicitly given; the induced graph is shown to be regular and the diagonal induced graph is proved to be almost strongly regular.
- The graphical approach provides a powerful tool in studying quantum network dynamics via their classical counterparts. Making use of existing understandings of classical consensus, we show how to carry out convergence speed optimization via convex programming. We also establish two necessary and sufficient conditions for exponential and asymptotic quantum consensus, respectively, for switching quantum interaction graphs.
- The possibility of quantum synchronization is also investigated, in the sense that the trajectory of each qubit (given by the reduced state under partial trace with respect to the space of other qubits) tends to the same trajectory. We show that quantum synchronization can be achieved if the network Hamiltonian admits an exact tensor product form (or Kronecker sum form) of identical Hamiltonians for each qubit. The trajectory synchronization of qubits serves as the quantum analogue of classical synchronization [36], [37].

The developments of the above quantum consensus results are inspired and heavily rely on the concepts introduced in [15]. We study qubit networks for the ease of presentation. Generalization to network of quantum nodes with identical but greater than two dimensional Hilbert spaces is straightforward. We remark that the proposed graphical approach applies directly also to the discrete-time quantum consensus dynamics [15], and thus the corresponding convergence rate characterization and optimization can be obtained using the results in [16]. We believe that our results add to the understanding of distributed control and state manipulation of quantum networks. The graphical approach proposed in the paper can also be useful for a larger class of quantum network control problems.

This rest of the paper is organized as follows. Section II presents some preliminaries including relevant concepts in linear algebra, graph theory and quantum systems. The n-qubit network model and its state evolution master equations are presented in Section III. Section IV is devoted to a systematic study of the relation between a quantum interaction graph and its induced graph. Section V establishes quantum synchronization conditions making use of the graphical approach. Section VI concludes this paper with a few remarks.

II. Preliminaries

In this section, we introduce some concepts and theory from linear algebra [25], graph theory [26], and quantum systems [7].

A. Matrix Vectorization and Geršgorin Theorem

Given a matrix $M \in \mathbb{C}^{m\times n}$, the vectorization of M, denoted by $\mathbf{vec}(M)$, is the $mn \times 1$ column vector $([M]_{11}, \ldots, [M]_{m1}, \ldots, [M]_{1n}, \ldots, [M]_{mn})^T$. We have $\mathbf{vec}(ABC) = (C^T \otimes A)\mathbf{vec}(B)$ for all matrices A, B, C with ABC well defined, where $\otimes$ stands for the Kronecker product. We always use I_ℓ to denote the $\ell \times \ell$ identity matrix, and $\mathbf{1}_\ell$ for the all one vector in $\mathbb{R}^\ell$.

The following is the Geršgorin disc Theorem which will be used in the proof of main results.

Lemma 1 [25, pp. 344]: Let $A = [a_{jk}] \in \mathbb{C}^{n\times n}$. Then all eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}.$$

B. Graph Theory Essentials

A simple undirected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ consists of a finite set $\mathrm{V} = \{1, \ldots, N\}$ of nodes and an edge set E, where an element $e = \{i, j\} \in \mathrm{E}$ denotes an *edge* between two distinct nodes $i \in \mathrm{V}$ and $j \in \mathrm{V}$. Two nodes $i, j \in \mathrm{E}$ are said to be *adjacent* if $\{i, j\}$ is an edge in E. The number of adjacent nodes of v is called its degree, denoted $\deg(v)$. The nodes that are adjacent with a node v as well as itself are called its neighbors. A graph G is called to be *regular* if all the nodes have the same degree. A path between two vertices v_1 and v_k in G is a sequence of distinct nodes $v_1 v_2 \ldots v_k$ such that for any $m = 1, \ldots, k-1$, there is an edge between v_m and v_{m+1}. A pair of distinct nodes i and j is called to be *reachable* from each other if there is a path between them. A node is always assumed to be reachable from itself. We call graph G *connected* if every pair of distinct nodes in V are reachable from each other. A subgraph of G associated with node set $\mathrm{V}^* \subseteq \mathrm{V}$, denoted as $\mathrm{G}|_{\mathrm{V}^*}$, is the graph $(\mathrm{V}^*, \mathrm{E}^*)$, where $\{i, j\} \in \mathrm{E}^*$ if and only if $\{i, j\} \in \mathrm{E}$ for $i, j \in \mathrm{V}^*$. A connected component (or just component) of G is a connected

[1]The continuous-time generalization of [15], [17] for quantum consensus with fixed but general quantum permutation interactions, was also independently presented in [21], where a necessary and sufficient condition was derived for reaching quantum symmetric consensus from a group-theoretical point of view.

subgraph induced by some $\mathrm{V}^* \subseteq \mathrm{V}$, which is connected to no additional nodes in $\mathrm{V} \setminus \mathrm{V}^*$.

The (weighted) Laplacian of G, denoted $L(\mathrm{G})$, is defined as

$$L(\mathrm{G}) = D(\mathrm{G}) - A(\mathrm{G})$$

where $A(\mathrm{G})$ is the $N \times N$ matrix given by $[A(\mathrm{G})]_{kj} = [A(\mathrm{G})]_{jk} = a_{kj}$ for some $a_{kj} > 0$ if $\{k, j\} \in \mathrm{E}$ and $[A(\mathrm{G})]_{kj} = 0$ otherwise, and $D(\mathrm{G}) = \mathrm{diag}(d_1, \ldots, d_N)$ with $d_k = \sum_{j=1, j\neq k}^{N} [A(\mathrm{G})]_{kj}$. It is well known that $L(\mathrm{G})$ is always positive semi-definite, and the following relation holds:

$$\mathrm{rank}\,(L(\mathrm{G})) = N - C_*(\mathrm{G}) \tag{1}$$

where $C_*(\mathrm{G})$ denotes the number of connected components of G.

C. Quantum Systems

1) Quantum Systems and the Master Equation: The state space associated with any isolated quantum system is a complex vector space with inner product, i.e., a Hilbert space. The system is completely described by its state vector, which is a unit vector in the system's state space. The state space of a composite quantum system is the tensor product of the state space of each component system. For an open quantum system, its state can be described by a positive Hermitian density operator (or density matrix) ρ satisfying $\mathrm{tr}(\rho) = 1$. In many situations, a master equation for the evolution of $\rho(t)$ is a suitable way to describe the dynamics of an open quantum system. One of the simplest cases is when a Markovian approximation can be applied under the assumption of a short environmental correlation time permitting the neglect of memory effects [23]. Markovian master equations have been widely used to model quantum systems with external inputs in quantum control [18]–[20], especially for Markovian quantum feedback [28]. Markovian master equations in the Lindblad form are described as [24], [28]

$$\dot{\rho}(t) = -\frac{\imath}{\hbar}\,[H, \rho(t)] + \sum_k \gamma_k \mathfrak{D}[L_k]\rho(t) \tag{2}$$

where H is the effective Hamiltonian as a Hermitian operator over the underlying Hilbert space, $\imath^2 = -1$, $\hbar$ is the reduced Planck constant, the non-negative coefficients γ_k specify the relevant relaxation rates, and

$$\mathfrak{D}[L_k]\rho = L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k.$$

2) Swapping Operators: In quantum systems, the two-dimensional Hilbert space forms the state-space of *qubits* (the most basic quantum system). For any Hilbert space $\mathcal{H}_*$, it is convenient to use $|\cdot\rangle$, known as the Dirac notion, to denote a unit (column) vector in $\mathcal{H}_*$ [7]. Moreover, $|\xi\rangle^\dagger$, i.e., the complex conjugate transpose of $|\xi\rangle$, is denoted as $\langle\xi|$.

Let $\mathcal{H}$ be a two-dimensional Hilbert space for qubits. The standard computational basis of $\mathcal{H}$ is denoted by $|0\rangle$ and $|1\rangle$. An n-qubits quantum network is the composite quantum system of n qubits in the set $\mathsf{V} = \{1, \ldots, n\}$, whose state space is the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, where $\otimes$ denotes the tensor product. The *swapping operator* between qubits i and j, denoted as U_{ij}, is defined by

$$U_{ij}\left(|q_1\rangle \otimes \cdots \otimes |q_i\rangle \otimes \cdots \otimes |q_j\rangle \otimes \cdots \otimes |q_n\rangle\right) = |q_1\rangle \otimes \cdots \otimes |q_j\rangle \otimes \cdots \otimes |q_i\rangle \otimes \cdots \otimes |q_n\rangle$$

for all $q_i \in \{0, 1\}$, $i = 1, \ldots, n$. In other words, the swapping operator U_{ij} switches the information held in qubits i and j without changing the states of other qubits.

Additionally, for any $|p\rangle, |q\rangle \in \mathcal{H}_*$, we use the notation $|p\rangle\langle q|$ to denote the operator over $\mathcal{H}_*$ defined by

$$(|p\rangle\langle q|)\,|\eta\rangle = \big\langle |q\rangle, |\eta\rangle \big\rangle |p\rangle,\ \forall |\eta\rangle \in \mathcal{H}_*$$

where $\langle\cdot,\cdot\rangle$ represents the inner product that the Hilbert space $\mathcal{H}_*$ is equipped with. In standard quantum mechanical notation, the inner product $\langle |p\rangle, |q\rangle\rangle$ is denoted as $\langle p|q\rangle$.

3) Partial Trace: Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the state spaces of two quantum systems A and B, respectively. Their composite system is described by a density operator ρ^{AB}. Let $\mathfrak{L}_A$, $\mathfrak{L}_B$, and $\mathfrak{L}_{AB}$ be the spaces of (linear) operators over $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_A \otimes \mathcal{H}_B$, respectively. Then the partial trace over system B, denoted by $\mathrm{Tr}_{\mathcal{H}_B}$, is an operator mapping $\mathfrak{L}_{AB}$ to $\mathfrak{L}_A$ defined by

$$\mathrm{Tr}_{\mathcal{H}_B}\left(|p_A\rangle\langle q_A| \otimes |p_B\rangle\langle q_B|\right) = |p_A\rangle\langle q_A| \mathrm{Tr}\left(|p_B\rangle\langle q_B|\right)$$

for all $|p_A\rangle$, $|q_A\rangle \in \mathcal{H}_A$, $|p_B\rangle$, $|q_B\rangle \in \mathcal{H}_B$.

The reduced density operator (state) for system A, when the composite system is in the state ρ^{AB}, is defined as $\rho^A = \mathrm{Tr}_{\mathcal{H}_B}(\rho^{AB})$. The physical interpretation of ρ^A is that ρ^A holds the full information of system A in ρ^{AB}. For a detailed introduction, we refer to [7].

