Periodic Behaviors for Discrete-Time Second-Order Multiagent Systems With Input Saturation Constraints

Tao Yang, *Member, IEEE*, Ziyang Meng, *Member, IEEE*, Dimos V. Dimarogonas, *Member, IEEE*, and Karl H. Johansson, *Fellow, IEEE*

***Abstract*—This brief considers the existence of periodic behaviors for discrete-time second-order multiagent systems with input saturation constraints. We first consider the case where the agent dynamics is a double integrator and then establish conditions on the feedback gains of the linear consensus control law for achieving periodic behaviors. This, in turn, shows that the previously established sufficient condition for reaching global consensus has a necessary aspect since these two conditions are exclusive. We further consider all other second-order agent dynamics and show that these multiagent systems under the linear consensus law exhibit periodic solutions provided the feedback gains satisfy certain conditions. Simulation results are used to validate the theoretical findings.**

***Index Terms*—Input saturation, multiagent systems, periodic behaviors.**

I. Introduction

In the multiagent systems literature, the consensus problem, where the goal is to achieve an asymptotic agreement on the agents' states, has been extensively studied. Various distributed control laws have been proposed to achieve consensus through neighboring information exchange, e.g., [1]–[4]. Recently, global consensus and semiglobal consensus for multiagent systems with input saturation constraints have been considered, e.g., [5]–[10]. In particular, it has been shown that the multiagent system with continuous-time double-integrator agent dynamics in the presence of input saturation constraints achieves global consensus under all locally linear consensus control laws [5]. On the other hand, we have previously shown that part of locally linear consensus control laws renders global consensus for the discrete-time counterpart [6]. However, whether global consensus is achieved when the other part of locally linear consensus laws is used is still unknown.

Manuscript received September 22, 2015; accepted December 13, 2015. Date of publication February 18, 2016; date of current version June 23, 2016. This brief was presented in part at the 52nd IEEE Conference on Decision and Control, Florence, Italy, December 10–13, 2013. This work was supported in part by the Knut and Alice Wallenberg Foundation, by the Swedish Research Council, and by the National Natural Science Foundation of China under Grant 61503249 and Grant 61403392. This brief was recommended by Associate Editor J. Lu.

T. Yang, D. V. Dimarogonas, and K. H. Johansson are with the ACCESS Linnaeus Center, School of Electrical Engineering, Royal Institute of Technology, 10044 Stockholm, Sweden (e-mail: taoyang@kth.se; dimos@kth.se; kallej@kth.se).

Z. Meng is with the State Key Laboratory of Precision Measurement Technology and Instruments, Department of Precision Instrument, Tsinghua University, Beijing 100084, China (e-mail: ziyangmeng@mail.tsinghua.edu.cn).

Color versions of one or more of the figures in this brief are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSII.2016.2532078

This motivates our study. In particular, we show that the multiagent system under the linear consensus control law exhibits periodic behaviors if the feedback gains satisfy certain conditions. This, in turn, implies that global consensus is not achieved. We further investigate the existence of the periodic behavior for all other second-order agent dynamics, including asymptotic stable, marginally stable, and unstable dynamics. We show that these multiagent systems under the linear consensus feedback law exhibit periodic behaviors provided that the feedback gains satisfy certain conditions.

The contribution of this brief is threefold: 1) Compared with the work in [11], where the existence of the periodic phenomenon has been shown for discrete-time single-integrator multiagent systems, the considered systems are of second order. Such an extension is not only challenging since the dynamics may diverge without control laws, but also useful since many physical systems can be modeled as second-order systems; 2) Compared with the work in [12], where only double-integrator agent dynamics has been considered, we consider all other second-order agent dynamics, including asymptotically stable, marginally stable, and unstable dynamics; and 3) this brief also extends the existence of periodic behaviors for individual discrete-time systems [13]–[15] to multiagent systems. Although this brief only studies the second-order multiagent systems, these systems are known as a key benchmark for the dynamical behavior of nonlinear multiagent systems. By fully understanding these systems, we make a key step in understanding the abilities of linear consensus control law for achieving periodic behaviors.

