How Agreement and Disagreement Evolve over Random Dynamic Networks

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***Abstract*—The dynamics of an agreement protocol interacting with a disagreement process over a common random network is considered. The model can represent the spreading of true and false information over a communication network, the propagation of faults in a large-scale control system, or the development of trust and mistrust in a society. At each time instance and with a given probability, a pair of network nodes interact. At random each of the nodes then updates its state towards the state of the other node (attraction), away from the other node (repulsion), or sticks to its current state (neglect). Agreement convergence and disagreement divergence results are obtained for various strengths of the updates for both symmetric and asymmetric update rules. Impossibility theorems show that a specific level of attraction is required for almost sure asymptotic agreement and a specific level of repulsion is required for almost sure asymptotic disagreement. A series of sufficient and/or necessary conditions are then established for agreement convergence or disagreement divergence. In particular, under symmetric updates, a critical convergence measure in the attraction and repulsion update strength is found, in the sense that the asymptotic property of the network state evolution transits from agreement convergence to disagreement divergence when this measure goes from negative to positive. The result can be interpreted as a tight bound on how much bad action needs to be injected in a dynamic network in order to consistently steer its overall behavior away from consensus.**

***Index Terms*—Dynamic networks, Opinion dynamics, Gossiping, Social networks, Consensus algorithms, Network science**

I. Introduction

A. Motivation

A GROWING number of applications are composed of a networked information structure executed over an underlying communication network. Examples include social networks over the Internet, control networks for the power grid, and information networks serving transportation systems. These networks are seldom centrally regulated, but have a strong component of distributed information processing and decision-making. While they are able to provide appropriate service to their users most of the time, open software and communication technologies together with large geographical distribution, make them more exposed to faulty components, software bugs, communication failures, and even purposeful injection of false data.

An interesting problem is to try to understand the amount of deficiencies that can be tolerated in the network before the global system performance is compromised. In this paper we tackle this challenging problem for a model inspired by agreement protocols, whose execution have been studied intensively over the last decade in a variety of other settings, including load balancing in parallel computing [6], [7], coordination of autonomous agents [8], [9], distributed estimation and signal processing [10], [11], and opinion dynamics in social networks [12]–[14].

B. Related Work

The network structure and the dynamics of the nodes in these networks are two fundamental components in network science [15], [16]. Probabilistic models for networks such as random graphs, provide an important and convenient means for modeling large-scale systems, and have found numerous applications in various fields of science. The classical Erdös–Rényi model, in which each edge exists randomly and independently of others with a given probability, was studied in [17]. The degree distribution of the Erdös–Rényi graph is asymptotically Poisson. Generalized models were proposed in [18] and [19], for which the degree distribution satisfies certain power law that better matches the properties of real-life networks such as the Internet. A detailed introduction to the structure of random networks can be found in [15], [20].

When information processing is executed on top of an underlying network, nodes are endowed with internal states that evolve as nodes interact. The dynamics of the node states depend on the particular problem under investigation. For instance, the boids model was introduced in [4] to model swarm behavior and animal groups, followed by Vicsek's model in [5]. Models of opinion dynamics in social networks were considered in [12], [13], [49] and the dynamics of communication protocols in [50]. Convergence to agreement for averaging algorithms have been extensively studied in the literature. Early results were developed in a general setting for studying the ergodicity of nonhomogeneous Markov chains [21], [22]. Deterministic models have been investigated in finding proper connectivity conditions that ensure consensus convergence [23]–[32]. Averaging algorithms over random graphs have also been considered [33]–[42].

In this paper, we use the asynchronous time model introduced in [45] to describe the randomized node interactions. Each node meets other nodes at independent time instances defined by a rate-one Poisson process, and then a pair of nodes is selected to meet at random determined by the underlying communication graph. Gossiping, in which each node is restricted to exchange data and decisions with at most one neighboring node at each time instance, has proven to be a robust and efficient way to implement distributed computations

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and signal processing [11], [47], [50]. We refer to [43]–[48], [50] for the convergence analysis for gossiping algorithms. The model we introduce and analyze in this paper can be viewed as an extension to the model discussed by Acemoglu et al. [49], who used a gossip algorithm to describe the spread of misinformation induced by forceful update in social networks. In this work we consider faulty and misbehaving nodes in gossip algorithms. While the distributed systems community has since long recognized the need to provide fault tolerant systems, e.g., [59], [60], efforts to provide similar results for randomized gossiping algorithms have so far been limited. This paper aims at providing such results.