III. Quantum Consensus and Synchronization Master Equations

A. Quantum Networks and Interaction Graphs

Consider a quantum network with n qubits. The qubits are indexed in the set $\mathsf{V} = \{1, \ldots, n\}$ and the state space of this n-qubit quantum network is denoted as the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, where $\mathcal{H}$ denotes a two-dimensional Hilbert space over $\mathbb{C}$. The density operator of the n-qubit network is denoted as ρ.

We define a quantum interaction graph over the n-qubit network as an undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, where each element in E, called a quantum edge, is an unordered pair of two distinct qubits denoted as $\{i, j\} \in \mathsf{E}$ with $i, j \in \mathsf{V}$. Let Ω denote the set of all quantum interaction graphs over node set $\mathsf{V} = \{1, \ldots, n\}$. Let $\sigma(\cdot) : [0, \infty) \mapsto \Omega$ be a piecewise constant function. The obtained time-varying graph is then denoted as $\mathsf{G}_{\sigma(t)} = (\mathsf{V}, \mathsf{E}_{\sigma(t)})$. We assume that there is a constant $\tau_D > 0$ as a lower bound between any two consecutive switching instants of $\sigma(t)$.

B. Dynamics

Let H be the (time-invariant) Hamiltonian of the n-qubit quantum network. In this paper, we propose and investigate the state evolution of the quantum network described by the following master equation:

$$\frac{d\rho}{dt} = -\frac{\imath}{\hbar}[H,\rho] + \sum_{\{j,k\}\in \mathsf{E}_{\sigma(t)}} \alpha_{jk}\left(U_{jk}\rho U_{jk}^{\dagger} - \rho\right) \quad (3)$$

where $\alpha_{jk} > 0$ is a constant marking the weight of edge $\{j,k\}$, and U_{jk} is the swapping operator between j and k.

The system (3) will be referred to as the *quantum synchronization master equation*. When we assume $H = 0$, the system (3) is reduced to

$$\frac{d\rho}{dt} = \sum_{\{j,k\}\in \mathsf{E}_{\sigma(t)}} \alpha_{jk}\left(U_{jk}\rho U_{jk}^{\dagger} - \rho\right). \quad (4)$$

We call the system (4) the *quantum consensus master equation*.

Remark 1: The Lindblad evolution (3) is a continuous-time analogue of the quantum gossip algorithm proposed in [15]. This continuous-time generalization to the discrete-time dynamics [15], [17] has also been independently investigated in [1], [21]. Compared to the results and analysis methodologies in [15], [17], [21], in this work we provide a new approach to investigate the connection between the proposed quantum consensus scheme and classical consensus dynamics. As will be shown in the following discussions, once this connection has been made clear, various results for classical consensus can then be adapted to establish convergence conditions under more relaxed conditions imposed on quantum interaction graphs.

Remark 2: The system (3) is related to the proposed realization of n-qubit quantum circuits by nearest-neighbor operations in [22], which showed that the ability to apply arbitrary Lindblad operators implies encoding of quantum circuits with polynomial overhead. In the system (3), the swapping operator U_{jk} represents external interactions between qubit j and qubit k through their local environment (cf., [22, Figure 1]), and the network Hamiltonian generates internal qubit interactions.

C. Objectives

A permutation of the set $\mathsf{V} = \{1,\ldots,n\}$ is a bijective map from V onto itself. We denote by π such a permutation. Particularly, a permutation π is called a swapping between j and k, denoted by π_{jk}, if $\pi(j) = k$, $\pi(k) = j$, and $\pi(s) = s$, $s \in \mathsf{V}\setminus\{j,k\}$. The set of all permutations of V forms a group, called the n'th permutation group and denoted by $\mathbf{P} = \{\pi\}$. There are $n!$ elements in $\mathbf{P}$. Given $\pi \in \mathbf{P}$, we define a unitary operator, U_π, over $\mathcal{H}^{\otimes n}$, by

$$U_\pi\left(|q_1\rangle \otimes \cdots \otimes |q_n\rangle\right) = \left|q_{\pi(1)}\right\rangle \otimes \cdots \otimes \left|q_{\pi(n)}\right\rangle$$

where $q_i = 0$ or 1 for all $i = 1,\ldots,n$. Define an operator over the density operators of $\mathcal{H}^{\otimes n}$, $\mathscr{P}_*$, by

$$\mathscr{P}_*(\rho) = \frac{1}{n!}\sum_{\pi\in\mathbf{P}} U_\pi \rho U_\pi^{\dagger}. \quad (5)$$

Introduced in [15], $\mathscr{P}_*(\rho)$ serves as the quantum average of the n-qubit network at the state ρ.

Let the initial time be $t_0 \geq 0$ and let $\rho(t_0)$ be the initial density operator of the quantum network. We make the following definition.

Definition 1:

(i) The system (4) reaches an asymptotic (symmetric-state) quantum consensus for initial time $t_0 \geq 0$ and initial state $\rho(t_0)$ if $\lim_{t\to\infty}\rho(t) = \mathscr{P}_*(\rho(t_0))$.

(ii) The system (4) reaches global asymptotic (symmetric-state) quantum consensus if quantum consensus is achieved for all $t_0 \geq 0$ and all initial density operators $\rho(t_0)$.

(iii) The system (4) reaches global exponential (symmetric-state) quantum consensus, if there exist $C(\rho(t_0)) > 0$ (which may depend on the initial state $\rho(t_0)$) and $\gamma > 0$ (which does not depend on $\rho(t_0)$) such that

$$\|\rho(t) - \mathscr{P}_*(\rho(t_0))\| \leq C(\rho(t_0))\, e^{-\gamma(t-t_0)},\ t \geq t_0$$

for all initial times $t_0 \geq 0$ and initial states $\rho(t_0)$.

Let

$$\rho^k(t) := \mathrm{Tr}_{\otimes_{j\neq k}\mathcal{H}_j}(\rho(t))$$

be the reduced state of qubit k at time t, $k = 1,\ldots,n$, defined by the partial trace over the remaining $n-1$ qubits' space $\otimes_{j\neq k}\mathcal{H}_j$. Here $\mathcal{H}_j$ denotes the two-dimensional Hilbert space corresponding to qubit j, $j \in \mathsf{V}$. Note that $\rho^k(t)$ contains all the information that qubit k holds in the composite state $\rho(t)$. Consistent with the classical definition of complex network synchronization [36], [37], we also introduce the following definition for quantum (reduced-state) synchronization.

Definition 2:

(i) The system (3) achieves global asymptotic quantum (reduced-state) synchronization if

$$\lim_{t\to\infty}\left(\rho^k(t) - \rho^m(t)\right) = 0,\ k,m \in \mathsf{V} \quad (6)$$

for all initial times t_0 and initial values $\rho(t_0)$.

(ii) The system (3) achieves global exponential quantum (reduced-state) synchronization if there are two constants $C(\rho(t_0)) > 0$ and $\gamma > 0$ such that

$$\left\|\rho^k(t) - \rho^m(t)\right\| \leq C(\rho(t_0))\, e^{-\gamma(t-t_0)},\ t \geq t_0 \quad (7)$$

for all $k,m \in \mathsf{V}$.

Note that along the Lindblad master equation (4), $\rho(t)$ will be preserved as positive, Hermitian, and with trace one, as long as $\rho(0)$ defines a proper density operator. While the convergence conditions to be derived in the paper do not depend on these properties held by the density operators. Therefore, throughout the rest of the paper, we assume that $\rho(t)$ lies in the general space $\mathbb{C}^{2^n\times 2^n}$.

IV. The Quantum Laplacian and Induced Graph

In this section, we explore the connection between the quantum consensus dynamics (4) and its classical analogue

through an induced (classical) graph from a graphical point of view. We introduce the quantum Laplacian matrix associated with a quantum interaction graph and show that the convergence to quantum consensus is fully governed by this quantum Laplacian. This inspired us to introduce the induced graph of the quantum interaction graph, and then equivalence is proved between quantum consensus over the interaction graph and classical consensus over the induced graph. We also establish some basic scaling and structural properties of the induced graph.

A. The Quantum Laplacian

We introduce quantum Laplacian associated with the interaction graph G as follows.

Definition 3: Let $\mathsf{G}=(\mathsf{V},\mathsf{E})$ be a quantum interaction graph. The quantum (non-weighted) Laplacian of G is defined as $L_{\mathsf{G}}:=\sum_{\{j,k\}\in\mathsf{E}}(I_{2^n}\otimes I_{2^n}-U_{jk}\otimes U_{jk})$.

Some properties of the quantum Laplacian can be clearly observed: L_{G} is real and symmetric, $L_{\mathsf{G}}\mathbf{1}_{2^{2n}}=0$, and all the off-diagonal entries of L_{G} are non-negative. Consequently, invoking the Geršgorin disc theorem (cf., Lemma 1) we know that all nonzero eigenvalues of L_{G} are positive, and we denote the smallest eigenvalue other than zero of L_{G} as $\lambda_2(L_{\mathsf{G}})$.

Consider the following quantum consensus master equation defined over the quantum interaction graph G:

$$\frac{d}{dt}\rho(t)=\sum_{\{j,k\}\in\mathsf{E}}\left(U_{jk}\rho(t)U_{jk}^{\dagger}-\rho\right). \tag{8}$$

Then (8) can be exactly written as

$$\frac{d}{dt}\mathbf{vec}(\rho)=-L_{\mathsf{G}}\mathbf{vec}(\rho) \tag{9}$$

under the vectorization $\rho(t)$.

There holds for the system (9) that $\mathbf{vec}(\rho(t))$ converges to a fixed point in the null space of L_{G} exponentially, with the convergence speed given by $\lambda_2(L_{\mathsf{G}})$. Moreover, different from classical definition of the Laplacian, the multiplicity of the zero eigenvalue of L_{G} is no longer one, even when the interaction graph G is connected. The following lemma provides a characterization of the null space of the quantum Laplacian.

Lemma 2: $\ker(L_{\mathsf{G}})=\{\mathbf{vec}(z):\mathscr{P}_*(z)=z\}$ if G is connected.

The proof of Lemma 2 can be found in Appendix A. In light of Lemma 2, it can be easily deduced that the system (8) reaches exponential quantum consensus as long as G is connected, with convergence rate $\lambda_2(L_{\mathsf{G}})$. This is consistent with the results in [15], [21].

B. The Induced Graph

For further investigations of the quantum Laplacian, we introduce the following definition.

Definition 4: The induced graph of the quantum interaction graph G, denoted by $\mathcal{G}=(\mathcal{V},\mathcal{E})$, is defined in that $\mathcal{V}=\{1,\ldots,2^{2n}\}$ and $\{r,s\}\in\mathcal{E}$, $r\neq s\in\mathcal{V}$ if and only if $[L_{\mathsf{G}}]_{rs}\neq 0$.