The remainder of this brief is organized as follows: In Section II, we present the motivation and formulate the considered problem. In Section III, we show that the multiagent system with the double-integrator dynamics under the linear consensus control law exhibits periodic solutions provided that the feedback gains satisfy certain conditions. In Section IV, we further show the existence of periodic behaviors for multiagent systems with all other second-order agent dynamics. Simulation examples are offered in Section V. Finally, Section VI concludes this brief.

II. Motivation and Problem Formulation

Consider a multiagent system of N identical discrete-time second-order linear systems, i.e.,

$$y_i(k+1) = Ay_i(k) + B\sigma\left(u_i(k)\right), \quad i \in \{1, \dots, N\} \tag{1}$$

where $y_i = [x_i; v_i] \in \mathbb{R}^2$, $u_i \in \mathbb{R}$, $\sigma(u_i)$ is the saturation function: $\sigma(u_i) = \mathrm{sgn}(u_i)\min\{1, |u_i|\}$, and the pair (A, B) describes the agent dynamics.

The network among agents is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with the set of agents $\mathcal{V} = \{1, \ldots, N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. We also assume that $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$. The set of neighboring agents of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} | a_{ij} > 0\}$. The linear consensus feedback control law with gain parameters α and β is given by

$$u_i(k) = \sum_{j\in\mathcal{N}_i} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_j(k) - y_i(k)). \tag{2}$$

For the case where the agent dynamics is a double integrator, i.e.,

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 1\end{bmatrix} \tag{3}$$

we have shown in [6] that the multiagent system (1) achieves global consensus under the linear consensus control law (2) if the feedback gain parameters satisfy the following condition:

$$0 < \sqrt{3}\alpha < \beta < \frac{3}{2\lambda_N} \tag{4}$$

where λ_N is the largest eigenvalue of the Laplacian matrix associated with the network.

It is natural to ask whether global consensus is still achieved if condition (4) is not satisfied. In this brief, we will show that if condition (4) is not fulfilled, the multiagent system may exhibit a nonconverging and nondiverging behavior, i.e., the periodic solution. In particular, we explicitly construct the linear consensus control law with appropriated gain parameters under which the multiagent system exhibits the periodic solution in the following sense.

Definition 1: A solution $y_i(k)$ of the multiagent system (1) under the linear consensus control law (2) is a periodic solution with period $T > 0$, if, for some initial states $y_i(0)$ for $i \in \{1, \ldots, N\}$, we have $y_i(k + T) = y_i(k)$ for all $i \in \{1, \ldots, N\}$ and for all $k = 0, 1, \ldots$.

III. Main Results

Our main result is given as follows.

Theorem 1: Consider the multiagent system (1) with the pair (A, B) given by (3) under the linear consensus control law (2). Suppose that $\mathcal{G}$ is connected. If the feedback gain parameters α and β satisfy

$$0 < \alpha < \beta < \tfrac{3}{2}\alpha \tag{5}$$

then there exist initial states such that the corresponding solution of the multiagent system is periodic with the period $T = 2m$, where

$$m \geq \frac{4(\alpha - \beta) + \frac{2}{\bar{a}}}{3\alpha - 2\beta} \tag{6}$$

$$\bar{a} = \min_{\substack{(i,j)\in\mathcal{E}\\ i\in S_e,\, j\in S_o}} a_{ij} \tag{7}$$

with S_e and S_o defined in the proof.

Proof: Since the graph is connected, without loss of generality, we assume that agent 1 is the root agent. We define the following sets based on whether the distance between agent $i \in \mathcal{V}$ and the root agent 1 is even or odd, i.e.,

$$S_e = \{i | d(i, 1) = 0, 2, \ldots\}, \; S_o = \{i | d(i, 1) = 1, 3, \ldots\} \tag{8}$$

where the distance between two nodes i and j, i.e., $d(i, j)$, is the number of edges of a path between i and j minimized over all possible paths.