C. Main Contribution

The main contribution of this paper is to provide conditions for agreement convergence and disagreement divergence over random networks. To study this problem, we use a model of asynchronous randomized gossiping. At each instance, two nodes meet with a given probability. When nodes meet, normally they should update as a weighted average (attraction). Besides that, we assume that nodes can misbehave in the sense that they can take a weighted combination with one negative coefficient (repulsion), or they can stick to their current state (neglect). The potential node misbehavior essentially results in model uncertainties in the considered algorithm. Each node follows one of the three update rules at random by given probabilities whenever it is selected to meet another node.

A fundamental question we answer is whether the network will converge to agreement (all nodes asymptotically reach the same value a.s.) or diverge to disagreement (all nodes disperse a.s.). We study both symmetric and asymmetric node updates [46]. Two general impossibility theorems are first proposed. Then, a series of sufficient and/or necessary conditions are established for the network to reach a.s. agreement convergence or disagreement divergence. In particular, under symmetric updates, a critical convergence measure is found in the sense that the asymptotic evolution of the network states transits from agreement to disagreement when this measure switches from negative to positive. This critical measure is in fact independent of the structure of the underlying communication graph. In other words, under the node dynamics considered in this paper, there is no difference if the underlying network is an Erdös–Rényi graph [17], a small-world graph [18], or a scale-free graph [19], for the network to reliably target an agreement.

D. Outline

The rest of the paper is organized as follows. In Section II, we introduce the network model, the considered algorithm, the problem formulation, together with some physical motivation for the model. Section III presents two general impossibility theorems on a.s. agreement and disagreement, respectively. In Section IV, we discuss the model in the absence of node repulsion and give conditions for a.s. agreement convergence for both symmetric and asymmetric update steps. Section V presents agreement and disagreement conditions for the general model. Finally, some concluding remarks are given in Section VI.

II. Problem Definition

In this section, we present the considered network model and define the problem of interest.

We first recall some basic definitions from graph theory [3] and stochastic matrices [1]. A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A digraph $\mathcal{G}$ is *weakly connected* if it is connected as a bidirectional graph when all the arc directions are ignored. A finite square matrix $M = [m_{ij}] \in \Re^{n\times n}$ is called *stochastic* if $m_{ij} \geq 0$ for all i, j and $\sum_j m_{ij} = 1$ for all i. A stochastic matrix M is *doubly stochastic* if also M^T is stochastic. Let $P = [p_{ij}] \in \Re^{n\times n}$ be a matrix with nonnegative entries. We can associate a unique digraph $\mathcal{G}_P = (\mathcal{V}, \mathcal{E}_P)$ with P on node set $\mathcal{V} = \{1, \dots, n\}$ such that $(j, i) \in \mathcal{E}_P$ if and only if $p_{ij} > 0$. We call $\mathcal{G}_P$ the *induced graph* of P.

A. Node Pair Selection

Consider a network with node set $\mathcal{V} = \{1, \dots, n\}$, $n \geq 3$. We use the asynchronous time model introduced in [45] to describe node interactions. Time is slotted. Let $x_i(k) \in \Re$ denote the state (value) of node i at the k'th slot. Then the network state is $x(k) = (x_1(k), \dots, x_n(k))^T \in \Re^n$. Node interactions are characterized by an $n \times n$ stochastic matrix $A = [a_{ij}]$. The meeting process is defined as follows.

Definition 1 (Node Pair Selection): Independent of time and node state, at time $k \geq 0$,

(i) A node $i \in \mathcal{V}$ is drawn with probability $1/n$;
(ii) Node i picks the pair (i, j) with probability a_{ij}.

For the the induced graph, $\mathcal{G}_A$, of the node pair selection matrix A, we use the following assumption.

A1. *(Underlying Connectivity)* $\mathcal{G}_A$ is weakly connected.

We denote $\mathcal{G}^*_A$ as the bidirectional graph obtained by ignoring all the directions for arcs in $\mathcal{G}_A$.

B. State Evolution

Suppose node i meets another node j at time k. Independent of time, node states, and pair selection process, their will be three events for the iterative update for node i.

