Network Synchronization With Nonlinear Dynamics and Switching Interactions

Tao Yang, Ziyang Meng, Guodong Shi, Yiguang Hong, and Karl Henrik Johansson

***Abstract*—This technical note considers the synchronization problem for networks of coupled nonlinear dynamical systems under switching communication topologies. Two types of nonlinear agent dynamics are considered. The first one is non-expansive dynamics [stable dynamics with a convex Lyapunov function $\varphi(\cdot)$] and the second one is dynamics that satisfies a global Lipschitz condition. For the non-expansive case, we show that various forms of joint connectivity for communication graphs are sufficient for networks to achieve global asymptotic φ-synchronization. We also show that φ-synchronization leads to state synchronization provided that certain additional conditions are satisfied. For the globally Lipschitz case, unlike the non-expansive case, joint connectivity alone is not sufficient for achieving synchronization. A sufficient condition for reaching global exponential synchronization is established in terms of the relationship between the global Lipschitz constant and the network parameters.**

***Index Terms*—Multi-agent systems, nonlinear agents, switching interactions, synchronization.**

I. Introduction

We consider the synchronization problem for a network of coupled nonlinear agents with agent set $\mathcal{V} = \{1, 2, \ldots, N\}$. Their interactions (communications in the network) are described by a time-varying directed graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, with $\sigma : [0, \infty) \to \mathcal{P}$ as a piecewise constant signal, where $\mathcal{P}$ is a finite set of all possible graphs over $\mathcal{V}$. The state of agent $i \in \mathcal{V}$ at time t is denoted as $x_i(t) \in \mathbb{R}^n$ and evolves as

$$\dot{x}_i = f(t, x_i) + \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(t)(x_j - x_i), \quad i \in \mathcal{V} \tag{1}$$

where $f(t, x_i) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and continuous in x_i representing the uncoupled inherent agent dynamics, $\mathcal{N}_i(\sigma(t)) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_{\sigma(t)}\}$ is the set of agent i's neighbors at time t, and $a_{ij}(t)$ is a piecewise continuous function marking the weight of edge (j, i) at time t. Note that $\mathcal{N}_i(\sigma(t))$ may be an empty set at certain time intervals.

Systems of the form (1) have attracted considerable attention. Most works focus on the case where the communication graph $\mathcal{G}_{\sigma(t)}$ is fixed, e.g., [1]–[7]. It is shown that for the case where $f(t, x_i)$ satisfies a Lipschitz condition, synchronization is achieved for a connected graph provided that the coupling strength is sufficiently large. However, for the case where the communication graph is time-varying, the synchronization problem becomes much more challenging and existing literature mainly focuses on a few special cases when $f(t, x_i)$ is linear, e.g., the single-integrator case [8]–[10], the double-integrator case [11], and the neutrally stable case [12], [13]. Other studies assume some particular structures for the communication graph [14]–[16]. In particular, in [14], the authors focus on the case where the adjacency matrices associated with all communication graphs are simultaneously triangularizable. The authors of [15] consider switching communication graphs that are weakly connected and balanced at all times. A more general case where the switching communication graph frequently has a directed spanning tree has been considered in [16]. These special structures on the switching communication graph are rather restrictive compared to joint connectivity where the communication can be lost at any time.

This technical note aims to investigate whether joint connectivity for switching communication graphs can yield synchronization for the nonlinear dynamics (1). We distinguish two cases depending on whether the nonlinear agent dynamics $f(t, x_i)$ is expansive or not. For the non-expansive case, we focus on the case where the nonlinear agent dynamics is stable with a convex Lyapunov function $\varphi(\cdot)$. We show that various forms of joint connectivity for communication graphs are sufficient for networks to achieve global asymptotic φ-synchronization, that is, the function φ of the agent state converges to a common value. We also show that φ-synchronization implies state synchronization provided that additional conditions are satisfied. For the expansive case, we focus on when the nonlinear agent dynamics is globally Lipschitz, and establish a sufficient condition for networks to achieve global exponential synchronization in terms of a relationship between the Lipschitz constant and the network parameters.

The remainder of this technical note is organized as follows. Section II presents the problem definition and main results. Section III provides all technical proofs. Finally, concluding remarks are offered in Section IV.

II. Problem Definition and Main Results

A. Problem Set-Up

In this technical note, we consider the network of coupled nonlinear agents (1). The communications are modeled by a time-varying directed graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$. The joint graph of $\mathcal{G}_{\sigma(t)}$ in the time interval $[t_1, t_2)$ with $t_1 < t_2 \leq \infty$ is denoted as $\mathcal{G}([t_1, t_2)) = \cup_{t \in [t_1, t_2)} \mathcal{G}(t) = (\mathcal{V}, \cup_{t \in [t_1, t_2)} \mathcal{E}_{\sigma(t)})$. For the communication graph, we introduce the following definition.

Manuscript received August 23, 2015; revised October 23, 2015; accepted November 2, 2015. Date of publication November 4, 2015; date of current version September 23, 2016. This work was supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council. Recommended by Associate Editor S. Zampieri.

T. Yang is with the Pacific Northwest National Laboratory, Richland, WA 99352 USA (e-mail: Tao.Yang@pnnl.gov).

Z. Meng is with the Department of Precision Instrument and the State Key Laboratory of Precision Measurement Technology and Instruments, Tsinghua University, Beijing 100084, China (e-mail: ziyangmeng@mail.tsinghua.edu.cn).

G. Shi is with the College of Engineering and Computer Science, The Australian National University, Canberra, ACT 0200, Australia (e-mail: guodong.shi@anu.edu.au).

Y. Hong is with the Key Laboratory of Systems and Control, Institute of Systems Science, Chinese Academy of Science, Beijing 100190, China (e-mail: yghong@iss.ac.cn).