Making use of (1) and noticing that L_{G} is the classical Laplacian of the induced graph $\mathcal{G}$, the following lemma follows from Lemma 2 as a preliminary property between a quantum interaction graph and its induced graph.

Lemma 3: If the quantum interaction graph G is connected, then its induced graph $\mathcal{G}$ has exactly

$$\dim\left(\left\{\mathbf{vec}(z):\mathscr{P}_*(z)=z,z\in\mathbb{C}^{2^n\times 2^n}\right\}\right)=\dim\left(\ker(L_{\mathsf{G}})\right)$$

connected components.

We let $X(t)=(x_1(t)\ldots x_{4^n}(t))^T:=\mathbf{vec}(\rho(t))$ so that the system (9) defines classical consensus dynamics over the induced graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ (cf., [5], [11]), where $x_i(t)\in\mathbb{C}$ stands for the state of node $i\in\mathcal{V}$ at time t. Let the initial time be $t_0=0$. We make the following definition.

Definition 5: Componentwise consensus over the graph $\mathcal{G}$ in the classical sense is achieved for the system (9) if

$$\lim_{t\to\infty}x_i(t)=\frac{\sum_{j\in\mathcal{R}_i}x_j(0)}{|\mathcal{R}_i|}$$

for all $i\in\mathcal{V}$, where $\mathcal{R}_i\subseteq\mathcal{V}$ denotes the set of nodes of the connected component in which node i lies.

It is well known that the system (9) reaching componentwise consensus is equivalent to [2]

$$\lim_{t\to\infty}\|X(t)\|_{L_{\mathsf{G}}}=0$$

where $\|X(t)\|_{L_{\mathsf{G}}}=X^T(t)L_{\mathsf{G}}X(t)$. On the other hand, we have from Lemma 2 that

$$\left\{\mathbf{vec}(z):\mathscr{P}_*(z)=z,z\in\mathbb{C}^{2^n\times 2^n}\right\}=\ker(L_{\mathsf{G}}).$$

As a result, the following conclusion holds providing a direct relation between quantum consensus and its classical analogue.

Theorem 1: Quantum consensus over G along (8) is equivalent to componentwise consensus in the classical sense over the induced graph $\mathcal{G}$ along (9).

Remark 3: Theorem 1 describes a form of quantum parallelism (cf., [7, Chapter 1.4.2]) in the sense that the original quantum consensus dynamics over n qubits, leads to independent consensus processes over disjoint subsets of nodes. As shown in Figs. 1 and 2, if the quantum interaction graph is well chosen, the state evolution can be of the same form for these different subsets of nodes, but starting from (in general) different initial values.

C. The Connected Components

We have seen from Theorem 1 that we can indeed investigate the connected components of the quantum induced graph $\mathcal{G}$ to obtain every detail of the quantum consensus master equation. Now we establish some basic properties of the connected components of the quantum induced graph.

1) The Reachable Nodes: We index the elements $\mathcal{V}=\{1,\ldots,2^{2n}\}$ under the standard computational basis of $\mathcal{H}^{\otimes n}$. Recall that $|0\rangle$ and $|1\rangle$ form a basis of $\mathcal{H}$. Let $|q_1\rangle\otimes\cdots\otimes$

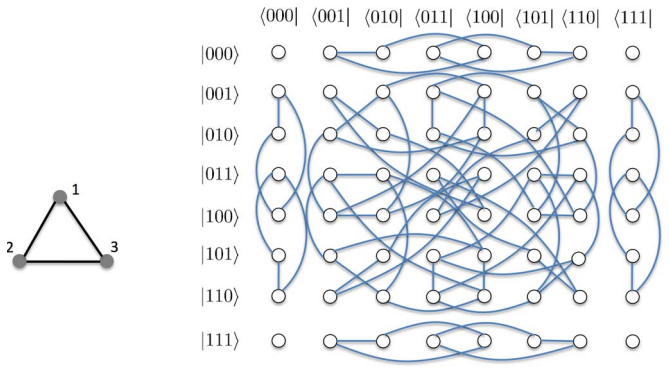


Fig. 1. The induced graph of the three-qubit quantum complete graph. There are 64 nodes in the induced graph, and they can be indexed as the elements in the basis $\mathbb{B}$.

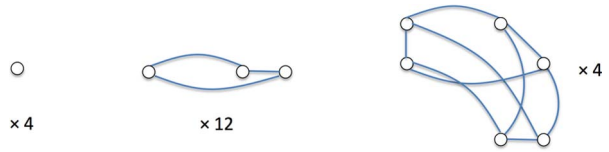


Fig. 2. The connected components of the induced graph for the three-qubits quantum complete graph. There are a total of 20 components, consisting of 4 components each with one node, 12 components each with three nodes, and the remaining 4 components each with six nodes. Note that all of these components are *regular* graphs in the sense that every node within the same component has the same degree.

$|q_n\rangle \in \mathcal{H}^{\otimes n}$ be denoted as $|q_1 \dots q_n\rangle$ for simplicity, where $\otimes$ represents the tensor product. Then, the following 2^n elements:

$$|q_1 \dots q_n\rangle : q_i \in \{0,1\},\ i = 1, \dots, n$$

form a basis of $\mathcal{H}^{\otimes n}$. We define

$$|q_1 \dots q_n\rangle\langle p_1 \dots p_n| : \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}$$

as a linear operator over $\mathcal{H}^{\otimes n}$ such that

$$\left(|q_1 \dots q_n\rangle\langle p_1 \dots p_n|\right)|\xi\rangle = \left(\langle p_1 \dots p_n|\xi\rangle\right)|q_1 \dots q_n\rangle$$

for all $|\xi\rangle \in \mathcal{H}^{\otimes n}$. We now obtain a basis for all linear operators over $\mathcal{H}^{\otimes n}$ (which is isomorphic to $\mathbb{C}^{2^n \times 2^n}$)

$$\mathbb{B} := \{|q_1 \dots q_n\rangle\langle p_1 \dots p_n| : q_i, p_i \in \{0,1\},\ i = 1, \dots, n\}.$$

Furthermore, associated with any $\pi \in \mathbf{P}$ with $\mathbf{P}$ being the permutation group over V, we define an operator $\mathscr{F}_\pi$ over $\mathcal{H}^{\otimes n} \times \mathcal{H}^{\otimes n}$ by

$$\mathscr{F}_\pi\left(|q_1 \dots q_n\rangle\langle p_1 \dots p_n|\right) = \left|q_{\pi(1)} \dots q_{\pi(n)}\right\rangle\left\langle p_{\pi(1)} \dots p_{\pi(n)}\right|$$

for all $|q_1 \dots q_n\rangle\langle p_1 \dots p_n| \in \mathbb{B}$. Particularly, when $\pi \in \mathbf{P}$ defines a swapping permutation π_{jk}, the corresponding $\mathscr{F}_\pi$ will be denoted as $\mathscr{F}_{\pi_{jk}}$. Then the following lemma holds with its proof given in Appendix B.

Lemma 4: For all $\rho \in \mathbb{C}^{2^n \times 2^n}$ and $\pi \in \mathbf{P}$, it holds that $U_\pi \rho U_\pi^\dagger = \mathscr{F}_\pi(\rho)$.

Each node in $\mathcal{V}$ corresponds to one entry in $\rho \in \mathbb{C}^{2^n \times 2^n}$ under vectorization. We identify the nodes in $\mathcal{V}$ as the elements in $\mathbb{B}$. For any $|q_1 \dots q_n\rangle\langle p_1 \dots p_n| \in \mathcal{V}$, we denote by $\mathcal{N}_{|q_1 \dots q_n\rangle\langle p_1 \dots p_n|}$ the set of nodes in $\mathcal{V}$ that are adjacent to $|q_1 \dots q_n\rangle\langle p_1 \dots p_n|$ in the induced graph $\mathcal{G}$. It is then clear from Lemma 4 that

$$\mathcal{N}_{|q_1 \dots q_n\rangle\langle p_1 \dots p_n|} = \left\{\left|q_{\pi_{jk}(1)} \dots q_{\pi_{jk}(n)}\right\rangle\left\langle p_{\pi_{jk}(1)} \dots p_{\pi_{jk}(n)}\right| \neq |q_1 \dots q_n\rangle\langle p_1 \dots p_n| :\ \pi_{jk} \in \mathsf{E}\right\}.$$

Noting that all the swapping permutations in

$$\{\pi_{jk} : \{j,k\} \in \mathsf{E}\}$$

form a generating subset of $\mathbf{P}$, the following lemma holds.

Lemma 5: Suppose G is connected. Then for any given node $|q_1 \dots q_n\rangle\langle p_1 \dots p_n| \in \mathcal{V}$

$$\mathcal{R}_{|q_1 \dots q_n\rangle\langle p_1 \dots p_n|} := \left\{\left|q_{\pi(1)} \dots q_{\pi(n)}\right\rangle\left\langle p_{\pi(1)} \dots p_{\pi(n)}\right| : \pi \in \mathbf{P}\right\}$$

is the set of nodes in $\mathcal{V}$ that are reachable from $|q_1 \dots q_n\rangle\langle p_1 \dots p_n|$ in the graph $\mathcal{G}$.

2) Several Counting Theorems: We now establish some scaling properties of the components of the induced graph. First of all the following theorem holds, with a detailed proof in Appendix C.

Theorem 2: Suppose G is connected. Then

(i) There are $\dim(\{\mathbf{vec}(z) : \mathscr{P}_*(z) = z, z \in \mathbb{C}^{2^n \times 2^n}\})$ connected components in $\mathcal{G}$. Different choices of G give the same node set partition of $\mathcal{V}$ along the connected components of their induced graphs.

(ii) Let $|\cdot|$ stand for the cardinality of a finite set. The degree of $|q_1 \dots q_n\rangle\langle p_1 \dots p_n| \in \mathcal{V}$ is computed as $|\mathcal{N}_{|q_1 \dots q_n\rangle\langle p_1 \dots p_n|}|$.

(iii) There are exactly four smallest components of $\mathcal{G}$, each of which contains only one node. The number of nodes in the largest components of $\mathcal{G}$ lies in the interval

$$\left[\max_{0 \le k \le n} \mathbf{C}_n^k, \left(\max_{0 \le k \le n} \mathbf{C}_n^k\right)^2\right]$$

where $\mathbf{C}_n^k$ is the combinatorial number of selecting k different elements out of n different choices.