(i) *(Attraction)* With probability α, node i updates as a weighted average with j, marked by event $\mathscr{A}_{ij}(k)$:
$$x_i(k+1) = x_i(k) + T_k\big(x_j(k) - x_i(k)\big) \qquad (1)$$
where $0 < T_k \leq 1$ is the average weight.

(ii) *(Neglect)* With probability β, node i sticks to its current state, marked by event $\mathscr{N}_{ij}(k)$:
$$x_i(k+1) = x_i(k). \qquad (2)$$

(iii) *(Repulsion)* With probability γ, node i updates as a weighted average with j, but with a negative coefficient, marked by $\mathscr{R}_{ij}(k)$:
$$x_i(k+1) = x_i(k) - S_k\big(x_j(k) - x_i(k)\big) \qquad (3)$$
where $S_k > 0$.

Naturally we assume $\alpha + \beta + \gamma = 1$. Node j's update is determined by the corresponding events $\mathscr{A}_{ji}(k)$, $\mathscr{N}_{ji}(k)$ and $\mathscr{R}_{ji}(k)$, which may depend on node i's update.

C. Problem

Introduce

$$H(k) \doteq \max_{i\in\mathcal{V}} x_i(k), \quad h(k) \doteq \min_{i\in\mathcal{V}} x_i(k)$$

as the maximum and minimum states among all nodes, respectively, and define $\mathcal{H}(k) \doteq H(k) - h(k)$ as the agreement measure. We make the following definition.

Definition 2: (i). Agreement convergence is achieved a.s. for initial time k_0 and initial value $x(k_0) \in \Re^n$ if

$$\mathbf{P}\Big(\limsup_{k\to\infty} \mathcal{H}(k) = 0\Big) = 1. \tag{4}$$

Global agreement convergence is achieved a.s. if (4) holds for all initial time and initial values.

(ii). Disagreement divergence is achieved a.s. for initial value $x(k_0) \in \Re^n$ if

$$\mathbf{P}\Big(\limsup_{k\to\infty} \mathcal{H}(k) > M\Big) = 1 \ for\ all\ M \geq 0. \tag{5}$$

D. Model Rationale

We illustrate and motivate the model introduced above through three application examples.

False Data Injection Attacks: Large distributed computing and control systems are vulnerable to cyber attacks [53]–[56]. An attacker may inject false data or malicious code in the network, to mislead the nodes or even change the overall behavior of the system. The model in this paper can represent a very simple system under a cyber attack. The attraction event $\mathscr{A}_{ij}$ corresponds to normal operation of the system, under which the nodes are supposed to reach consensus. The neglect event $\mathscr{N}_{ij}$ can represent a denial-of-service attack, which block node i from updating its state based on information from its neighbor j. The injection of malicious code in node i changing its update law is modeled by the repulsion update. State agreement or disagreement indicates the failure or success of the attack. Our results in this paper allow us to explicitly characterize how large attacks a network can withstand.

Fault-Tolerant Systems: "An important goal in distributed system design is to construct the system in such a way that it can automatically recover from partial failures without seriously affecting the overall performance," as pointed out in [57]. In our model the events $\mathscr{N}_{ij}$ and $\mathscr{R}_{ij}$ can represent node faults during a randomized computation process or in the coordination of a multi-robot system. For example, the magnitude of the repulsion parameter S_k can indicate how severe a fault is. Our results show that a networked systems can sometimes be robust to quite severe faults. It is also shown that in certain cases the topology of the network does not play an essential role but the persistence and the size are more important.

Social Networks: Distributed averaging has been widely used to characterize opinion dynamics in social networks, e.g., [12]–[14], [49], [51]. The state x_i of node i is in these models the opinion of an individual. The individuals meet and exchange opinions. The attraction event $\mathscr{A}_{ij}$ models the trust of node i to node j. Whenever $\mathscr{A}_{ij}(k)$ happens, node i believes in node j and therefore takes an attraction update step. The parameter T_k measures the level of trust. The neglect event $\mathscr{N}_{ij}$ models the mistrust of node i to node j, which results in that i simply ignores j and sticks to its current opinion. The repulsion event $\mathscr{R}_{ij}$ models the antagonism of node i to node j. In this case, node i takes the opposite direction to the attraction to keep a large distance to the opinion of node j. In this way, our model characterizes the influence of node relations to the convergence of the opinion in social networks. The idea follows the discussions on the possibilities of spread of misinformation and persistent disagreement in [49], [51]. In addition, our model also allows for opinion divergence, as indicated in the definition of disagreement divergence.