K. H. Johansson is with the ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, 10044 Stockholm, Sweden (e-mail: kallej@kth.se).

Digital Object Identifier 10.1109/TAC.2015.2497907

Definition 1: (i) $\mathcal{G}_{\sigma(t)}$ is *uniformly jointly strongly connected* if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is strongly connected for any $t \geq 0$.

(ii) Assume that $\mathcal{G}_{\sigma(t)}$ is undirected for all $t \geq 0$. $\mathcal{G}_{\sigma(t)}$ is *infinitely jointly connected* if $\mathcal{G}([t, \infty))$ is connected for any $t \geq 0$.

We assume that there are constants $0 < a_* \leq a^*$ such that $a_* \leq a_{ij}(t) \leq a^*$ for all $t \geq 0$. We also make a standard dwell time assumption [17] on the switching signal $\sigma(t)$: there is a lower bound $\tau_D > 0$ between two consecutive switching time instants of $\sigma(t)$. We denote $x = [x_1^T, x_2^T, \ldots, x_N^T]^T \in \mathbb{R}^{nN}$ and assume that the initial time is $t = t_0 \geq 0$, and the initial state $x(t_0) = (x_1^T(t_0), \ldots, x_N^T(t_0))^T \in \mathbb{R}^{nN}$.

In this technical note, we are interested in the following synchronization problems.

Definition 2: The multi-agent system (1) achieves global asymptotic φ-synchronization, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, if for any initial state $x(t_0)$, there exists a constant $d_\star(x(t_0))$, such that $\lim_{t\to\infty} \varphi(x_i(t)) = d_\star(x(t_0))$ for any $i \in \mathcal{V}$ and any $t_0 \geq 0$.

Definition 3: (i) The multi-agent system (1) achieves global asymptotic synchronization if $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ for any $i, j \in \mathcal{V}$, any $t_0 \geq 0$ and any $x(t_0) \in \mathbb{R}^{nN}$.

Multi-agent system (1) achieves global exponential synchronization if there exist $\gamma \geq 1$ and $\lambda > 0$ such that

$$\max_{\{i,j\}\in\mathcal{V}\times\mathcal{V}} \|x_i(t) - x_j(t)\|^2 \leq \gamma e^{-\lambda(t-t_0)} \times \max_{\{i,j\}\in\mathcal{V}\times\mathcal{V}} \|x_i(t_0) - x_j(t_0)\|^2, \quad t \geq t_0 \tag{2}$$

for any $t_0 \geq 0$ and any $x(t_0) \in \mathbb{R}^{nN}$.

Remark 1: φ-synchronization is a type of output synchronization where the output of agent $i \in \mathcal{V}$ is chosen to be $\varphi(x_i)$. It is related to but different from χ-synchronization [18], [19] since φ is a function of an individual agent state while χ is a function of all agent states.

B. Non-Expansive Dynamics

In this section, we focus on when the nonlinear agent dynamics is non-expansive as indicated by the following assumption.

Assumption 1: $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable positive definite convex function satisfying

(1). $\lim_{\|\eta\|\to\infty} \varphi(\eta) = \infty$;
(2). $\langle \nabla\varphi(\eta), f(t, \eta)\rangle \leq 0$ for any $\eta \in \mathbb{R}^n$ and any $t \geq 0$.

The following lemma shows how Assumption 1 enforces non-expansive dynamics.

Lemma 1: Let Assumption 1 hold. Along the multi-agent dynamics (1), $\max_{i\in\mathcal{V}} \varphi(x_i(t))$ is non-increasing for all $t \geq 0$.

We now state main results for the non-expansive case.

Theorem 1: Let Assumption 1 hold. The multi-agent system (1) achieves global asymptotic φ-synchronization if $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected.

Theorem 2: Let Assumption 1 hold. Assume that $\mathcal{G}_{\sigma(t)}$ is undirected for all $t \geq t_0$. The multi-agent system (1) achieves global asymptotic φ-synchronization if $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected.

Remark 2: For the linear time-varying case $f(t, x) = A(t)x$, if there exists a matrix $P = P^T > 0$ such that

$$PA(t) + A^T(t)P \leq 0, \quad \forall\, t \geq 0 \tag{3}$$

then $\varphi(x) = x^T P x$ for $x \in \mathbb{R}^n$ satisfies Assumption 1. For the linear time-invariant case $f(t, x) = Ax$, the condition (3) is equivalent to that the matrix A is neutrally stable [13].

C. φ-Synchronization vs. State Synchronization

The following result establishes conditions under which φ-synchronization implies state synchronization.

Theorem 3: Let $\mathcal{G}_{\sigma(t)} \equiv \mathcal{G}$ with $\mathcal{G}$ being a fixed, strongly connected digraph under which the multi-agent system (1) achieves global asymptotic φ-synchronization for some positive definite function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Let Assumption 1 hold. Moreover, assume that

1) $f(t, \eta)$ is bounded for any $t \geq 0$ and any $\eta \in \mathbb{R}^n$.
2) $c_1\|\eta\|^2 \leq \varphi(\eta) \leq c_2\|\eta\|^2$ for some $0 < c_1 \leq c_2$;
3) $\varphi(\cdot)$ is strongly convex.

Then the multi-agent system (1) achieves global asymptotic synchronization.

D. Lipschitz Dynamics

We consider also the case when the nonlinear agent dynamics is possibly expansive. We focus on when the dynamics satisfies the following global Lipschitz condition.

Assumption 2: There exists a constant $L > 0$ such that

$$\|f(t, \eta) - f(t, \zeta)\| \leq L\, \|\eta - \zeta\|, \quad \forall \eta, \zeta \in \mathbb{R}^n,\ \forall\, t \geq 0. \tag{4}$$

Our main result for this case is given below.