We prove the theorem by explicitly constructing periodic solutions with an even period $T = 2m$. The periodic solution that we will construct is such that the input sequences (2) for all the agents are always in saturation, and for $i \in S_e$

$$u_i(k) \geq 1,\; k=0, \ldots, m-1,\; u_i(k) \leq -1,\; k = m, \ldots, 2m-1 \tag{9}$$

and for $i \in S_o$

$$u_i(k) \leq -1,\; k=0, \ldots, m-1,\; u_i(k) \geq 1,\; k=m, \ldots, 2m-1. \tag{10}$$

In what follows, we will show that (9) and (10) are satisfied for certain m and initial states $x_i(0)$ and $v_i(0)$ for $i \in \mathcal{V}$ and that the solution is periodic with period $T = 2m$ for these initial states. The proof has three steps.

Step 1: It is sufficient to show that $x_i(T) = x_i(0)$ and $v_i(T) = v_i(0)$ for all $i \in \mathcal{V}$. It follows from (1), (9), and (10) that $v_i(T) = v_i(0)$ for all $i \in \mathcal{V}$. It is also easy to obtain that $x_i(2m) = x_i(0) + 2mv_i(0) + m^2$ for $i \in S_e$ and $x_i(2m) = x_i(0) + 2mv_i(0) - m^2$ for $i \in S_o$. Thus, to have $x_i(T) = x_i(0)$ for all $i \in \mathcal{V}$, we must have that

$$\begin{cases} v_i(0) = -m/2, & i \in S_e\\ v_i(0) = \frac{m}{2}, & i \in S_o. \end{cases} \tag{11}$$

Step 2: We show that the $2m$ inequalities in either (9) or (10) can be reduced to two inequalities by appropriately choosing initial states $x_i(0)$ for some $i \in \mathcal{V}$.

Step 2.1: For agent $j \in S_o$, we have

$$\begin{aligned} u_j(k) &= \sum_{i\in\mathcal{N}_j} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_i(k) - y_j(k))\\ &= \sum_{i\in\mathcal{N}_j\cap S_e} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_i(k) - y_j(k))\\ &\quad + \sum_{i\in\mathcal{N}_j\cap S_o} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_i(k) - y_j(k)). \end{aligned}$$

Choosing $x_i(0) = x_j(0)$ for $i \in S_o$ if $(i, j) \in \mathcal{E}$ and using $v_i(0) = v_j(0)$ for all $i, j \in S_o$ gives

$$u_j(k) = \sum_{i\in\mathcal{N}_j\cap S_e} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_i(k) - y_j(k)). \tag{12}$$

Similarly, for agent $i \in S_e$, choosing $x_j(0) = x_i(0)$ for $j \in S_e$ if $(i, j) \in \mathcal{E}$ yields

$$u_i(k) = \sum_{j\in\mathcal{N}_i\cap S_o} a_{ij} \begin{bmatrix}\alpha & \beta\end{bmatrix} (y_j(k) - y_i(k)).$$

Step 2.2: Let us now focus on any edge $(i, j) \in \mathcal{E}$, such that $i \in S_e$ and $j \in S_o$. We first note that $0 < \alpha < \beta$ from (5) implies that $\beta > \alpha - (1/2)k\alpha$ for $k = 0, \ldots, m-1$, which yields $-(\alpha m/2) + \beta > (1/2)\alpha(-m - k + 2)$. Since $m - k - 1 \geq 0$, multiplying the above inequality on both sides with $m - k - 1$ yields

$$-\frac{\alpha m}{2}(m - k - 1) + \beta(m - k - 1) \geq \alpha \left[\frac{k(k-1)}{2} - \frac{(m-1)(m-2)}{2}\right].$$

This is equivalent to that

$$a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(m-1)-y_j(m-1)) \geq a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k)-y_j(k)) \quad (13)$$

for $k=0,\ldots,m-1$, since $v_i(0)=-(m/2)$ for $i\in S_e$, $v_j(0)=m/2$ for $j\in S_o$, and $a_{ij}\geq 0$.

Step 2.3: Since inequality (13) holds for each $i\in\mathcal{N}_j\cap S_e$, where $j\in S_o$, adding them up together with (12) yields

$$u_j(m-1)\geq u_j(k),\ k=0,\ldots,m-1,\quad j\in S_o.$$

Hence, for $j\in S_o, u_j(m-1)\leq -1$ implies that $u_j(k)\leq -1$ for all $k=0,\ldots,m-1$.