III. Impossibility Theorems

In this section, we discuss the impossibilities of agreement convergence or disagreement divergence. First a general impossibility theorem for agreement convergence is established as follows on the sequence $\{T_k\}_0^\infty$.

Theorem 1: Global agreement convergence can be achieved a.s. only if either $\sum_{k=0}^\infty T_k = \infty$ or $\sum_{k=0}^\infty (1 - T_k) = \infty$. In fact, if either $\sum_{k=0}^\infty T_k < \infty$ or $\sum_{k=0}^\infty (1 - T_k) < \infty$ holds, then for almost all initial values, we have $\mathbf{P}\big(\limsup_{k\to\infty} \mathcal{H}(k) = 0\big) = 0$ when k_0 is sufficiently large.

Proof. The proof relies on the following well-known lemma.

Lemma 1: Let $\{b_k\}_0^\infty$ be a sequence of real numbers with $b_k \in [0,1)$ for all k. Then $\sum_{k=0}^\infty b_k = \infty$ if and only if $\prod_{k=0}^\infty (1 - b_k) = 0$.

Now suppose $\sum_{k=0}^\infty T_k < \infty$. Then $\exists K_0 \geq 0$ s.t. $T_k < 1/2, k \geq K_0$. Let node pair (i,j) be selected at time $k \geq K_0$. There are two cases.

(i) Neither $x_i(k)$ nor $x_j(k)$ reaches the minimum value. Then straightforwardly we have $h(k+1) \leq h(k)$.

(ii) One of the two nodes, say i, reaches the minimum value. In this case, we have $x_i(k+1) \leq h(k) + T_k\mathcal{H}(k)$ if $\mathscr{A}_{ij}(k)$ happens, and $h(k+1) \leq h(k)$ otherwise.

Thus, we obtain

$$\mathbf{P}\Big(h(k+1) \leq h(k) + T_k\mathcal{H}(k),\ k \geq K_0\Big) = 1. \tag{6}$$

A similar analysis leads to that

$$\mathbf{P}\Big(H(k+1) \geq H(k) - T_k\mathcal{H}(k),\ k \geq K_0\Big) = 1. \tag{7}$$

We see from (6) and (7) that

$$\mathbf{P}\Big(\mathcal{H}(k+1) \geq \big(1 - 2T_k\big)\mathcal{H}(k),\ k \geq K_0\Big) = 1. \tag{8}$$

Thus, according to (8), we conclude

$$\mathbf{P}\Big(\mathcal{H}(m) \geq \rho_*\mathcal{H}(K_0)\Big) = 1$$

for all $m \geq K_0$, where $\rho_* \doteq \prod_{k=K_0}^\infty \big(1 - 2T_k\big)$ is a constant satisfying $0 < \rho_* < 1$ based on Lemma 1. This implies

$$\mathbf{P}\Big(\limsup_{k\to\infty} \mathcal{H}(k) > 0\Big) \geq \mathbf{P}\Big(\mathcal{H}(m) > 0,\ m \geq K_0\Big) = 1$$

for all initial conditions with $k_0 \geq K_0$ and $\mathcal{H}(k_0) > 0$. It is obvious to see that $\big\{x = (x_1 \dots x_n)^T \in \Re^n : x_1 = \dots = x_n\big\}$ is a set with measure zero in $\Re^n$. The desired conclusion follows.

Moreover, the conclusion for the other case $\sum_{k=0}^{\infty}(1-T_k)<\infty$ follows from a symmetric argument. This completes the proof. $\square$

The corresponding impossibility theorem for disagreement divergence is presented as follows.

Theorem 2: Disagreement divergence can be achieved a.s. only if $\prod_{k=0}^{\infty}(1+2S_k)=\infty$.

Proof. It is straightforward to see that

$$\mathbf{P}\Big(\mathcal{H}(k+1)\leq\big(1+2S_k\big)\mathcal{H}(k)\Big)=1 \tag{9}$$

for all k. The desired conclusion follows immediately. $\square$

IV. Attraction vs. Neglect

In this section, we focus on the role of node attraction for the network to reach agreement convergence. We consider the case when repulsion events never take place, as indicated in the following assumption.

A2. *(Repulsion-Free)* $\mathbf{P}(\mathscr{R}_{ij}(k))=0$ for all (i,j) and k.

We study symmetric and asymmetric node dynamics, respectively.