Theorem 4: Let Assumption 2 hold. Assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. The multi-agent system (1) achieves global exponential synchronization if $L < \rho_*/2$, where ρ_* is a constant depending on the network parameters.

Remark 3: Assumption 2 and its variants have been considered in the literature for fixed communication graphs, e.g., [1]–[7]. Compared with the existing literature, we here study a more challenging case, where the communication graphs are time-varying. Unlike the fixed case where the global Lipschitz condition is sufficient to guarantee synchronization, Theorem 4 established a sufficient synchronization condition related to the Lipschitz constant and the network parameters.

III. Proofs of Main Results

In this section, we provide proofs of the main results.

A. Proof of Lemma 1

Recall that the upper Dini derivative of a function $h(t) : (a, b) \to \mathbb{R}$ at t is defined as $D^+h(t) = \limsup_{s\to 0^+}((h(t+s) - h(t))/s)$. The following lemma from [10], [20] is useful for the proof.

Lemma 2: Let $V_i(t, x)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, N)$ be continuously differentiable and $V(t, x) = \max_{i=1,\ldots,N} V_i(t, x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \ldots, N\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t, then $D^+V(t, x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t, x(t))$.

Denote $\mathcal{I}(t) = \{i \in \mathcal{V} : \max_{j\in\mathcal{V}} \varphi(x_j(t)) = \varphi(x_i(t))\}$. We first note that the convexity property of $\varphi(\cdot)$ implies that [21, pp. 69]

$$\langle \nabla\varphi(\eta), \zeta - \eta\rangle \leq \varphi(\zeta) - \varphi(\eta), \quad \forall\, \eta, \zeta \in \mathbb{R}^n. \tag{5}$$

It then follows from Lemma 2, Assumption 1(ii) and (5) that:

$$D^+ \max_{i\in\mathcal{V}} \varphi(x_i(t)) = \max_{i\in\mathcal{I}(t)} \left\langle \nabla\varphi(x_i), f(t, x_i) + \sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(t)(x_j - x_i) \right\rangle \leq \max_{i\in\mathcal{I}(t)} \sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(t)\left(\varphi(x_j) - \varphi(x_i)\right) \leq 0$$

where the last inequality follows from $\varphi(x_j) \leq \varphi(x_i)$. This proves the lemma.

B. Proof of Theorem 1

It follows from Lemma 1 that for any initial state $x(t_0) \in \mathbb{R}^{nN}$, there exists a constant $d_* = d_\star(x(t_0)) \geq 0$, such that $\lim_{t\to\infty} \max_{i\in\mathcal{V}} \varphi(x_i) = d_\star$. We shall show that $d_\star$ is the required constant in Definition 2 of φ-synchronization.

We first note that, by Lemma 1, for all $i \in \mathcal{V}$, there exist constants $0 \leq \alpha_i \leq \beta_i \leq d_\star$, such that

$$\liminf_{t\to\infty} \varphi(x_i(t)) = \alpha_i, \quad \limsup_{t\to\infty} \varphi(x_i(t)) = \beta_i.$$

Also note that it follows from $\lim_{t\to\infty} \max_{i\in\mathcal{V}} \varphi(x_i(t)) = d_\star$ that for any $\varepsilon > 0$, there exists $T_1(\varepsilon) > 0$ such that:

$$\varphi(x_i(t)) \in [0, d_\star + \varepsilon], \quad \forall\, i \in \mathcal{V},\ \forall\, t \geq T_1(\varepsilon). \tag{6}$$

The proof of Theorem 1 is based on a contradiction argument and relies on the following lemma.

Lemma 3: Let Assumption 1 hold. Assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. If there exists an agent $k_0 \in \mathcal{V}$ such that $0 \leq \alpha_{k_0} < d_\star$, then there exists $0 < \bar{\rho} < 1$ and $\bar{t}$ such that for all $i \in \mathcal{V}$, $\varphi(x_i(\bar{t} + (N-1)T_0)) \leq \bar{\rho} M_0 + (1-\bar{\rho})(d_\star + \varepsilon)$, where

$$T_0 \triangleq T + 2\tau_D \tag{7}$$

with T given in Definition 1(i) and τ_D is the dwell time.

Proof: Let us first define $M_0 \triangleq ((\alpha_{k_0} + \beta_{k_0})/2) < d_\star$. Then there exists an infinite time sequence $t_0 < \tilde{t}_1 < \cdots < \tilde{t}_k < \cdots$ with $\lim_{k\to\infty} \tilde{t}_k = \infty$ such that $\varphi(x(\tilde{t}_k)) = M_0$ for all $k = 1, 2, \ldots$. Let then $\tilde{t}_k$ be greater than or equal to $T_1(\varepsilon)$, and denote it by $\tilde{t}_{k_0}$.

We now prove the lemma by estimating an upper bound of the scalar function $\varphi(x_i)$ agent by agent. The proof is based on a generalization of the method proposed in the proof of [22, Lemma 4.3] but with substantial differences on the dynamics and the Lyapunov function. Moreover, the convexity of $\varphi(\cdot)$ plays an important role.

Step 1. Focus on agent k_0. By using Assumption 1(ii), (5), and (6), we obtain that for all $t \geq \tilde{t}_{k_0}$

$$\begin{aligned}\frac{d}{dt}\varphi(x_{k_0}(t)) &= \left\langle \nabla\varphi(x_{k_0}), f(t, x_{k_0}) + \sum_{j\in\mathcal{N}_{k_0}(\sigma(t))} a_{k_0 j}(t)(x_j - x_{k_0}) \right\rangle \\ &\leq \sum_{j\in\mathcal{N}_{k_0}(\sigma(t))} a_{k_0 j}(t)\left(\varphi(x_j) - \varphi(x_{k_0})\right) \\ &\leq a^*(N-1)\left(d_\star + \varepsilon - \varphi(x_{k_0})\right).\end{aligned}$$

It then follows that for all $t \geq \tilde{t}_{k_0}$:

$$\varphi(x_{k_0}(t)) \leq e^{-\lambda_1(t-\tilde{t}_{k_0})}\varphi\left(x_{k_0}\left(\tilde{t}_{k_0}\right)\right) + \left(1 - e^{-\lambda_1(t-\tilde{t}_{k_0})}\right)(d_\star + \varepsilon) \tag{8}$$

where $\lambda_1 = a^*(N-1)$.