A similar argument shows that

$$u_j(2m-1)\leq u_j(k),\ k=m,\ldots,2m-1, j\in S_o.$$

Hence, for $j\in S_o, u_j(2m-1)\geq 1$ implies that $u_j(k)\geq 1$ for all $k=m,\ldots,2m-1$.

Similarly, we can show that for $i\in S_e, u_i(m-1)\geq 1$ implies that $u_i(k)\geq 1$ for $k=0,\ldots,m-1$, and that $u_i(2m-1)\leq -1$ implies that $u_i(k)\leq -1$ for $k=m,\ldots,2m-1$.

To summarize, if there is an edge connecting agents within S_e or S_o, we set their initial states to be the same, i.e.,

$$x_i(0)=x_j(0)\ \text{for}\ (i,j)\in\mathcal{E},\ \text{if}\ i,j\in S_e,\ \text{or}\ i,j\in S_o. \quad (14)$$

Then, inequality (9) or (10) is reduced to $u_j(m-1)\leq -1$ and $u_j(2m-1)\geq 1$ for $j\in S_o$ or $u_i(m-1)\geq 1$ and $u_i(2m-1)\leq -1$ for $i\in S_e$.

Step 3: It is clear that these two inequalities are satisfied provided that for each edge $(i,j)\in\mathcal{E}$, where $i\in S_e$ and $j\in S_o$, the following two conditions:

$$a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(m-1)-y_j(m-1)) = a_{ij}\{\alpha[x_i(0)-x_j(0)-2m+2]+\beta(m-2)\}\leq -1$$
$$a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(2m-1)-y_j(2m-1)) = a_{ij}\{\alpha[x_i(0)-x_j(0)+m-2]-\beta(m-2)\}\geq 1$$

are satisfied. They are equivalent to

$$\frac{1}{a_{ij}}+(\beta-\alpha)(m-2)\leq\alpha(x_i(0)-x_j(0)) \leq 2\alpha(m-1)-\beta(m-2)-\frac{1}{a_{ij}}. \quad (15)$$

We see that suitable $x_i(0)$ and $x_j(0)$, where $i\in S_e$, $j\in S_o$, and $(i,j)\in\mathcal{E}$, exist if

$$\frac{1}{a_{ij}}+(\beta-\alpha)(m-2)\leq 2\alpha(m-1)-\beta(m-2)-\frac{1}{a_{ij}}. \quad (16)$$

For $m>2$, (16) is equivalent to $\beta\leq((3m-4)/(2m-4))\alpha-(1/(a_{ij}(m-2)))$. If we take the value of m to be sufficiently large, we obtain that

$$\beta\leq\lim_{m\to+\infty}\left[\frac{3m-4}{2m-4}\alpha-\frac{1}{a_{ij}(m-2)}\right]=\frac{3}{2}\alpha.$$

Therefore, for any α and β that satisfy (5), if condition (6) holds, then (16) is satisfied.

From the given analysis, we see that the solution of the multiagent system is periodic with period $T=2m$, where m satisfies (6), for initial states satisfying (11), (14), and (15). ■

Remark 1: For the multiagent system (1) in the absence of input saturation constraints, where the agent dynamics is a double integrator and the undirected network is connected, it is shown in [16, Corollary 1] that the following condition:

$$0<\alpha<\beta<\frac{\alpha}{2}+\frac{2}{\lambda_N} \quad (17)$$

is a necessary and sufficient condition for achieving consensus. Note that condition (5) overlaps (17) provided that $\alpha<(2/\lambda_N)$. In view of this, the existence of the periodic solution shown in Theorem 1 is clearly due to the input saturation constraints.

Remark 2: Condition (5) implies that $0<\sqrt{3}\alpha<\beta<(3/2\lambda_N)$, which is a sufficient condition for achieving global consensus in the presence of input saturation constraints [6, Th. 2], has a necessary aspect since these conditions are exclusive in the sense that $(3\alpha/2)<\sqrt{3}\alpha$.

Remark 3: In [13] and [14], the periodic behaviors have been considered for individual discrete-time systems. Theorem 1 extends this result to multiagent systems for the double-integrator case. In particular, the specified input is explicitly designed based on the saturation and the linear consensus control law.