Remark 4: Note that $\max_{0 \le k \le n} \mathbf{C}_n^k$ is achieved at $k = \lfloor (n+1)/2 \rfloor$, where $\lfloor b \rfloor$ denotes the greatest integer no larger than b for a given $b \in \mathbb{R}$. Invoking the famous Stirling's formula it is known that

$$\max_{0 \le k \le n} \mathbf{C}_n^k \sim \frac{2^n}{\sqrt{\pi n/2}}.$$

Therefore, based on Theorem 2, we know that the size of the largest component, asymptotically (as n tends to infinity) lies in

$$\left[\frac{2^n}{\sqrt{\frac{\pi n}{2}}}, \frac{4^n}{\frac{\pi n}{2}}\right].$$

Let $\mathbf{K}_n$ denote the complete graph with n nodes. The following theorem establishes some tight bounds of the node degree for the induced graph, whose proof is in Appendix D.

Theorem 3:

(i) If $n \bmod 4 = 0$, then $\deg(v) \leq 3n^2/8$ for all $v \in \mathcal{V}$;
(ii) If $n \bmod 4 = 1$, then $\deg(v) \leq (3n^2-3)/8$ for all $v \in \mathcal{V}$;
(iii) If $n \bmod 4 = 2$, then $\deg(v) \leq (3n^2-4)/8$ for all $v \in \mathcal{V}$;
(iv) If $n \bmod 4 = 3$, then $\deg(v) \leq (3n^2-3)/8$ for all $v \in \mathcal{V}$.

Moreover, there exist nodes with degrees at these upper bounds when $\mathsf{G} = \mathbf{K}_n$.

Remark 5: Theorem 3 indicates that the maximum degree of the induced graph asymptotically tends to $3n^2/8$ as n tends to infinity. While the maximum component is of the size at least $2^n/\sqrt{\pi n/2}$ from Remark 4. As a result, the largest components of the induced graph tend to be rather *sparse* as n becomes large.

3) Component Structure: We now investigate the structure of the components. We focus on the case when the quantum interaction graph is the complete graph.

Recall that an undirected graph is *regular* if all nodes in the graph have the same degree [26]. We further introduce the following definition [27].

Definition 6: Let G be a simple, undirected regular graph with N nodes and node degree k. We call G strongly regular if there are two integers λ and μ such that

(i) Every two adjacent nodes have λ neighbors in common;
(ii) Every two non-adjacent nodes have μ neighbors in common.

We also introduce the quantum induced graph on the diagonal entries as a subgraph of $\mathcal{G}$.

Definition 7: The quantum diagonal induced graph, denoted $\mathcal{G}_{\rm diag} = (\mathcal{V}_{\rm diag}, \mathcal{E}_{\rm diag})$, is the subgraph generated by the node set $\mathcal{V}_{\rm diag} := \{|p_1 \dots p_n\rangle\langle p_1 \dots p_n| : p_i \in \{0,1\}\}$ in the graph $\mathcal{V}$.

With Lemma 5, there are no edges between $\mathcal{V}_{\rm diag}$ and $\mathcal{V} \setminus \mathcal{V}_{\rm diag}$ in the graph $\mathcal{G}$. The quantum diagonal induced graph $\mathcal{G}_{\rm diag}$ therefore fully characterizes the dynamics of the diagonal entries of the density operator. The physical interpretation of the diagonal entries is that

$$[\rho]_{|p_1 \dots p_n\rangle\langle p_1 \dots p_n|}$$

represents the probability of finding the system at the state $|p_1 \dots p_n\rangle\langle p_1 \dots p_n|$ when performing measurement to the quantum network under the standard basis [7].

The following theorem provides a structural characterization of the induced graph. The proof can be found in Appendix E.

Theorem 4: Suppose $\mathsf{G} = \mathbf{K}_n$. Then

(i) Every connected component of the induced graph $\mathcal{G}$ is regular;
(ii) Every connected component of the diagonal induced graph $\mathcal{G}_{\rm diag}$ is almost strongly regular in the sense that
  a) every two adjacent nodes in $\mathcal{G}_{\rm diag}$ have $n-2$ neighbors in common;
  b) every two non-adjacent nodes in $\mathcal{G}_{\rm diag}$ have either zero or one neighbor in common.

Remark 6: The exponentially increasing dimension with respect to the number of components is a fundamental obstacle for understanding and analyzing large-scale quantum systems. Theorems 1, 2, 3, and 4 illustrate the possibility of splitting the dimensions into decoupled smaller pieces (e.g., Remark 4, the dimension is reduced by a factor which is at least $2/\pi n$) by graphical analysis, and then combinatorial analysis would be able to uncover deeper characterizations. The nature of quantum systems engineered by sparse Lindblad operators, or quantum systems with sparse Hamiltonians, suggests potential applicability of the methodology to more studies of quantum multi-body systems [31], [32].

D. Discussions

1) Why Swapping Operators?: We now provide a brief discussion to illustrate that the choice of swapping operators in the quantum consensus dynamics (4), is very natural from classical consensus dynamics [11]. A group-theoretic point of view for their relationships is also provided in [15].

Consider a classical graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ with $\mathrm{V} = \{1, \dots, N\}$. Let $x_i(t) \in \mathbb{R}$ be the state of node i in V. Denote $x(t) = (x_1(t) \ \dots \ x_N(t))^T$. Let every edge's weight be one, and let $L_{\rm G}$ be the Laplacian in the classical sense of the graph G. Then a classical average consensus process is defined by [5], [11]

$$\frac{d}{dt}x(t) = -L_{\rm G}x(t). \tag{10}$$

We introduce a classical swapping operator (matrix) along the edge $\{i,j\} \in \mathrm{E}$, denoted by $\widetilde{U}_{ij} \in \mathbb{R}^{N\times N}$, in the way that

$$\widetilde{U}_{ij}(z_1 \dots z_i \dots z_j \dots z_N)^T = (z_1 \dots z_j \dots z_i \dots z_N)^T \tag{11}$$

for all $(z_1 \dots z_N)^T \in \mathbb{R}^m$. Then physically $\widetilde{U}_{ij}$ switches the i'th and j'th entries with the rest unchanged, and is therefore a classical version of the quantum swapping U_{ij}. In fact $\widetilde{U}_{ij}$ is a permutation matrix. It is interesting to note the following equality:

$$L_{\rm G} = -\sum_{\{i,j\}\in \mathrm{E}} (\widetilde{U}_{ij} - I_N). \tag{12}$$

Plugging (12) into (10), we obtain the following equivalent form of (10):

$$\frac{d}{dt}x(t) = \sum_{\{i,j\}\in \mathrm{E}} \left(\widetilde{U}_{ij}x(t) - x(t)\right). \tag{13}$$

It is now clear that the system (4) is a formal quantum version of the system (13), noting that in the quantum case the swapping operator U_{ij} maps a density operator ρ to $U_{ij}\rho U_{ij}^\dagger$. This is to say, the connection between the quantum consensus and its classical prototype, is inherent within their structures, and the realization of quantum consensus seeking via swapping operators is remarkably natural.

Remark 7: As a matter of fact, the quantum consensus state, defined in (5) (originally introduced in [15]), is formally of the same form as the classical average noticing

$$\frac{1}{N!}\sum_{\pi\in\mathbf{P}}\widetilde{U}_\pi z=\frac{1}{N!}\cdot((N-1)!)\,\mathbf{1}_N^T z\mathbf{1}_N=\frac{\sum_{i=1}^n z_i}{N}\mathbf{1}_N \quad (14)$$

for all $z=(z_1\cdots z_N)^T\in\mathbb{C}^N$, where $\widetilde{U}_\pi$ denotes the classical permutation. We have now seen that the classical average (14) and the quantum average (5) are closely connected.

2) Convergence Speed Optimization: If each edge $\{i,j\}\in\mathsf{E}$ is associated with a weight α_{ij}, we can correspondingly define the weighted quantum Laplacian $L_{\mathsf{G}}(\alpha):=\sum_{\{j,k\}\in\mathsf{E}}\alpha_{jk}(I_{2^n}\otimes I_{2^n}-U_{jk}\otimes U_{jk})$ with $\alpha=(\alpha_{jk}:\{j,k\}\in\mathsf{E})$.

The speed of convergence to a quantum consensus for

$$\frac{d\rho}{dt}=\sum_{\{j,k\}\in\mathsf{E}}\alpha_{jk}\left(U_{jk}\rho U_{jk}^\dagger-\rho\right) \quad (15)$$

is thus given by the smallest non-zero eigenvalue of $L_{\mathsf{G}}(\alpha)$, denoted $\lambda_2(L_{\mathsf{G}}(\alpha))$.

As a continuous-time and quantum analogue of [3], we can therefore optimally distribute a certain amount, say $W_0>0$, of edge weights onto the edges so that the fastest convergence rate can be achieved

$$\begin{aligned}&\text{maximize}\quad \lambda_2\left(L_{\mathsf{G}}(\alpha)\right)\\ &\text{subject to}\quad \sum_{\{i,k\}\in\mathsf{E}}\alpha_{jk}\le W_0.\end{aligned} \quad (16)$$

Following similar argument as in [3], we know that $\lambda_2(L_{\mathsf{G}}(\alpha))$ is a concave function of α. Therefore, the fastest convergence can be obtained by solving (16) via standard convex programming methods.

We conclude this section with a few remarks. In this section we have provided a graphical approach for studying the quantum consensus master equation. We introduce the quantum Laplacian and the quantum induced graph, and show that quantum consensus over the interaction graph is equivalent to componentwise classical consensus over the induced graph, with convergence rate given by the smallest eigenvalue of the quantum Laplacian. We establish some basic properties of the induced graph in terms its scaling and structure. Such a fundamental connection makes the majority of graphical developments in classical network systems directly applicable to quantum networks. The proposed graphical approach certainly also applies to discrete-time quantum dynamics, e.g., [15].

V. Quantum Synchronization

In this section, we establish synchronization conditions for the Lindblad equation (3). First of all, making use of the graphical approach developed in the previous section, we establish two necessary and sufficient quantum consensus conditions for the system (4) in light of existing results on classical consensus. Next, we show that for a class of network Hamiltonians, quantum consensus of the system (4) implies synchronization of the system (3). Finally, we discuss the connection between the quantum synchronization results and their classical analogue and present a numerical example.

A. Quantum Consensus Conditions

The following theorem establishes consensus conditions of the system (4).

Theorem 5:

(i) The system (4) achieves global exponential quantum consensus if and only if there exists a constant $T>0$ such that $\mathsf{G}([t,t+T)):=(\mathsf{V},\bigcup_{t\in[t,t+T)}\mathsf{E}_{\sigma(t)})$ is connected for all $t\ge 0$.

(ii) The system (4) achieves global asymptotic quantum consensus if and only if $\mathsf{G}([t,\infty)):=(\mathsf{V},\bigcup_{t\in[t,\infty)}\mathsf{E}_{\sigma(t)})$ is connected for all $t\ge 0$.