A. Symmetric Update

This subsection focuses on the condition when the nodes' updates are symmetric when two nodes meet, as indicated in the following assumption.

A3. *(Symmetric Attraction)* The events $\mathscr{A}_{ij}(k)=\mathscr{A}_{ji}(k)$ a.s. for all (i,j) and k.

The main result for the symmetric update model is as follows.

Proposition 1: Suppose **A1**, **A2** and **A3** hold. Global agreement convergence is achieved a.s. if $\sum_{k=0}^{\infty}T_k(1-T_k)=\infty$.

Proof. With **A2** and **A3**, the considered gossip algorithm can be expressed as $x(k+1)=\Phi(k)x(k)$, where $\Phi(k)$ is the random matrix satisfying

$$\begin{aligned}&\mathbf{P}\Big(\Phi(k)=\Phi_{\langle ij\rangle}\doteq I-T_k(e_i-e_j)(e_i-e_j)^T\Big)\\&=\frac{\alpha}{n}(a_{ij}+a_{ji}),\qquad i\neq j\end{aligned} \tag{10}$$

with $e_m=(0\dots0\ 1\ 0\dots0)^T$ denoting the $n\times1$ unit vector whose m'th component is 1. Define $L(k)=\sum_{i=1}^{n}|x_i(k)-x_{\rm ave}|^2$, where $x_{\rm ave}=\sum_{i=1}^{n}x_i(k_0)/n$ is the average of the initial values and $|\cdot|$ represents the Euclidean norm of a vector or the absolute value of a scalar.

It is easy to verify that for every possible sample and fixed instant k, $\Phi_{\langle ij\rangle}$ defined in (10), is a symmetric, and doubly stochastic matrix, i.e., $\Phi_{\langle ij\rangle}\mathbf{1}=\mathbf{1}$ and $\mathbf{1}^T\Phi_{\langle ij\rangle}=\mathbf{1}^T$. Therefore, we have

$$\begin{aligned}&\mathbf{E}\Big(L(k+1)\big|x(k)\Big)\\&=\mathbf{E}\Big(\big(x(k)-x_{\rm ave}\mathbf{1}\big)^T\Phi(k)^T\Phi(k)\big(x(k)-x_{\rm ave}\mathbf{1}\big)\big|x(k)\Big)\\&=\big(x(k)-x_{\rm ave}\mathbf{1}\big)^T\mathbf{E}\big(\Phi^2(k)\big)\big(x(k)-x_{\rm ave}\mathbf{1}\big)\\&\leq\lambda_2\Big(\mathbf{E}\big(\Phi^2(k)\big)\Big)L(k),\end{aligned} \tag{11}$$

where $\lambda_2(M)$ for a stochastic matrix M denotes the largest eigenvalue in magnitude excluding the eigenvalue at one.

Noticing that

$$\begin{aligned}&\Big(I-T_k(e_i-e_j)(e_i-e_j)^T\Big)^2\\&=I-2T_k(1-T_k)(e_i-e_j)(e_i-e_j)^T,\end{aligned} \tag{12}$$

we see from (10) that

$$\begin{aligned}&\mathbf{P}\Big(\Phi^2(k)=I-2T_k(1-T_k)(e_i-e_j)(e_i-e_j)^T\Big)\\&=\frac{\alpha}{n}(a_{ij}+a_{ji}),\qquad i\neq j.\end{aligned}$$

This leads to

$$\mathbf{E}\big(\Phi^2(k)\big)=I-2T_k(1-T_k)\frac{\alpha}{n}\big(D-(A+A^T)\big), \tag{13}$$

where $D={\rm diag}(d_1\dots d_n)$ with $d_i=\sum_{j=1}^{n}(a_{ij}+a_{ji})$.