IV. Periodic Behaviors for Other Agent Dynamics

In this section, we further investigate the existence of periodic behaviors for multiagent systems with all controllable second-order agent dynamics. Without loss of generality, we assume that the pair (A,B) is in the following controllable canonical form:

$$A=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix} \quad (18)$$

since, otherwise, the system can be transferred into this form via a nonsingular state transformation [17, Th. 9.2].

Our main result for this case is given as follows.

Theorem 2: Consider the multiagent system (1) with the pair (A,B) given by (18) under the linear consensus control law (2). Suppose that $\mathcal{G}$ is connected and that A has no eigenvalues at ± 1 and $\pm j$. The feedback gain parameters α and β satisfy

$$(1-a_0+a_1)\alpha+(1-a_0-a_1)\beta\geq\frac{(1-a_0)^2+a_1^2}{2\bar{a}} \quad (19a)$$
$$(1-a_0-a_1)\alpha-(1-a_0+a_1)\beta\geq\frac{(1-a_0)^2+a_1^2}{2\bar{a}} \quad (19b)$$

where $\bar{a}$ is defined by (7). Then, there exist initial states such that the corresponding solution of the multiagent system is periodic with period $T=4$.

Proof: A periodic solution with period $T=4$ is such that the input sequences (2) for all the agents are always in saturation. Moreover, it holds that

$$\begin{cases}u_i(k)\geq 1, & k=0,1,\\ u_i(k)\leq -1, & k=2,3,\end{cases}\ i\in S_e \quad (20)$$

$$\begin{cases}u_i(k)\leq -1, & k=0,1,\\ u_i(k)\geq -1, & k=2,3,\end{cases}\ i\in S_o. \quad (21)$$

In what follows, we will show that (20) and (21) are satisfied for certain initial states $y_i(0), i \in \mathcal{V}$ and that the solution is periodic with period $T = 4$. Again, the proof is carried out in three steps.

Step 1: It follows from (1) and (20) that for $i \in S_e, y_i(4) = A^4 y_i(0) + A^3 B + A^2 B - AB - B$. Thus, to have $y_i(0) = y_i(4)$ for $i \in S_e$, we need

$$\begin{aligned} y_i(0) &= (I - A^4)^{-1}(A^3B + A^2B - AB - B) \\ &= -(I + A^2)^{-1}(I + A)B \end{aligned}$$

where we have used the assumption on the eigenvalues of A.

By plugging in matrices A and B, given in (18), into this equation, we obtain that

$$y_i(0) = -\frac{1}{(1-a_0)^2 + a_1^2}\begin{bmatrix} 1 - a_0 + a_1 \\ 1 - a_0 - a_1 \end{bmatrix} \text{ for } i \in S_e. \tag{22}$$

Similarly, to have $y_i(0) = y_i(4)$ for $i \in S_o$, we need

$$y_i(0) = \frac{1}{(1-a_0)^2 + a_1^2}\begin{bmatrix} 1 - a_0 + a_1 \\ 1 - a_0 - a_1 \end{bmatrix} \text{ for } i \in S_o. \tag{23}$$

Step 2: In this step, we show that the four inequalities in either (20) or (21) can be reduced to two inequalities. For agent $j \in S_o$, we have

$$\begin{aligned} u_j(k) &= \sum_{i \in \mathcal{N}_j} a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k) - y_j(k)) \\ &= \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k) - y_j(k)) \\ &\quad + \sum_{i \in \mathcal{N}_j \cap S_o} a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k) - y_j(k)). \end{aligned}$$

Taking into account that $x_i(0) = x_j(0)$ and $v_i(0) = v_j(0)$ for all $i, j \in S_o$, we obtain

$$u_j(k) = \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k) - y_j(k)). \tag{24}$$

From (24) and the initial states given by (22) and (23), it is easy to verify that for $j \in S_o$, $u_j(k + 2) \geq 1$ are equivalent to $u_j(k) \leq -1$ for $k = 0, 1$. Similarly, for $i \in S_e, u_i(k + 2) \leq -1$ is equivalent to $u_i(k) \geq 1$ for $k = 0, 1$. Thus, inequalities (20) and (21) are equivalent to the following inequalities: $u_i(0) \geq 1$ and $u_i(1) \geq 1$ for $i \in S_e$ and $u_j(0) \leq -1$ and $u_j(1) \leq -1$ for $j \in S_o$.