The proof of Theorem 5 is based on the connection between quantum consensus and classical consensus from a graphical point of view, and has been put in Appendix F. These results are essentially consistent with the results for consensus seeking over classical networks [4]–[13]. We remark that under the conditions of Theorems 5, the convergence rates can be explicitly computed making use of the analysis in [13], for both cases. We also remark that for simplicity of presentation we assume the edge weights α_{jk} to be a constant. Generalization to the case where α_{jk} is time-varying or even state-dependent is straightforward using existing works in the literature on classical consensus convergence, e.g., [13].

Remark 8: Theorem 5 provides a generalization to the result in [21] for switching quantum interaction graphs. In fact, from its proof it is clear that the convergence rate can be obtained utilizing the results in [13] under the given conditions.

B. From Consensus to Synchronization

Let the initial time be $t_0=0$ and denote $\rho_*=\mathscr{P}_*(\rho(0))$. Introduce

$$\tilde{\rho}(t)=e^{\imath Ht/\hbar}\rho(t)e^{-\imath Ht/\hbar}.$$

Suppose $[H,U_\pi]=0$ for all $\pi\in\mathbf{P}$. Then some simple calculations lead to the fact that the evolution of $\tilde{\rho}(t)$ satisfies

$$\frac{d\tilde{\rho}}{dt}=\sum_{\{j,k\}\in\mathsf{E}}\alpha_{jk}\left(U_{jk}\tilde{\rho}U_{jk}^\dagger-\tilde{\rho}\right). \quad (17)$$

Substituting the results in Theorem 5, we immediately obtain

$$\lim_{t\to\infty}\left[\rho(t)-e^{-\imath Ht/\hbar}\rho_* e^{\imath Ht/\hbar}\right]=0 \quad (18)$$

when the same connectivity conditions hold in Theorem 5 for the switching quantum interaction graph.

Define $\rho_*^k(t):=\mathrm{Tr}_{\otimes_{j\ne k}\mathcal{H}_j}(e^{-\imath Ht/\hbar}\rho_* e^{\imath Ht/\hbar})$ for all $k\in\mathsf{V}$. The following lemma can be established from the definition of the partial trace [7] (or, directly applying [15, Theorem 1]).

Lemma 6: Suppose $[H,U_\pi]=0$ for all $\pi\in\mathbf{P}$. Then $\rho_*^k(t)=\rho_*^m(t)$ for all $k,m\in\mathsf{V}$ and all t.

As a result, the following theorem holds.

Theorem 6: Suppose $[H, U_\pi] = 0$ for all $\pi \in \mathbf{P}$.

(i) If $\mathsf{G}([t,\infty)) := (\mathsf{V}, \bigcup_{t\in[t,\infty)} \mathsf{E}_{\sigma(t)})$ is connected for all $t \geq 0$, then the system (3) achieves global asymptotical quantum (reduced-state) synchronization.

(ii) If there exists a constant $T > 0$ such that $\mathsf{G}([t, t+T)) := (\mathsf{V}, \bigcup_{t\in[t,t+T)} \mathsf{E}_{\sigma(t)})$ is connected for all $t \geq 0$, then the system (3) achieves global exponential quantum (reduced-state) synchronization.

The following lemma, with its proof given in Appendix G, presents two classes of Hamiltonians satisfying the condition $[H, U_\pi] = 0$ for all $\pi \in \mathbf{P}$. Denote the Kronecker sum $H_0^{\oplus n} = \sum_{i=1}^n I^{\otimes(i-1)} \otimes H_0 \otimes I^{\otimes(n-i)}$, where H_0 is a Hermitian operator over $\mathcal{H}$.

Lemma 7: Let H_0 be a Hermitian operator over $\mathcal{H}$. If either $H = H_0^{\otimes n}$ or $H = H_0^{\oplus n}$ holds, then $[H, U_\pi] = 0$ for all $\pi \in \mathbf{P}$.

Remark 9: If $H = H_0^{\oplus n}$, then there holds $e^{\imath Ht/\hbar} = e^{\imath H_0 t/\hbar} \otimes \cdots \otimes e^{\imath H_0 t/\hbar}$ and $e^{-\imath Ht/\hbar} = e^{-\imath H_0 t/\hbar} \otimes \cdots \otimes e^{-\imath H_0 t/\hbar}$. Consequently, it can be further deduced that

$$\begin{aligned}\rho_*^k(t) &= \mathrm{Tr}_{\otimes_{j\neq k}\mathcal{H}_j}\left(e^{-\imath Ht/\hbar}\rho_* e^{\imath Ht/\hbar}\right)\\ &= e^{-\imath H_0 t/\hbar}\left(\mathrm{Tr}_{\otimes_{j=1}^{n-1}\mathcal{H}_j}(\rho_*)\right)e^{\imath H_0 t/\hbar}. \end{aligned} \tag{19}$$

from the definition of the partial trace [7].

C. Discussions

It is worth noticing that the quantum synchronization results established in Theorem 6, is exactly the quantum analogues of the classical studies on the synchronization of coupled oscillators [36]–[38]. Fundamental results have been derived for the classical notion of synchronization for the following dynamics [36]–[38]:

$$\frac{d}{dt}x_i(t) = Ax_i(t) + \sum_{j=1}^N W_{ij}\left(x_j(t) - x_i(t)\right),\ i = 1, \ldots, N \tag{20}$$

where $x_i \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times m}$, $W_{ij} \geq 0$. Here $x_i(t)$ represents the state of the i'th oscillator, A is the inherent mode of the dynamics of the oscillators, and an interaction graph is induced by $[W_{ij}]$. Note that it is critical that all of the oscillators share an identical inherent dynamics for synchronization of the system (20). Therefore, it becomes clear that the condition $H = H_0^{\oplus n}$ plays the same role in imposing identical inherent dynamics for the qubits. The system (3) becomes the quantum equivalence of the system (20) when such a condition holds, and the behavior of the system trajectories in the two systems are indeed consistent [36]. On the other hand, for the case with $H = H_0^{\otimes n}$, the tensor product of Hamiltonians introduces internal interactions among the qubits. Synchronization of the qubits' reduced states is still reached since these internal interactions *cooperate* with the (external) swapping interactions in such a way that H is invariant under permutations. It is however difficult to write down the explicit trajectory of each qubit's reduced state as a function of H_0 in this case, and the synchronization orbit is certainly no longer the one determined by H_0 for the most choices of H_0.

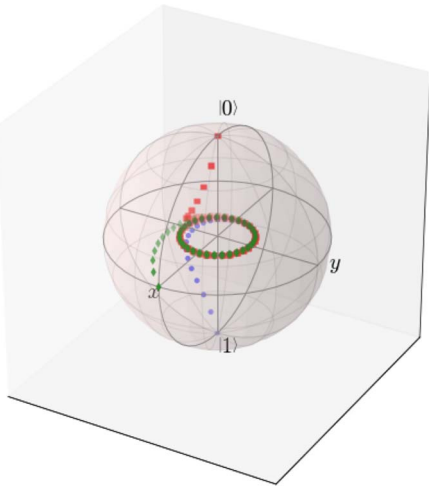


Fig. 3. An illustration of the quantum synchronization: The orbits of the three qubits asymptotically converge to the same trajectory for the proposed master equation.

Remark 10: Note that when the nodes' inherent self-dynamics are not identical in the classical synchronization dynamics (20), it is well-known in the literature that it will be extremely difficult and often impossible to achieve synchronization for the system (20) [38]. Now that it becomes clear from above discussion that the condition that either $H = H_0^{\otimes n}$ or $H = H_0^{\oplus n}$ in the quantum master equation plays the same role in enforcing identical inherent self-dynamics, quantum synchronization will in general be difficult to reach without such conditions.

D. Numerical Example

In this subsection, we present a simple numerical example to illustrate the above quantum synchronization result.

We consider three qubits indexed in $\mathsf{V} = \{1, 2, 3\}$. Their interaction graph is fixed as the complete graph, i.e., $\mathsf{E} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Let $\alpha_{12} = \alpha_{13} = \alpha_{23} = 1$. The initial network state is chosen to be

$$\rho_0 = \frac{1}{2}|100\rangle\langle 100| + \frac{1}{2}|100\rangle\langle 101| + \frac{1}{2}|101\rangle\langle 101| + \frac{1}{2}|101\rangle\langle 100|.$$

The network Hamiltonian is chosen to be $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

is one of the Pauli matrices.

We first plot the evolution of the reduced states of the three qubits on one Bloch sphere. Clearly their orbits asymptotically tend to the same trajectory determined by the Hamiltonian σ_z (cf., Fig. 3).

Next, recall that the trace distance between two density operator ρ_1, ρ_2 over the same Hilbert space, denoted by $\|\rho_1 - \rho_2\|_{\mathrm{Tr}}$, is defined as

$$\|\rho_1 - \rho_2\|_{\mathrm{Tr}} = \frac{1}{2}\mathrm{Tr}\sqrt{(\rho_1 - \rho_2)^\dagger(\rho_1 - \rho_2)}.$$

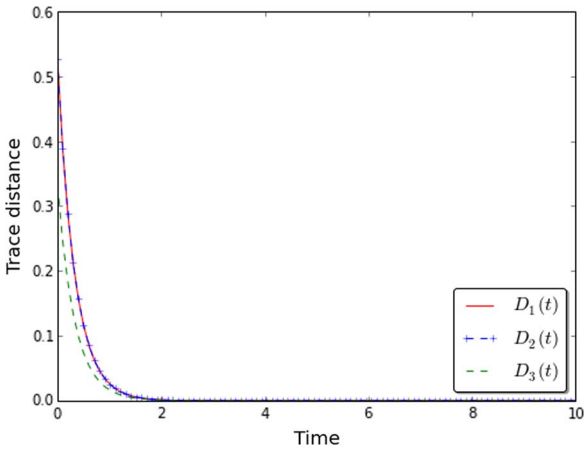


Fig. 4. An illustration of the quantum synchronization: Exponential convergence to the synchronization orbit. Note that two of the three qubits' distance functions exactly agree with each other so there are only two curves distinguishable in this plot.

We then plot the trace distances between the reduced states and the synchronization orbit

$$D_k(t) := \left\| \rho^k(t) - \mathrm{Tr}_{\otimes_{j=1}^2 \mathcal{H}_j} \left(e^{-\imath Ht/\hbar} \rho_* e^{\imath Ht/\hbar} \right) \right\|_{\mathrm{Tr}}$$

for $k = 1,\ 2,\ 3$, as a function of t, where $\rho_* = (1/3!) \sum_{\pi \in \mathbf{P}_3} U_\pi \rho_0 U_\pi^\dagger$ is the quantum average with $\mathbf{P}_3$ denoting the permutation group with order three. Clearly they all converge to zero with an exponential rate (cf., Fig. 4).