Step 2. Consider agent $k_1 \neq k_0$ such that $(k_0, k_1) \in \mathcal{E}_{\sigma(t)}$ for $t \in [\tilde{t}_{k_0}, \tilde{t}_{k_0} + T_0)$. The existence of such an agent can be shown as follows. Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, it is not hard to see that there exists an agent $k_1 \neq k_0 \in \mathcal{V}$ and $t_1 \geq \tilde{t}_{k_0}$ such that $(k_0, k_1) \in \mathcal{E}_{\sigma(t)}$ for $t \in [t_1, t_1 + \tau_D) \subseteq [\tilde{t}_{k_0}, \tilde{t}_{k_0} + T_0)$.

From (8), we obtain that for all $t \in [\tilde{t}_{k_0}, \tilde{t}_{k_0} + (N-1)T_0]$

$$\varphi(x_{k_0}(t)) \leq \kappa_0 \triangleq \rho M_0 + (1-\rho)(d_\star + \varepsilon) \tag{9}$$

where $\rho = e^{-\lambda_1(N-1)T_0} = e^{-a^*(N-1)^2 T_0}$.

We can similarly bound $\varphi(x_{k_1}(t))$ by considering two different cases: (i) $\varphi(x_{k_1}(t)) > \varphi(x_{k_0}(t))$ for all $t \in [t_1, t_1 + \tau_D)$ and (ii) there exists a time instant $\bar{t}_1 \in [t_1, t_1 + \tau_D)$ such that $\varphi(x_{k_1}(\bar{t}_1)) \leq \varphi(x_{k_0}(\bar{t}_1)) \leq \kappa_0$. For both cases, we obtain that for all $t \in [t_1 + \tau_D, \tilde{t}_{k_0} + (N-1)T_0]$

$$\varphi(x_{k_1}(t)) \leq \varphi_1 M_0 + (1-\varphi_1)(d_\star + \varepsilon).$$

where $\varphi_1 = (1-\mu)\rho^2$.

It then follows from (9) and $0 < \varphi_1 < \rho < 1$ that for all $t \in [t_1 + \tau_D, \tilde{t}_{k_0} + (N-1)T_0]$:

$$\varphi(x_j(t)) \leq \varphi_1 M_0 + (1-\varphi_1)(d_\star + \varepsilon), \quad j \in \{k_0, k_1\}. \tag{10}$$

Step 3. Consider agent $k_2 \notin \{k_0, k_1\}$ such that there exists an edge from the set $\{k_0, k_1\}$ to the agent k_2 in $\mathcal{E}_{\sigma(t)}$ for $t \in [t_2, t_2 + \tau_D) \subseteq [\tilde{t}_{k_0} + T_0, \tilde{t}_{k_0} + 2T_0)$. The existence of such an agent k_2 and t_2 follows similarly from the argument in Step 2.

Similarly, we can bound $\varphi(x_{k_2}(t))$ by considering two different cases and obtain that for all $t \in [t_2 + \tau_D, \tilde{t}_{k_0} + (N-1)T_0]$

$$\varphi(x_{k_2}(t)) \leq \varphi_2 M_0 + (1-\varphi_2)(d_\star + \varepsilon) \tag{11}$$

where $\varphi_2 = ((1-\mu)\rho^2)^2$.

By combining (10) and (11), and using $0 < \varphi_2 < \varphi_1 < 1$, we obtain that for all $t \in [t_2 + \tau_D, \tilde{t}_{k_0} + (N-1)T_0]$

$$\varphi(x_j(t)) \leq \varphi_2 M_0 + (1-\varphi_2)(d_\star + \varepsilon), \quad j \in \{k_0, k_1, k_2\}.$$

Step 4. By repeating the above process on time intervals $[\tilde{t}_{k_0} + 2T_0, \tilde{t}_{k_0} + 3T_0), \ldots, [\tilde{t}_{k_0} + (N-2)T_0, \tilde{t}_{k_0} + (N-1)T_0)$, we eventually obtain that for all $i \in \mathcal{V}$

$$\varphi\left(x_i\left(\tilde{t}_{k_0} + (N-1)T_0\right)\right) \leq \varphi_{N-1} M_0 + (1-\varphi_{N-1})(d_\star + \varepsilon).$$

where $\varphi_{N-1} = ((1-\mu)\rho^2)^{N-1}$. The result of the lemma then follows by choosing $\bar{\rho} = \varphi_{N-1}$ and $\bar{t} = \tilde{t}_{k_0}$. ■

We are now ready to prove Theorem 1 by contradiction. Suppose that there exists an agent $k_0 \in \mathcal{V}$ such that $0 \leq \alpha_{k_0} < d_\star$. It then follows from Lemma 3 that $\varphi(x_i(\bar{t} + (N-1)T_0)) < d_\star$ for all $i \in \mathcal{V}$, provided that $\varepsilon < (\bar{\rho}(d_\star - M_0)/(1-\bar{\rho}))$. This contradicts the fact that $\lim_{t\to\infty} \max_{i\in\mathcal{V}} \varphi(x_i) = d_\star$. Thus, there does not exist an agent $k_0 \in \mathcal{V}$ such that $0 \leq \alpha_{k_0} < d_\star$. Hence, $\lim_{t\to\infty} \varphi(x_i(t)) = d_\star$ for all $i \in \mathcal{V}$.