Step 3: These two inequalities are satisfied for each agent provided that for each edge $(i, j) \in \mathcal{E}$, where $i \in S_e$ and $j \in S_o$, the following two conditions:

$$\begin{aligned} &a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(0) - y_j(0)) \\ &= -\tfrac{2a_{ij}}{(1-a_0)^2 + a_1^2}[(1 - a_0 + a_1)\alpha + (1 - a_0 - a_1)\beta] \leq -1 \\ &a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(1) - y_j(1)) \\ &= -\tfrac{2a_{ij}}{(1-a_0)^2 + a_1^2}[(1 - a_0 - a_1)\alpha - (1 - a_0 + a_1)\beta] \leq -1 \end{aligned}$$

are satisfied. It is easy to see that this is the case if the feedback gain parameters α and β satisfy (19).

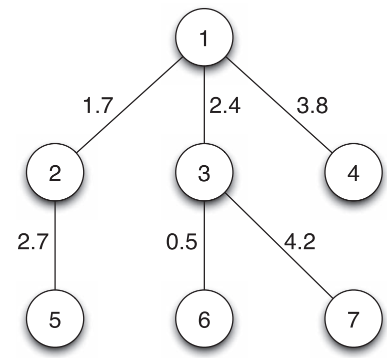

Fig. 1. Network with seven agents.

From the above analysis, it follows that the solution of the multiagent system is periodic with period $T = 4$, for initial states satisfying (22) and (23). ■

Remark 4: Note that the periodic behavior presented in Theorem 2 has a period $T = 4$, which is independent of the network topology, whereas the feedback gain parameters for achieving this periodic behavior depend on the minimal edge weights of the underlying graph as given by (19). This is in contrast to the double-integrator case in Theorem 1, where the feedback gain parameters for achieving periodic behaviors do not depend on the network topology; however, the periodic T depends on the minimal edge weights.

Remark 5: In [13, Corollary 21.10], it has been shown that for the time-invariant system $x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$, if A^K has no unity eigenvalue; then, for every K-periodic input signal $u(k)$, there exists an x_0 such that the corresponding solution is K periodic. In view of this, Theorem 2 extends this result to multiagent systems for the second-order dynamics. In particular, A^4 has no unity eigenvalue because of the assumption that A has no eigenvalues at ± 1 and $\pm j$. Moreover, a four-periodic input is constructed based on the saturation and the linear consensus control law with the gain parameters satisfying (19).

V. Illustrative Examples

In this section, we present examples to illustrate the results. The network consists of $N = 7$ agents, and the topology is given by the undirected weighted graph depicted in Fig. 1.

A. *Double-Integrator Case*

For the double-integrator case, we choose the feedback gain parameters $\alpha = 1$ and $\beta = 1.2$ such that the sufficient condition for achieving a periodic behavior (5) is satisfied. It is easy to see that $\bar{a} = a_{36} = 0.5$, and therefore, we choose $m = 11$ such that (6) is satisfied. From the proof of Theorem 1, we see that the multiagent system exhibits a periodic solution with $T = 22$ if the initial states satisfy (11) and (15) with $m = 11$, i.e., $v_i(0) = -5.5$ for $i \in S_e = \{1, 5, 6, 7\}$, $v_i(0) = 5.5$ for $i \in S_o = \{2, 3, 4\}$, $2.3882 \leq x_1(0) - x_2(0) \leq 8.6118$, $2.2167 \leq x_1(0) - x_3(0) \leq 8.7833$, $2.0632 \leq x_1(0) - x_4(0) \leq 8.9368$, $2.1704 \leq x_5(0) - x_2(0) \leq 8.8296$, $3.8000 \leq x_6(0) - x_3(0) \leq 7.2000$, and $2.0381 \leq x_7(0) - x_3(0) \leq 8.9619$. We then choose $x_1(0) = 21$, $x_2(0) = 16$, $x_3(0) = 17$, $x_4(0) = 15$, $x_5(0) = 23$, $x_6(0) = 22$, and $x_7(0) = 24$ such that the above conditions are satisfied. With these initial states, the multiagent system with input saturation constraints exhibits a periodic behavior of period 22 as shown in Fig. 2, where the state trajectories for agents 1, 2, 5, and 7 are given.