VI. Conclusion

We have investigated consensus and synchronization problems for a quantum network with n qubits. The state evolution of the quantum network equipped with continuous-time swapping operators, is described by a Lindblad master equation. These swapping operators also introduce an underlying interaction graph. A graphical method bridging the proposed quantum consensus scheme and classical consensus dynamics was presented, by studying an induced graph (with 2^{2n} nodes) of the quantum interaction graph (with n qubits). We provided several fundamental relations between a quantum graph and its induced classical graph. Two necessary and sufficient conditions for exponential and asymptotic quantum consensus were obtained, respectively, for switching quantum interaction graphs. We also presented quantum synchronization conditions, in the sense that the reduced states of all qubits tend to a common trajectory. We showed that this is exactly the quantum analogue of classical synchronization of coupled oscillators.

The consensus and synchronization problems for the quantum network considered in this paper can be taken as a special class of stabilization problems in quantum control [30]–[35] where the control actions are realized by swapping operators. We believe the results presented in the current paper add some novel understandings regarding the control and state manipulation of quantum networks in a distributed manner. The graphical approach proposed may serve as a systematic and useful tool for analyzing distributed quantum dynamics. In future, it is also worth investigating new algorithms for other consensus/synchronization states in quantum networks and developing control methods for stabilizing the states of quantum networks.

Appendix A
Proof of Lemma 2

The following equalities hold:

$$\begin{aligned}
\ker(L_{\mathsf{G}}) &= \left\{ \mathbf{vec}(z) : \sum_{\{j,k\} \in \mathsf{E}} \left(U_{jk} z U_{jk}^\dagger - z \right) = 0 \right\} \\
&\overset{a)}{=} \left\{ \mathbf{vec}(z) : U_{jk} z U_{jk}^\dagger = z, \{j,k\} \in \mathsf{E} \right\} \\
&\overset{b)}{=} \left\{ \mathbf{vec}(z) : U_\pi z U_\pi^\dagger = z, \pi \in \mathbf{P} \right\} \\
&\overset{c)}{=} \left\{ \mathbf{vec}(z) : \mathscr{P}_*(z) = z \right\}. \qquad (22)
\end{aligned}$$

Here $a)$ is based on Lemma 5.2 in [29]; $b)$ holds from the fact that G is a connected graph so that the swapping permutations along the edges among qubits consist of a generating set of the group $\mathbf{P}$ (cf. [15, Proposition 8 and Lemma 1]). Regarding equality $c)$, on one hand it is straightforward to see that

$$\left\{ \mathbf{vec}(z) : U_\pi z U_\pi^\dagger = z, \pi \in \mathbf{P} \right\} \subseteq \left\{ \mathbf{vec}(z) : \mathscr{P}_*(z) = z \right\}.$$

On the other hand, if $\mathscr{P}_*(z) = z$, then

$$U_\pi z U_\pi^\dagger = U_\pi \left(\mathscr{P}_*(z) \right) U_\pi^\dagger = \mathscr{P}_*(z) = z$$

since $\pi \mathbf{P} = \mathbf{P}$ for any $\pi \in \mathbf{P}$. Thus we also have

$$\left\{ \mathbf{vec}(z) : \mathscr{P}_*(z) = z \right\} \subseteq \left\{ \mathbf{vec}(z) : U_\pi z U_\pi^\dagger = z, \pi \in \mathbf{P} \right\}.$$

This proves the desired lemma. ■

Appendix B
Proof of Lemma 4

Since the two operators

$$\rho \to U_\pi \rho U_\pi^\dagger$$
$$\rho \to \mathscr{F}_\pi(\rho)$$

are both linear, we just need to verify the equality for each element in the basis $\mathbb{B}$.

The following holds:

$$\begin{aligned}
&\left(U_\pi |q_1 \dots q_n\rangle\langle p_1 \dots p_n| U_\pi^\dagger \right) |\xi\rangle \\
&\quad = \langle p_1 \dots p_n | U_\pi^\dagger | \xi \rangle U_\pi |q_1 \dots q_n\rangle \\
&\quad = \left(\langle p_{\pi(1)} \dots p_{\pi(n)} | \xi \rangle \right) |q_{\pi(1)} \dots q_{\pi(n)}\rangle \\
&\quad = \left(|q_{\pi(1)} \dots q_{\pi(n)}\rangle\langle p_{\pi(1)} \dots p_{\pi(n)}| \right) |\xi\rangle \\
&\quad = \mathscr{F}_\pi \left(|q_1 \dots q_n\rangle\langle p_1 \dots p_n| \right) |\xi\rangle \qquad (23)
\end{aligned}$$

for any $|\xi\rangle \in \mathcal{H}^{\otimes n}$. This proves the desired lemma. ■

APPENDIX C
PROOF OF THEOREM 2

(i) The number of connected components of $\mathcal{G}$ has been derived in Lemma 3. The fact that the sizes of $\mathcal{G}$'s connected components do not depend on the form of G, as long as G is connected, can be simply deduced from Lemma 5.

(ii) The conclusion holds directly from the proof of Lemma 5.

(iii) First of all note that the following four nodes $|0\dots0\rangle\langle0\dots0|$, $|0\dots0\rangle\langle1\dots1|$, $|1\dots1\rangle\langle0\dots0|$, $|1\dots1\rangle\langle1\dots1|$ are always isolated in $\mathcal{G}$ since both $|0\dots0\rangle$ and $|1\dots1\rangle$ are invariant under any permutation $\pi\in\mathbf{P}$. Furthermore, it is easy to see that for a node

$$|q_1\dots q_n\rangle\langle p_1\dots p_n|\in\mathcal{V}$$

to be isolated, it must be the case that both $|q_1\dots q_n\rangle$ and $|p_1\dots p_n\rangle$ are invariant under any permutation $\pi\in\mathbf{P}$. This proves that the four isolated nodes presented above are the only four isolated nodes in $\mathcal{G}$.

Finally, we establish the upper and lower bounds to the number of nodes in the largest component. The following claim holds.

Claim: $|\{|q_{\pi(1)}\dots q_{\pi(n)}\rangle,\pi\in\mathbf{P}\}|=\mathbf{C}_n^r$ with $r=\sum_{k=1}^n q_k$.

For any $|q_1\dots q_n\rangle$ and $|p_1\dots p_n\rangle$ with $\sum_{k=1}^n q_k=\sum_{k=1}^n p_k$, we can always find a permutation $\pi^\sharp\in\mathbf{P}$ such that $|q_1\dots q_n\rangle=|p_{\pi^\sharp(1)}\dots p_{\pi^\sharp(n)}\rangle$. As a result, $\{|q_{\pi(1)}\dots q_{\pi(n)}\rangle,\pi\in\mathbf{P}\}$ has $\mathbf{C}_n^r$ elements. This proves the claim.

From Lemma 5, as long as either $|q_1\dots q_n\rangle\neq|q_{\pi(1)}\dots q_{\pi(n)}\rangle$ or $|p_1\dots p_n\rangle\neq|p_{\pi(1)}\dots p_{\pi(n)}\rangle$ holds, π will generate a reachable node for $|q_1\dots q_n\rangle\langle p_1\dots p_n|$. Then the upper and lower bounds for the size of the largest component in $\mathcal{G}$ follows immediately.

The proof is now complete. ■

APPENDIX D
PROOF OF THEOREM 3

The argument is based on a combinatorics analysis on the choice of nodes under the basis $\mathbb{B}$. We present the detailed proof for Cases (i) and (iii). The remaining two cases can be proved via the same techniques, and whose details are therefore omitted.

(i) Let $n=2m$ with some positive integer $m\geq1$ and take a node $v\in\mathcal{V}$. Without loss of generality, we assume v takes the form

$$\left|\underbrace{0\dots0}_{2\chi}\underbrace{1\dots1}_{2m-2\chi}\right\rangle\langle p_1\dots p_{2m}|$$

where $p_j\in\{0,1\}$ and $0\leq\chi\leq m$. It is clear that a quantum link $\{j,k\}\in\mathsf{E}$ (i.e., operator π_{jk}) generates a neighbor of node v only for the following three cases:

a) $j\leq\chi$ and $k\geq\chi+1$, or $k\leq\chi$ and $j\geq\chi+1$;

b) $j\leq\chi$ and $k\leq\chi$ with $p_j\neq p_k$;

c) $j\geq\chi+1$ and $k\geq\chi+1$ with $p_j\neq p_k$.

Consequently, direct combinatorial calculations lead to

$$\begin{aligned}\deg(v)&\leq\chi^2+(m-\chi)^2+2\chi(2m-2\chi)\\&=-2\chi^2+2m\chi+m^2\\&\leq\frac{3m^2}{2}.\end{aligned}\tag{24}$$

Moreover, the upper bound $3m^2/2$ is reached when $\mathsf{G}=\mathbf{K}_n$, m is even (i.e., $n \bmod 4=0$), and v is of the form with $\chi=m/2$

$$\left|\underbrace{0\dots0}_{2\chi}\underbrace{1\dots1}_{2m-2\chi}\right\rangle\left\langle\underbrace{0\dots0}_{\chi}\underbrace{1\dots1}_{\chi}\underbrace{0\dots0}_{m-\chi}\underbrace{1\dots1}_{m-\chi}\right|.$$

This proves (i).