C. Proof of Theorem 2

The proof relies on the following lemma.

Lemma 4: Let Assumption 1 hold. Assume that $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected. If there exists an agent $k_0 \in \mathcal{V}$ such that $0 \leq \alpha_{k_0} < d_\star$, then there exist $0 < \tilde{\rho} < 1$ and $\tilde{t}$ such that

$$\varphi\left(x_i(\tilde{t} + \tau_D)\right) \leq \tilde{\rho} M_0 + (1-\tilde{\rho})(d_\star + \varepsilon), \quad \forall\, i \in \mathcal{V}.$$

The proof of Lemma 4 is similar to that of Lemma 3 and based on estimating an upper bound for the scalar quantity $\varphi(x_i)$ agent by agent. However, the method by which we obtained the sequence of agents $k_0, k_1, \ldots, k_{N-1}$ in the proof of Theorem 1 does not apply under the condition that $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected. We can however apply the strategy in [22]. We omit the details and refer to [23] for a complete proof.

The remaining proof of Theorem 2 follows from a contradiction argument and Lemma 4 in the same way as the proof of Theorem 1.

D. Proof of Theorem 3

If the multi-agent system (1) reaches asymptotic φ-synchronization, i.e., $\lim_{t\to\infty}\varphi(x_i(t)) = d_\star$ for all $i \in \mathcal{V}$, then for any $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that

$$d_\star - \epsilon \leq \varphi(x_i(t)) \leq d_\star + \epsilon, \quad \forall i \in \mathcal{V},\ \forall t \geq T_\epsilon. \tag{12}$$

If $d_\star = 0$ the desired conclusion holds trivially, i.e., $\lim_{t\to\infty} x_i(t) = 0$ for all $i \in \mathcal{V}$ due to the positive definiteness of $\varphi(\cdot)$. In the remainder of the proof we assume $d_\star > 0$. We shall prove the result by contradiction. Suppose that state synchronization is not achieved. Then there exist two agents $i_0, j_0 \in \mathcal{V}$ such that $\limsup_{t\to\infty}\|x_{i_0}(t) - x_{j_0}(t)\| > 0$. In other words, there exist an infinite time sequence $t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to\infty} t_k = \infty$, and a constant $\delta > 0$ such that $\|x_{i_0}(t_k) - x_{j_0}(t_k)\| = 2^{N-1}\sqrt{\delta/c_1}$ for all $k \geq 1$, where c_1 is given in condition (ii) of Theorem 3. We divide the following analysis into three steps.

Step 1. In this step, we prove the following crucial claim.

Claim. For any t_k, there are two agents $i_*, j_* \in \mathcal{V}$ with $(i_*, j_*) \in \mathcal{E}$ such that $\|x_{i_*}(t_k) - x_{j_*}(t_k)\| \geq \sqrt{\delta/c_1}$.

We establish this claim via a recursive analysis. If either $(i_0, j_0) \in \mathcal{E}$ or $(j_0, i_0) \in \mathcal{E}$ then the result follows trivially. Otherwise we pick up another agent k_0 satisfying that there is an edge between k_0 and $\{i_0, j_0\}$. This k_0 always exists since $\mathcal{G}$ is strongly connected. Then either $\|x_{k_0}(t_k) - x_{i_0}(t_k)\| \geq 2^{N-2}\sqrt{\delta/c_1}$ or $\|x_{k_0}(t_k) - x_{j_0}(t_k)\| \geq 2^{N-2}\sqrt{\delta/c_1}$ must hold. Thus, we have again either established the claim, or we can continue to select another agent different from i_0, j_0, and k_0 and repeat the argument. Since we have a finite number of agents, the desired claim holds.

Furthermore, since there is a finite number of agent pairs, without loss of generality, we assume that the given agent pair i_*, j_* does not vary for different t_k (otherwise we can always select an infinite subsequence of t_k for the following discussions).

Step 2. In this step, we establish a lower bound of $\|x_{i_*}(t) - x_{j_*}(t)\|^2$ for a small time interval after a particular t_k satisfying $t_k > T_\epsilon$. From (12) and $\lim_{\|x\|\to\infty}\varphi(x) = \infty$ given in Assumption 1(i), we see that $x_i(t)$ and $\nabla\varphi(x_i(t) - x_j(t))$ are bounded for all $i, j \in \mathcal{V}$ and for all $t \geq T_\epsilon$. It then follows from condition (i) of Theorem 3 and (1) that:

$$\begin{aligned}
&\left|\frac{d}{dt}\varphi\left(x_{i_*} - x_{j_*}\right)\right| \\
&\quad \leq \left|\left\langle \nabla\varphi\left(x_{i_*} - x_{j_*}\right), f\left(t, x_{i_*}\right) - f\left(t, x_{j_*}\right)\right\rangle\right| \\
&\qquad + \left|\left\langle \nabla\varphi\left(x_{i_*} - x_{j_*}\right), \sum_{k_1 \in \mathcal{N}_{i_*}(\sigma(t))} a_{i_* k_1}(t)\left(x_{k_1} - x_{i_*}\right)\right\rangle\right| \\
&\qquad + \left|\left\langle \nabla\varphi\left(x_{i_*} - x_{j_*}\right), \sum_{k_2 \in \mathcal{N}_{j_*}(\sigma(t))} a_{j_* k_2}(t)\left(x_{k_2} - x_{j_*}\right)\right\rangle\right| \leq L_*
\end{aligned}$$

for all $t \geq t_k$ and some $L_* > 0$.