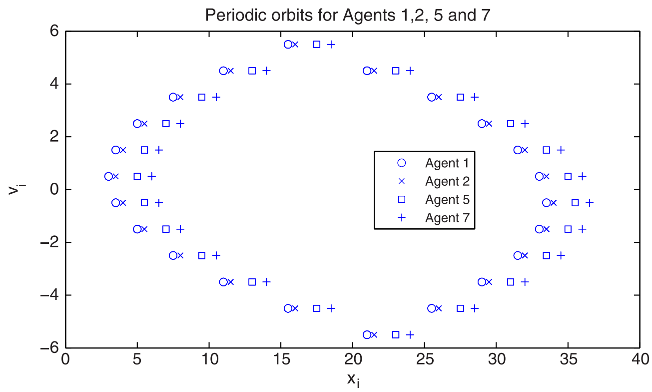

Fig. 2. Periodic solutions of period 22 for the double-integrator case.

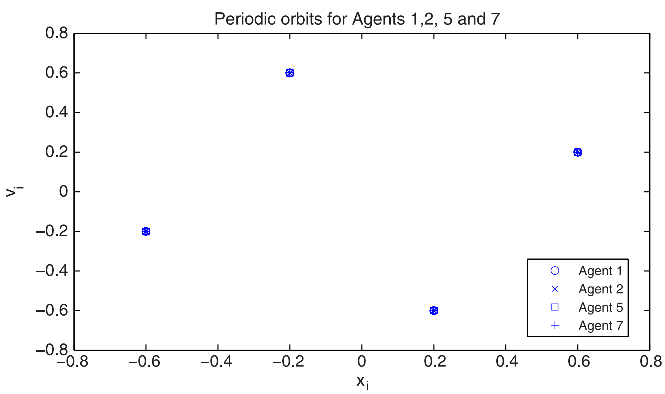

Fig. 3. Periodic behavior of period four for the unstable case.

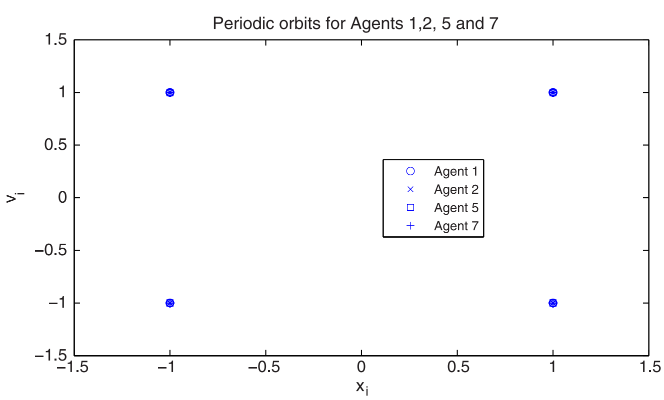

Fig. 4. Periodic behavior of period four for the marginally stable case.

B. Unstable Case

Next, we consider the case where the agent dynamics is given by (18). We begin by considering where $a_0 = 0$ and $a_1 = -2$, which results in an unstable eigenvalue at 2. For this case, (19) becomes $(-\beta + 5)/3 \leq \alpha \leq 3\beta - 5$. We then choose the feedback gain parameters $\alpha = 1$ and $\beta = 3$. According to the proof of Theorem 2, the multiagent system exhibits a periodic solution of period $T = 4$ if the initial states satisfy (22) and (23), that is, $x_i(0) = 0.2, v_i(0) = -0.6$ for $i \in S_e = \{1, 5, 6, 7\}$ and $x_i(0) = -0.2, v_i(0) = 0.6$ for $i \in S_o = \{2, 3, 4\}$. Fig. 3 shows that the multiagent system exhibits a periodic behavior with $T = 4$.