(ii) Again let $n=2m$ with some positive integer $m\geq1$. We study the case when v takes the form

$$\left|\underbrace{0\dots0}_{2\chi+1}\underbrace{1\dots1}_{2m-2\chi-1}\right\rangle\langle p_1\dots p_{2m}|$$

where $p_j\in\{0,1\}$ and $2\chi+1\leq2m$. Via similar analysis we have

$$\begin{aligned}\deg(v)&\leq\chi(\chi+1)+(m-\chi)(m-\chi-1)\\&\quad+(2\chi+1)(2m-2\chi-1)\\&=-2\chi^2+2(m-1)\chi+m^2+m-1\\&\leq\frac{3m^2-1}{2}.\end{aligned}\tag{25}$$

The upper bound $(3m^2-1)/2$ is reached when $\mathsf{G}=\mathbf{K}_n$, m is odd (i.e., $n \bmod 4=2$), and v is of the form with $\chi=(m-1)/2$

$$\left|\underbrace{0\dots0}_{2\chi+1}\underbrace{1\dots1}_{2m-2\chi-1}\right\rangle\left\langle\underbrace{0\dots0}_{\chi}\underbrace{1\dots1}_{\chi+1}\underbrace{0\dots0}_{m-\chi}\underbrace{1\dots1}_{m-\chi-1}\right|.$$

This proves (iii). ■

APPENDIX E
PROOF OF THEOREM 4

(i) Let $|q_1\dots q_n\rangle\langle p_1\dots p_n|$ and $|q_1'\dots q_n'\rangle\langle p_1'\dots p_n'|$ be two nodes in $\mathcal{V}$ belonging to a common component, where q_i,p_i,q_i',q_i' take values from {0,1}. From Lemma 5, we know that we can find a permutation $\pi_*\in\mathbf{P}$ such that

$$|q_1'\dots q_n'\rangle\langle p_1'\dots p_n'|=\left|q_{\pi_*(1)}\dots q_{\pi_*(n)}\right\rangle\left\langle p_{\pi_*(1)}\dots p_{\pi_*(n)}\right|.\tag{26}$$

Now suppose π_{jk} generates a link to node $|q_1\dots q_n\rangle\langle p_1\dots p_n|$ in the induced graph, i.e., $|q_1\dots q_n\rangle\langle p_1\dots p_n|\neq|q_{\pi_{jk}(1)}\dots q_{\pi_{jk}(n)}\rangle\langle p_{\pi_{jk}(1)}\dots p_{\pi_{jk}(n)}|$. We define a swapping permutation $\pi^\natural$ by

$$\pi^\natural=\pi_{\pi_*(j)\pi_*(k)}.$$

In other words, $\pi^\natural$ flips the state of qubits $\pi_*(j)$ and $\pi_*(k)$. This gives us

$$\begin{aligned}&|q'_1\dots q'_n\rangle\langle p'_1\dots p'_n|\\&\quad=\left|q_{\pi_*(1)}\dots q_{\pi_*(n)}\right\rangle\left\langle p_{\pi_*(1)}\dots p_{\pi_*(n)}\right|\\&\quad\neq\left|q_{\pi^\natural\pi_*(1)}\dots q_{\pi^\natural\pi_*(n)}\right\rangle\left\langle p_{\pi^\natural\pi_*(1)}\dots p_{\pi^\natural\pi_*(n)}\right|\\&\quad=\left|q'_{\pi^\natural(1)}\dots q'_{\pi^\natural(n)}\right\rangle\left\langle p'_{\pi^\natural(1)}\dots p'_{\pi^\natural(n)}\right|.\end{aligned}\tag{27}$$

Consequently, $\pi^\natural$, as an edge in G since $\mathsf{G}=\mathbf{K}_n$, also generates a link to node $|q'_1\dots q'_n\rangle\langle p'_1\dots p'_n|$ in the induced graph. Noting that the positions of $|q_1\dots q_n\rangle\langle p_1\dots p_n|$ and $|q'_1\dots q'_n\rangle\langle p'_1\dots p'_n|$ are symmetric in the above argument, we have constructed a bijection between the adjacent nodes of $|q_1\dots q_n\rangle\langle p_1\dots p_n|$ and those of $|q'_1\dots q'_n\rangle\langle p'_1\dots p'_n|$. This proves the desired conclusion.

(ii) From the proof of Theorem 2 we know that

$$\mathcal{R}_{|p_1\dots p_n\rangle\langle p_1\dots p_n|}=\left\{|p'_1\dots p'_n\rangle\langle p'_1\dots p'_n|:\sum_{k=1}^n p'_k=\sum_{k=1}^n p_k\right\}.\tag{28}$$

For two nodes $v=|p_1\dots p_n\rangle\langle p_1\dots p_n|$ and $v'=|p'_1\dots p'_n\rangle\langle p'_1\dots p'_n|$ in the same component of the diagonal induced graph, we introduce

$$H(v,v')=\sum_{k=1}^n|p_k-p'_k|.$$

Proof of Condition a): let $v=|p_1\dots p_n\rangle\langle p_1\dots p_n|$ and $v'=|p'_1\dots p'_n\rangle\langle p'_1\dots p'_n|$ be two adjacent nodes in the diagonal induced graph. As a result, we have $H(v,v')=2$ and $\sum_{k=1}^n p'_k=\sum_{k=1}^n p_k=L$ for some integer $L\leq n$. The following claim holds.

Claim: There are $n-2$ common neighbors for v and v'.

Since $H(v,v')=2$, without loss of generality, we write $v=|01p_3\dots p_n\rangle\langle 01p_3\dots p_n|$ and $v'=|10p_3\dots p_n\rangle\langle 01p_3\dots p_n|$. If $p_3=0$, then it is straightforward to see that

$$|001p_4\dots p_n\rangle\langle 001p_4\dots p_n|$$

is a common neighbor of v and v'. Similarly if $p_3=1$, a common neighbor of v and v' is given as

$$|110p_4\dots p_n\rangle\langle 001p_4\dots p_n|.$$

Continuing the argument to $p_4,\dots,p_n$ we can find $n-2$ common neighbors for v and v'. Apart from these $n-2$ common neighbors, either v or v' however has only two more neighbors as themselves. This proves the claim.

Proof of Condition b): let $v=|p_1\dots p_n\rangle\langle p_1\dots p_n|$ and $v'=|p'_1\dots p'_n\rangle\langle p'_1\dots p'_n|$ be two non-adjacent nodes in the same component. This means that $H(v,v')>2$. From (28) we know that $H(v,v')$ must be an even number. Thus, $H(v,v')\geq 4$. On the other hand, let $v^\flat:=|p^\flat_1\dots p^\flat_n\rangle\langle p^\flat_1\dots p^\flat_n|$ be a common neighbor of v and v'. Then $H(v,v^\flat)=2$ and $H(v',v^\flat)=2$, which yields $H(v,v')\leq 4$. Consequently, we can easily conclude that v and v' have exactly one common neighbor if $H(v,v')=4$, and they have no common neighbor if $H(v,v')>4$.

The proof is now complete. ■

APPENDIX F
PROOF OF THEOREM 5

The proof is based on the graphical approach developed in Section IV. Under vectorization, the system (4) is equivalent to the following vector form:

$$\frac{d}{dt}\mathbf{vec}\left(\rho(t)\right)=-L\left(\sigma(t)\right)\mathbf{vec}\left(\rho(t)\right)\tag{29}$$

where by definition

$$L\left(\sigma(t)\right):=\sum_{\{j,k\}\in\mathsf{E}_{\sigma(t)}}\alpha_{jk}(I_{2^n}\otimes I_{2^n}-U_{jk}\otimes U_{jk}).$$

We denote the induced graph of the quantum interaction graph $\mathsf{G}_{\sigma(t)}=(\mathsf{V},\mathsf{E}_{\sigma(t)})$, as $\mathcal{G}_{\sigma(t)}=(\mathcal{V},\mathcal{E}_{\sigma(t)})$. The following lemmas hold.

Lemma 8: Let $T>0$ be a constant. Then $\mathcal{G}([t,t+T))$ has $m^\natural=\dim(\{\mathbf{vec}(z):\mathscr{P}_*(z)=z\})$ connected components if $\mathsf{G}([t,t+T))$ is connected.

Proof: Noticing the fact that $\mathcal{G}([t,t+T))$ is the induced graph of $\mathsf{G}([t,t+T))$ following Definition 4, the desired lemma holds directly from Lemma 3. ■

Lemma 9: Suppose $\mathsf{G}([0,\infty))$ is connected. Then the system (29) defines $m^\natural$ classical consensus processes over $m^\natural$ disjoint subsets of nodes in $\mathcal{V}$.

Proof: We will show it using Lemma 8. If $\mathsf{G}([0,\infty))$ is connected, then $\mathcal{G}([0,\infty))$ has $m^\natural$ connected components. This means that for any two nodes belonging to different connected components of $\mathcal{G}([0,\infty))$, there is never an edge between them for the system (29). This implies the desired conclusion. ■

We now denote the $m^\natural$ disjoint subsets of nodes in $\mathcal{V}$, each defining the node set of one component of $\mathcal{G}([0,\infty))$ when $\mathsf{G}([0,\infty))$ is connected, as $\mathcal{V}_1,\dots,\mathcal{V}_{m^\natural}$. Correspondingly, we denote by

$$\mathcal{G}^o_{\sigma(t)}=\left(\mathcal{V}_o,\mathcal{E}^o_{\sigma(t)}\right),\ o=1,\dots,m^\natural$$

the subgraph that is associated with $\mathcal{V}_o$ in the graph $\mathcal{G}_{\sigma(t)}$. We give another technical lemma.

Lemma 10: Suppose $\mathsf{G}([0,\infty))$ is connected. Then

(i) The system (4) reaches global (exponential, or asymptotic) quantum consensus if and only if the system (29) reaches classical global (exponential or asymptotic) consensus over all node subsets $\mathcal{V}_o$, $o=1,\dots,m^\natural$.

(ii) Let $T>0$ be a constant. Then $\mathcal{G}^o([t,t+T)):=(\mathcal{V}_o,\bigcup_{t\in[t,t+T)}\mathcal{E}^o_{\sigma(t)})$ is connected for all $o=1,\dots,m^\natural$ if and only if $\mathsf{G}([t,t+T))$ is connected.

Proof:

(i) First of all we fix the initial time as $t_0=0$ and the initial value for $\rho(0)$, and show the equivalence between quantum consensus and classical consensus. The fact that classical consensus is reached for the system (29) means that

$$\lim_{t\to\infty}x_i(t)=\frac{\sum_{j\in\mathcal{V}_o}x_j(0)}{|\mathcal{V}_o|},\ i\in\mathcal{V}_o,\ o=1,\dots,m^\natural$$

where again we use the notation $X(t) = (x_1(t) \dots x_{4^n}(t))^T := \mathbf{vec}(\rho(t))$, since each $L(\sigma(t))$ is always symmetric. This in turn implies that

$$\lim_{t\to\infty} \|X(t)\|_{L_{\mathsf{G}}} = 0$$

for an arbitrary connected G. Thus, quantum consensus is equivalent to classical consensus for this fixed initial condition.

Next, it is clear that $\rho(0)$ taking value from all legitimate density operators makes $X_o(0) = (x_k(0) : k \in \mathcal{V}_o)^T$ possibly take value from a unit ball in $\mathbb{R}^{|\mathcal{V}_o|}$. This implies that global quantum consensus for the system (4) is equivalent to global consensus for the system (29).

Finally, the convergence rate equivalence (exponential, or asymptotic), is obvious since $m^\natural$ defines a finite number.