Without loss of generality we assume that $\epsilon \leq 1$. Then L_* will be independent of ϵ. By plugging in the fact that $\|x_{i_*}(t_k) - x_{j_*}(t_k)\| \geq \sqrt{\delta/c_1}$ and using the condition (ii) of Theorem 3, we obtain that

$$\|x_{i_*}(t) - x_{j_*}(t)\|^2 \geq \frac{\delta}{2c_2}, \quad t \in \left[t_k, t_k + \frac{\delta}{2L_*}\right]. \tag{13}$$

Step 3. We first note that the strong convexity of $\varphi(\cdot)$ implies that [21, pp. 459] there exists an $m > 0$ such that

$$\langle\nabla\varphi(\eta), \zeta - \eta\rangle \leq \varphi(\zeta) - \varphi(\eta) - \frac{m}{2}\|\eta - \zeta\|^2, \quad \forall \eta, \zeta \in \mathbb{R}^n. \tag{14}$$

By using Assumption 1(ii) and (14), we obtain for $t \geq T_\epsilon$

$$\begin{aligned}
\frac{d}{dt}\varphi(x_{j_*}(t)) \leq\ & a_{j_* i_*}(t)|\varphi(x_{i_*}) - \varphi(x_{j_*})| - \frac{m}{2}a_{j_* i_*}(t)\|x_{i_*}(t) - x_{j_*}(t)\|^2 \\
& + \sum_{k \in \mathcal{N}_{j_*}(\sigma(t))\setminus\{i_*\}} a_{j_* k}(t)\left|\varphi(x_k) - \varphi\left(x_{j_*}\right)\right|.
\end{aligned} \tag{15}$$

By using (12), (13), (15), and condition (ii) of Theorem 3, we obtain that for $t \in [t_k, t_k + (\delta/2L_*)]$

$$\frac{d}{dt}\varphi\left(x_{j_*}(t)\right) \leq 2(N-1)a^*\epsilon - \frac{a_* m\delta}{4c_2}$$

which yields

$$\varphi\left(x_{j_*}\left(t_k + \frac{\delta}{2L_*}\right)\right) \leq d_\star + \epsilon + \left[2(N-1)a^*\epsilon - \frac{a_* m\delta}{4c_2}\right]\frac{\delta}{2L_*}.$$

It is then straightforward to see that

$$\varphi\left(x_{j_*}\left(t_k + \frac{\delta}{2L_*}\right)\right) < d_\star - \frac{a_* m\delta^2}{32c_2 L_*}$$

if we take

$$\epsilon < \min\left\{\frac{a_* m\delta^2}{32c_2 L_*}, \frac{a_* m\delta}{16(N-1)a^* c_2}\right\}.$$

However, this contradicts the definition of φ-synchronization since t_k is arbitrarily chosen. This completes the proof and the desired conclusion holds.

E. Proof of Theorem 4

The proof is based on the convergence analysis of the scalar quantity

$$V(t, x(t)) = \max_{\{i,j\} \in \mathcal{V}\times\mathcal{V}} V_{ij}\left(t, x(t)\right) \tag{16}$$

where

$$V_{ij}\left(t, x(t)\right) = \frac{1}{2}e^{-2L(t-t_0)}\left\|x_i(t) - x_j(t)\right\|^2, \quad \forall\{i,j\} \in \mathcal{V}\times\mathcal{V}. \tag{17}$$

Unlike the contradiction argument used in the proof of Theorem 1, where the convergence rate is unclear, here we explicitly characterize the convergence rate. The proof relies on the following lemmas.

Lemma 5: Let Assumption 4 hold. Along the multi-agent dynamics (1), $V(t, x(t))$ is non-increasing for all $t \geq 0$.

Proof: Let $\overline{\mathcal{V}}_1 \times \overline{\mathcal{V}}_2$ be the set containing all the node pairs that reach the maximum at time t, i.e., $\overline{\mathcal{V}}_1(t) \times \overline{\mathcal{V}}_2(t) = \{\{i,j\} \in \mathcal{V} \times \mathcal{V} : V_{ij}(t) = V(t)\}$. It is not hard to obtain that

$$\begin{aligned} D^+V = \max_{\{i,j\}\in\overline{\mathcal{V}}_1\times\overline{\mathcal{V}}_2} \Bigg\{ & e^{-2L(t-t_0)}(x_i - x_j)^T \left(f(t,x_i) - f(t,x_j)\right) \\ & + e^{-2L(t-t_0)}(x_i - x_j)^T \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(t)(x_{k_1} - x_i) \\ & - e^{-2L(t-t_0)}(x_i - x_j)^T \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(t)(x_{k_2} - x_j) \\ & - Le^{-2L(t-t_0)}\|x_i - x_j\|^2 \Bigg\} \\ \leq \frac{1}{2} \max_{\{i,j\}\in\overline{\mathcal{V}}_1\times\overline{\mathcal{V}}_2} \Bigg\{ & e^{-2L(t-t_0)} \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(t) \\ & \times \left(\|x_j - x_{k_1}\|^2 - \|x_i - x_j\|^2\right) \\ & + e^{-2L(t-t_0)} \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(t) \\ & \times \left(\|x_i - x_{k_2}\|^2 - \|x_i - x_j\|^2\right) \Bigg\} \\ \leq \max_{\{i,j\}\in\overline{\mathcal{V}}_1\times\overline{\mathcal{V}}_2} \Bigg\{ & \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(t)\left(V_{jk_1} - V_{ij}\right) \\ & + \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(t)\left(V_{ik_2} - V_{ij}\right) \Bigg\} \leq 0 \end{aligned} \tag{18}$$

where the first equality follows from Lemma 2 and (1), the first inequality follows from (4) and $\pm ab \leq ((a^2 + b^2)/2)$ for all $a, b \in \mathbb{R}$, and the second inequality follows from (17). ■

Lemma 6: Let Assumption 4 hold. Assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. Then there exists $0 < \tilde{\beta} < 1$ such that

$$V_{ij}\left(\overline{N}T_0, x(\overline{N}T_0)\right) \leq \tilde{\beta}V^*, \quad \forall\{i,j\} \in \mathcal{V} \times \mathcal{V}$$

where $\overline{N} = N - 1$, T_0 given by (7) and $V^* = V(t_0, x(t_0))$.