C. Marginally Stable Case

We then consider the case where $a_0 = 1$ and $a_1 = -1$, which results in eigenvalues $(1/2) \pm (\sqrt{3}/2)j$ on the unit circle. For this case, (19) becomes $-\beta + 1 \leq \alpha \leq \beta - 1$. We then choose the feedback gain parameters $\alpha = 0.5$ and $\beta = 2$. Fig. 4 shows that the multiagent system exhibits a periodic behavior with $T = 4$ for initial states $x_i(0) = 1$, $v_i(0) = -1$ for $i \in S_e = \{1, 5, 6, 7\}$ and $x_i(0) = -1$, $v_i(0) = 1$ for $i \in S_o = \{2, 3, 4\}$, which satisfy (22) and (23).

VI. CONCLUSION

In this brief, we have considered second-order discrete-time multiagent systems with input saturation constraints. We showed that these multiagent systems under linear consensus control laws exhibit periodic behaviors. We explicitly characterized conditions on feedback gain parameters and initial states for achieving periodic behaviors. An interesting future direction is to extend the results to high-order multiagent systems.

REFERENCES

[1] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Dept. Elect. Eng. Comput. Sci., MIT, Cambridge, MA, USA, 1984.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[3] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[4] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[5] Z. Meng, Z. Zhao, and Z. Lin, "On global leader-following consensus of identical linear dynamic systems subject to actuator saturation," *Syst. Control Lett.*, vol. 62, no. 2, pp. 132–142, Feb. 2013.

[6] T. Yang, Z. Meng, D. V. Dimarogonas, and K. H. Johansson, "Global consensus for discrete-time multi-agent systems with input saturation constraints," *Automatica*, vol. 50, no. 2, pp. 499–506, Feb. 2014.

[7] T. Yang, A. A. Stoorvogel, H. F. Grip, and A. Saberi, "Semi-global regulation of output synchronization for heterogeneous networks of non-introspective, invertible agents subject to actuator saturation," *Int. J. Robust Nonlinear Control*, vol. 24, no. 3, pp. 548–566, Feb. 2014.

[8] H. Su, Z. Chen, J. Lam, and Z. Lin, "Semi-global leader-following consensus of linear multi-agent systems with input saturation via low gain feedback," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 7, pp. 1881–1889, Jul. 2013.

[9] Z. Zhao and Z. Lin, "Semi-global leader-following consensus of multiple linear systems with position and rate limited actuators," *Int. J. Robust Nonlinear Control*, vol. 25, no. 13, pp. 2083–2100, Sep. 2015.

[10] Y. Su and J. Huang, "Cooperative semi-global robust output regulation for a class of nonlinear uncertain multi-agent systems," *Automatica*, vol. 50, no. 4, pp. 1053–1065, Apr. 2014.

[11] Y. Chen, J. Lü, F. Han, and X. Yu, "On the cluster consensus of discrete-time multi-agent systems," *Syst. Control Lett.*, vol. 60, no. 7, pp. 517–523, Jul. 2011.

[12] T. Yang, Z. Meng, D. V. Dimarogonas, and K. H. Johansson, "Periodic behaviors in multi-agent systems with input saturation constraints," in *Proc. 52nd IEEE Conf. Decision Control*, Florence, Italy, Dec. 2013, pp. 4467–4472.

[13] W. Rugh, *Linear System Theory*, 2nd ed. Englewood Cliffs, NJ, USA: Prentice-Hall, 1996.

[14] T. Yang, A. A. Stoorvogel, and A. Saberi, "Dynamic behavior of the discrete-time double integrator with saturated locally stabilizing linear state feedback laws," *Int. J. Robust Nonlinear Control*, vol. 23, no. 17, pp. 1899–1931, Nov. 2013.

[15] T. Yang, A. A. Stoorvogel, A. Saberi, and K. H. Johansson, "Further results on saturated globally stabilizing linear state feedback control laws for single-input neutrally stable planar systems," in *Proc. Eur. Control Conf.*, Zurich, Switzerland, Jul. 2013, pp. 2728–2733.

[16] D. Xie and S. Wang, "Consensus of second-order discrete-time multi-agent systems with fixed topology," *J. Math. Anal. Appl.*, vol. 387, no. 1, pp. 8–16, Mar. 2012.

[17] J. Lygeros and F. Ramponi, Lecture Notes on Linear System Theory, 2013. [Online]. Available: http://control.ee.ethz.ch/~ifalst/docs/Notes/LectureNotes.pdf