(ii) Noticing the definition of connected component and Lemma 8, the desired conclusion follows immediately. ■

It is straightforward to see that $\mathsf{G}([0,\infty))$ must be connected so that quantum consensus convergence becomes possible for the n-qubit network. Based on Theorem 4.1 in [13], global exponential consensus is achieved for the component $\mathcal{V}_o$ if and only if there exists $T > 0$ such that $\mathcal{G}^o([t, t+T))$ is connected for all t. Theorem 5.2 in [13] showed that global asymptotic consensus is achieved for the component $\mathcal{V}_o$ if and only if $\mathcal{G}^o([t,\infty))$ is connected for all t. As a result, utilizing Lemma 10 on the equivalence between quantum consensus and classical consensus, Theorem 5 immediately holds. This concludes the proof. ■

APPENDIX G
PROOF OF LEMMA 7

We only prove the lemma for case (i) and the other case follows from a similar argument. Take $\pi \in \mathbf{P}$. The following holds:

$$\begin{aligned}
&[H_0 \otimes \cdots \otimes H_0] U_\pi \left(|q_1 \dots q_n\rangle\right) \\
&\quad = \left|H_0 q_{\pi(1)}\right\rangle \otimes \cdots \otimes \left|H_0 q_{\pi(n)}\right\rangle \\
&\quad = U_\pi \left(|H_0 q_1\rangle \otimes \cdots \otimes |H_0 q_n\rangle\right) \\
&\quad = U_\pi [H_0 \otimes \cdots \otimes H_0] \left(|q_1 \dots q_n\rangle\right) \qquad (30)
\end{aligned}$$

for all $|q_1 \dots q_n\rangle \in \mathcal{H}^{\otimes n}$. This immediately implies $[H, U_\pi] = 0$ and the desired conclusion thus holds. ■

ACKNOWLEDGMENT

The authors thank the Associate Editor and the anonymous reviewers for their useful and constructive comments, which helped them improve the presentation of the work considerably, and Dr. Shuangshuang Fu for her generous help in preparing the numerical example.

REFERENCES

[1] G. Shi, D. Dong, I. R. Petersen, and K. H. Johansson, "Consensus of quantum networks with continuous-time Markovian dynamics," in *Proc. 11th World Congress Intell. Control Autom.*, Shenyang, China, Jun. 29–Jul. 4, 2014, pp. 307–312.

[2] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[3] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Syst. Control Lett.*, vol. 53, pp. 65–78, 2005.

[4] J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control*, vol. 31, no. 9, pp. 803–812, 1986.

[5] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[6] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, 2005.

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[8] X. Wang, P. Pemberton-Ross, and S. G. Schirmer, "Symmetry and subspace controllability for spin networks with a single-node control," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 1945–1956, 2012.

[9] Y. Kato and N. Yamamoto, "Structure identification and state initialization of spin networks with limited access," *New J. Phys.*, vol. 16, 2014, Art. ID 023024.

[10] J. Gough and M. R. James, "The series product and its application to quantum feedforward and feedback networks," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2530–2544, 2009.

[11] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, pp. 1520–1533, 2004.

[12] W. Ren and R. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, 2005.

[13] G. Shi and K. H. Johansson, "Robust consensus for continuous-time multi-agent dynamics," *SIAM J. Control Optim.*, vol. 51, no. 5, pp. 3673–3691, 2013.

[14] R. Sepulchre, A. Sarlette, and P. Rouchon, "Consensus in non-commutative spaces," in *Proc. 49th IEEE Conf. Decision Control*, Atlanta, GA, USA, Dec. 15–17, 2010, pp. 6596–6601.

[15] L. Mazzarella, A. Sarlette, and F. Ticozzi, "Consensus for quantum networks: From symmetry to gossip iterations," *IEEE Trans. Autom. Control*, vol. 60, no. 1, pp. 158–172, 2015.

[16] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inform. Theory*, vol. 52, no. 6, pp. 2508–2530, 2006.

[17] L. Mazzarella, F. Ticozzi, and A. Sarlette, "From consensus to robust randomized algorithms: A symmetrization approach," quant-ph, arXiv 1311.3364, 2013.

[18] F. Ticozzi and L. Viola, "Quantum Markovian subsystems: Invariance, attractivity and control," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2048–2063, 2008.

[19] F. Ticozzi and L. Viola, "Analysis and synthesis of attractive quantum Markovian dynamics," *Automatica*, vol. 45, pp. 2002–2009, 2009.

[20] D. Dong, I. R. Petersen, and H. Rabitz, "Sampled-data design for robust control of a single qubit," *IEEE Trans. Autom. Control*, vol. 58, pp. 2654–2659, 2013.

[21] F. Ticozzi, L. Mazzarella, and A. Sarlette, "Symmetrization for quantum networks: A continuous-time approach," in *Proc. 21st Int. Symp. Math. Theory Netw. Syst. (MTNS)*, Groningen, The Netherlands, Jul. 2014, pp. 1685–1690.

[22] F. Verstraete, M. M. Wolf, and J. I. Cirac, "Quantum computation and quantum-state engineering driven by dissipation," *Nature Phys.*, vol. 5, pp. 633–636, 2009.

[23] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, 1st ed. Oxford, U.K.: Oxford Univ. Press, 2002.

[24] G. Lindblad, "On the generators of quantum dynamical semigroups," *Comm. Math. Phys.*, vol. 48, no. 2, pp. 119–130, 1976.

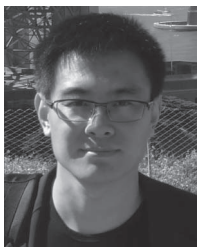
[25] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[26] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY, USA: Springer-Verlag, 2001.

[27] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*. New York, NY, USA: Springer-Verlag, 2012.

[28] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[29] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, "Information preserving structures: A general framework for quantum zero-error information," *Phys. Rev. A*, vol. 82, 2010, Art. ID 062306.
[30] C. Altafini and F. Ticozzi, "Modeling and control of quantum systems: An introduction," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 1898–1917, 2012.
[31] L. Erdős and B. Schlein, "Quantum dynamics with mean field interactions: A new approach," *J. Stat. Phys.*, vol. 134, no. 5–6, pp. 859–870, 2009.
[32] J. Sirker, N. P. Konstantinidis, F. Andraschko, and N. Sedlmayr, "Locality and thermalization in closed quantum systems," *Phys. Rev. A*, vol. 89, 2014, Art. ID 042104.
[33] R. van Handel, J. K. Stockton, and H. Mabuchi, "Feedback control of quantum state reduction," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 768–780, 2005.
[34] M. Mirrahimi and R. van Handel, "Stabilizing feedback controls for quantum systems," *SIAM J. Control Optim.*, vol. 46, no. 2, pp. 445–467, 2007.
[35] B. Qi, H. Pan, and L. Guo, "Further results on stabilizing control of quantum systems," *IEEE Trans. Autom. Control*, vol. 58, no. 5, pp. 1349–1354, 2013.
[36] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Trans. Circuits Syst.*, vol. 42, pp. 430–447, 1995.
[37] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phy. Rev. Lett.*, vol. 80, pp. 2019–2112, 1998.
[38] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. Singapore: World Scientific, 2007.



Guodong Shi (M'15) received the Ph.D. degree in systems theory from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2010.

From 2010 to 2014 he was a Postdoctoral Researcher at the ACCESS Linnaeus Centre, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden. Since 2014 he has been with the College of Engineering and Computer Science, The Australian National University, Canberra, Australia, as a Lecturer and Future Engineering Research Leadership Fellow.

Dr. Shi was selected in the Triennial IFAC Young Author Prize Finalist in 2011, and received the Best Paper Award in Control Theory from the 11th World Congress on Intelligent Control and Automation in 2014.

Daoyi Dong (SM'11) received the B.E. degree in automatic control and the Ph.D. degree in engineering from the University of Science and Technology of China, Hefei, China, in 2001 and 2006, respectively.

Currently, he is a Senior Lecturer at the University of New South Wales, Canberra, Australia. He was with the Institute of Systems Science, Chinese Academy of Sciences, Beijing, China, and with the Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou, China. He had visiting positions at Princeton University, Princeton, NJ, USA, RIKEN, Wako, Japan, and The University of Hong Kong, Hong Kong. His research interests include quantum control, multiagent systems and intelligent control.

Dr. Dong received the International Collaboration Award and the Australian Post-Doctoral Fellowship from the Australian Research Council, and the Best Paper Award on Control Theory at The 11th World Congress on Intelligent Control and Automation (WCICA). He serves as an Associate Editor of IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS.

Ian R. Petersen (F'00) was born in Victoria, Australia. He received the Ph.D. degree in electrical engineering from the University of Rochester, Rochester, NY, USA, in 1984.

From 1983 to 1985, he was a Postdoctoral Fellow at the Australian National University. In 1985, he joined UNSW Canberra where he is currently Scientia Professor and an Australian Research Council Laureate Fellow in the School of Engineering and Information Technology. He has served as an Associate Editor for *Systems and Control Letters*, *Automatica*, and *SIAM Journal on Control and Optimization*. Currently, he is an Editor for *Automatica*. His main research interests are in robust control theory, quantum control theory and stochastic control theory.

Dr. Petersen has served as an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and is currently an Associate Editor for the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY. He is a Fellow of IFAC and the Australian Academy of Science.

Karl Henrik Johansson (F'13) received the M.Sc. and Ph.D. degrees in electrical engineering from Lund University, Lund, Sweden.

He is Director of the ACCESS Linnaeus Centre and Professor at the School of Electrical Engineering, KTH Royal Institute of Technology, Sweden. He is a Wallenberg Scholar and has held a six-year Senior Researcher Position with the Swedish Research Council. He is also heading the Stockholm Strategic Research Area ICT The Next Generation. He has held visiting positions at UC Berkeley (1998–2000) and California Institute of Technology (2006–2007). His research interests are in networked control systems, hybrid and embedded system, and applications in transportation, energy, and automation systems.

Dr. Johansson received the Best Paper Award of the IEEE International Conference on Mobile Ad-hoc and Sensor Systems in 2009 and the Best Theory Paper Award of the World Congress on Intelligent Control and Automation in 2014. In 2009, he was awarded Wallenberg Scholar, as one of the first ten scholars from all sciences, by the Knut and Alice Wallenberg Foundation. He was awarded an Individual Grant for the Advancement of Research Leaders from the Swedish Foundation for Strategic Research in 2005. He received the triennial Young Author Prize from IFAC in 1996 and the Peccei Award from the International Institute of System Analysis, Austria, in 1993. He received Young Researcher Awards from Scania in 1996 and from Ericsson in 1998 and 1999. He has been a member of the IEEE Control Systems Society Board of Governors and the Chair of the IFAC Technical Committee on Networked Systems. He has been on the Editorial Boards of several journals, including *Automatica*, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and *IET Control Theory and Applications*. He is currently on the Editorial Board of IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS and the *European Journal of Control*. He has been Guest Editor for special issues, including two issues of IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He was the General Chair of the ACM/IEEE Cyber-Physical Systems Week 2010 in Stockholm and IPC Chair of many conferences. He has served on the Executive Committees of several European research projects in the area of networked embedded systems.