Proof: The proof is based on the convergence analysis of $V_{ij}(t, x(t)$ for all agent pairs $\{i,j\} \in \mathcal{V} \times \mathcal{V}$ in several steps, which is similar to the proof of Lemma 3. Without loss of generality, we assume that $t_0 = 0$. We also sometimes denote $V(t, x(t))$ and $V_{ij}(t, x(t))$ as V and V_{ij}, respectively, for notational simplification.

Step 1. We begin by considering any agent $i_1 \in \mathcal{V}$. Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, we know that i_1 is the root and that there exists a time t_1 and an agent $i_2 \in \mathcal{V} \setminus \{i_1\}$ such that $(i_1, i_2) \in \mathcal{E}$ during $t \in [t_1, t_1 + \tau_D) \subset [0, T_0]$.

We first note that it follows from (16) and Lemma 5 that for all $t \in [0, \overline{N}T_0]$

$$V_{ij}(t, x(t)) \leq V(t, x(t)) \leq V^*, \quad \forall\{i,j\} \in \mathcal{V} \times \mathcal{V}. \tag{19}$$

Taking the derivative of V_{ij} along the trajectories of (1), we obtain that for all $t \in [t_1, t_1 + \tau_D)$

$$\begin{aligned} \dot{V}_{i_1i_2} = & - Le^{-2L(t-t_0)}\|x_{i_1} - x_{i_2}\|^2 + e^{-2L(t-t_0)}(x_{i_1} - x_{i_2})^T \\ & \times \Bigg\{ \sum_{k_1\in\mathcal{N}_{i_1}(\sigma(t))} a_{i_1k_1}(t)\left(x_{k_1} - x_{i_1}\right) \\ & - \sum_{k_2\in\mathcal{N}_{i_2}(\sigma(t))} a_{i_2k_2}(t)(x_{k_2} - x_{i_2}) + (f(t,x_{i_1}) - f(t,x_{i_2})) \Bigg\} \\ \leq & \sum_{k_1\in\mathcal{N}_{i_1}(\sigma(t))} a_{i_1k}(t)\left(V_{i_2k_1} - V_{i_1i_2}\right) - a_{i_2i_1}(t)V_{i_1i_2} \\ & + \sum_{k_2\in\mathcal{N}_{i_2}(\sigma(t))\setminus\{i_1\}} a_{i_2k_2}(t)\left(V_{i_1k_2} - V_{i_1i_2}\right) \\ \leq & (N-1)a^*\left(V^* - V_{i_1i_2}\right) - a_*V_{i_1i_2} + (N-2)a^*\left(V^* - V_{i_1i_2}\right) \\ = & -\alpha\left(V_{i_1i_2} - \frac{(2N-3)a^*}{\alpha}V^*\right) \end{aligned}$$

where $\alpha = (2N-3)a^* + a_*$. The first inequality follows from (4) and (17), while the second inequality follows from (19).

It thus follows that:

$$V_{i_1i_2}\left(t_1 + \tau_D, x(t_1 + \tau_D)\right) \leq \hat{\alpha}_1V^* \tag{20}$$

where $\hat{\alpha}_1 = 1 - (a_*/\alpha)(1 - e^{-\alpha\tau_D}) \in (0,1)$.

Similarly we obtain that for all $t \in [t_1 + \tau_D, \overline{N}T_0]$, $\dot{V}_{i_1i_2} \leq \overline{\alpha}(V^* - V_{i_1i_2})$, where $\overline{\alpha} = 2(N-1)a^*$. It then follows from (20) that:

$$V_{i_1i_2}\left(t, x(t)\right) \leq \alpha_1^*V^*, \quad \forall\, t \in [t_1 + \tau_D, \overline{N}T_0] \tag{21}$$

where $\alpha_1^* = 1 - (1 - e^{-\alpha\tau_D})(a_*/\alpha)e^{-\overline{\alpha}\overline{N}T_0} \in (0,1)$.

Step 2. Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, we know that that there exists a time instant t_2 and an arc from $h \in \mathcal{V}_1 \triangleq \{i_1, i_2\}$ to $i_3 \in \mathcal{V} \setminus \mathcal{V}_1$ during $[t_2, t_2 + \tau_D) \subset [T_0, 2T_0]$.

We then estimate an upper bound for V_{hi_3} by considering two different cases: $h = i_1$ and $h = i_2$. We eventually obtain that for all $t \in [t_2 + \tau_D, \overline{N}T_0]$

$$V_{hi_3}\left(t, x(t)\right) \leq \left(1 - \beta_*^2\right)V^*, \quad \forall\, h \in \mathcal{V}_1. \tag{22}$$

where

$$\beta_* = (1 - e^{-\alpha\tau_D})\frac{a_*}{\alpha}e^{-\overline{\alpha}\overline{N}T_0} \in (0,1). \tag{23}$$

It then follows from (21), (22), and $1 - \beta_*^2 > 1 - \beta_* = \alpha_1^*$ that for all $t \in [2T_0, \overline{N}T_0]$:

$$V_{i_1k}\left(t, x(t)\right) \leq \left(1 - \beta_*^2\right)V^*, \quad \forall\, k \in \mathcal{V}_2 \setminus \{i_1\}$$

where $\mathcal{V}_2 \triangleq \{i_1, i_2, i_3\}$.

Step 3. By continuing the above process, we obtain that for all $k \in \mathcal{V} \setminus \{i_1\}$

$$V_{i_1k}\left(\overline{N}T_0, x(\overline{N}T_0)\right) \leq \left(1 - \beta_*^{\overline{N}}\right)V^*. \tag{24}$$

Step 4. Since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, (24) holds for any $i_1 \in \mathcal{V}$. By using the same analysis, we eventually obtain that for all $i, j \in \mathcal{V}$, $V_{ij}(\overline{N}T_0, x(\overline{N}T_0)) \leq (1 - \beta_*^{\overline{N}})V^*$. Hence the result follows by choosing $\tilde{\beta} = 1 - \beta_*^{\overline{N}}$ with β_* given by (23). ■

We are now ready to prove Theorem 4. By using Lemma 6 and (16), we obtain that $V(t, x(t)) \leq \tilde{\beta}^{\lfloor t/\overline{N}T_0 \rfloor}V^* \leq (1/\tilde{\beta})e^{-\rho_*t}V^*$, where

$\lfloor t/\overline{N}T_0 \rfloor$ denotes the largest integer that is not greater than $(t/\overline{N}T_0)$ and $\rho_* = (1/\overline{N}T_0)\ln(1/\tilde{\beta})$.

It then follows from (16) and (17) that:

$$\max_{\{i,j\}\in\mathcal{V}\times\mathcal{V}} \|x_i(t)-x_j(t)\|^2 \leq \frac{1}{\tilde{\beta}} e^{-(\rho_*-2L)t} \max_{\{i,j\}\in\mathcal{V}\times\mathcal{V}} \|x_i(0)-x_j(0)\|^2.$$

Hence, global exponential synchronization is achieved with $\gamma = (1/\tilde{\beta})$ and $\lambda = \rho_* - 2L$ provided that $\rho_* > 2L$. This concludes the proof of the desired theorem.

IV. Conclusion

In this technical note, synchronization problems for networks with nonlinear agent dynamics and switching topologies have been investigated. Two types of nonlinear dynamics were considered: non-expansive and globally Lipschitz. For the non-expansive case, we found that the convexity of the Lyapunov function plays a crucial rule in the analysis and showed that the uniformly joint strong connectivity is sufficient for achieving global asymptotic φ-synchronization. When communication graphs are undirected, the infinitely joint connectivity is a sufficient synchronization condition. Moreover, we established conditions under which φ-synchronization implies state synchronization. For the globally Lipschitz case, we found that joint connectivity alone is not sufficient to achieve synchronization but established a sufficient synchronization condition. The proposed condition reveals the relationship between the Lipschitz constant and the network parameters. The results of this technical note can be extended to leader-follower networks, see [23] for details. An interesting future direction is to study the synchronization problem for coupled non-identical nonlinear dynamics under general switching topologies.

References

[1] C. W. Wu and L. O. Chua, "Application of graph theory to the synchronization in an array of coupled nonlinear oscillators," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 42, no. 8, pp. 494–497, Aug. 1995.

[2] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. Singapore: World Scientific: Singapore, 2007.

[3] V. Belykh, I. Belykh, and M. Hasler, "Connection graph stability method for synchronized coupled chaotic systems," *Physica D*, vol. 195, no. 1/2, pp. 159–187, Aug. 2004.

[4] P. DeLellis, M. D. Bernardo, and G. Russo, "On QUAD, Lipschitz, contracting vector fields for consensus and synchronization of networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 3, pp. 576–583, Mar. 2011.

[5] W. Yu, G. Chen, and M. Cao, "Consensus in directed networks of agents with nonlinear dynamics," *IEEE Trans. Autom. Control*, vol. 56, no. 6, pp. 1436–1441, Jun. 2011.

[6] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Consensus in multi-agent systems with coupling delays and switching topology," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2976–2982, Dec. 2011.

[7] H. Liu, M. Cao, and C. W. Wu, "Coupling strength allocation for synchronization in complex networks using spectral graph theory," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 5, pp. 1520–1530, May 2014.

[8] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[9] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[10] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous-time coupled nonlinear systems," *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 288–307, 2007.

[11] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[12] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, vol. 45, no. 11, pp. 2557–2562, Nov. 2009.

[13] Y. Su and J. Huang, "Stability of a class of linear switching systems with applications to two consensus problem," *IEEE Trans. Autom. Control*, vol. 57, no. 6, pp. 1420–1430, Jun. 2012.

[14] J. Zhao, D. J. Hill, and T. Liu, "Synchronization of complex dynamical networks with switching topology: A switched system point of view," *Automatica*, vol. 45, no. 11, pp. 2502–2511, Nov. 2009.

[15] J. Qin, H. Gao, and W. Zheng, "Exponential synchronization of complex networks of linear systems and nonlinear oscillators: A unified analysis," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 61, no. 2, pp. 499–511, Mar. 2014.

[16] G. Wen, Z. Duan, G. Chen, and W. Yu, "Consensus tracking of multi-agent systems with Lipschitz-type node dynamics and switching topologies," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 2, pp. 499–511, Feb. 2014.

[17] D. Liberzon and A. S. Morse, "Basic problem in stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, Oct. 1999.

[18] J. Cortés, "Distributed algorithms for reaching consensus on general functions," *Automatica*, vol. 44, no. 3, pp. 726–737, Mar. 2008.

[19] X. Wang and Y. Hong, "Distributed finite-time χ-consensus algorithms for multi-agent systems with variable coupling topology," *J. Syst. Sci. Complexity*, vol. 23, no. 2, pp. 209–218, 2010.

[20] J. Danskin, "The theory of max-min, with applications," *SIAM J. Appl. Math.*, vol. 14, no. 6, pp. 641–664, 1996.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge Univ. Press, 2004.

[22] G. Shi, K. H. Johansson, and Y. Hong, "Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 610–622, Mar. 2013.

[23] T. Yang, Z. Meng, G. Shi, Y. Hong, and K. H. Johansson, "Network synchronization with nonlinear dynamics and switching interactions," *Systems and Control*, pp. 1–27, 2015. [Online]. Available: http://arxiv.org/abs/1401.6541.