Note that $D-(A+A^T)$ is actually the (weighted) Laplacian of the graph $\mathcal{G}_{A+A^T}$. With assumption **A1**, $\mathcal{G}_{A+A^T}$ is a connected graph, and therefore, based on the well-known property of Laplacian matrix of connected graphs [3], we have $\lambda_2^*>0$, where λ_2^* is the second smallest eigenvalue of $D-(A+A^T)$. On the other hand, since A is a stochastic matrix, it is straightforward to see that $\sum_{j=1,j\neq i}a_{ij}+a_{ji}\leq n$ for all $i=1,\dots,n$. According to Gershgorin's circle theorem, every eigenvalue λ_i^* of $D-(A+A^T)$ is bounded by $2n$. Therefore,

$$2T_k(1-T_k)\frac{\alpha}{n}\lambda_i^*\leq4T_k(1-T_k)\leq4\Big(\frac{T_k+(1-T_k)}{2}\Big)^2=1$$

for all $\lambda_i^*\in\sigma\big(D-(A+A^T)\big)$, where $\sigma(\cdot)$ denotes the spectrum of a matrix. Now we conclude that for all k,

$$\lambda_2\Big(\mathbf{E}\big(\Phi^2(k)\big)\Big)=1-\frac{2T_k(1-T_k)\alpha}{n}\lambda_2^*. \tag{14}$$

With (11) and (14), we obtain

$$\mathbf{E}\Big(L(k+1)\Big)\leq\prod_{i=k_0}^{k}\Big(1-\frac{2T_k(1-T_k)\alpha}{n}\lambda_2^*\Big)L(k_0), \tag{15}$$

Therefore, based on Lemma 1 and Fatou's lemma, we have

$$\mathbf{E}\Big(\lim_{k\to\infty}L(k)\Big)\leq\lim_{k\to\infty}\mathbf{E}\Big(L(k)\Big)=0,$$

if $\sum_{k=0}^{\infty}T_k(1-T_k)=\infty$, where $\lim_{k\to\infty}L(k)$ exists simply from the fact that the sequence is non-increasing. This immediately implies

$$\mathbf{P}\Big(\lim_{k\to\infty}x_i(k)=x_{\rm ave}\Big)=1.$$

The proof is finished. $\square$

There is an interesting connection between the impossibility statement Theorem 1 and Proposition 1. Let us consider a special case when T_k is monotone. Combining Theorem 1 and Proposition 1, the following conclusion becomes clear.

Theorem 3: Suppose **A1**, **A2** and **A3** hold. Assume that either $T_{k+1}\leq T_k$ or $T_{k+1}\geq T_k$ for all k. Then $\sum_{k=0}^{\infty}T_k(1-T_k)=\infty$ is a threshold condition regarding global a.s. agreement convergence:

(i) $\mathbf{P}\big(\limsup_{k\to\infty}\mathcal{H}(k)=0\big)=0$ for almost all initial conditions with k_0 sufficiently large if $\sum_{k=0}^{\infty}T_k(1-T_k)<\infty$;

(ii) $\mathbf{P}\big(\limsup_{k\to\infty}\mathcal{H}(k)=0\big)=1$ for all initial conditions if $\sum_{k=0}^{\infty}T_k(1-T_k)=\infty$.

B. Asymmetric Update

In this subsection, we investigate the case when the node updates are asymmetric, as indicated by the following assumption.

A4. *(Asymmetric Attraction)* $\mathbf{P}(\mathscr{A}_{ij}(k)\bigcap\mathscr{A}_{ji}(k))=0$ for all (i,j) and k.

We present the main result for the asymmetric update model as follows.

Proposition 2: Suppose **A1**, **A2** and **A4** hold. Then global agreement convergence is achieved a.s. if

$$\sum_{k=0}^{\infty}\Big[\prod_{s=k(n-1)}^{(k+1)(n-1)-1}T_s\big(1-T_s\big)\Big]=\infty.$$

Proof. Take $k_*\geq 0$. Denote $a_*=\min\{a_{ij}:\ a_{ij}>0\}$ as the lower bound of the nonzero entries of A. Suppose i_0 is some node satisfying $x_{i_0}(k_*)=h(k_*)$.

Let i_1 be a node which is connected to i_0 in graph $\mathcal{G}_A^*$. We see that such i_1 exists based on the weak connectivity assumption **A1**. With assumptions **A2** and **A4**, we have

$$\mathbf{P}\Big(\text{pair } (i_0,i_1) \text{ or } (i_1,i_0) \text{ selected, and } \mathscr{A}_{i_1i_0} \text{ happens}\Big)\geq\frac{a_*}{n}\alpha.$$

Moreover, if $\mathscr{A}_{i_1i_0}$ happens, we have

$$\begin{aligned}&x_{i_1}(k_*+1)\\&=T_{k_*}x_{i_0}(k_*)+(1-T_{k_*})x_{i_1}(k_*)\\&\leq T_{k_*}h(k_*)+(1-T_{k_*})H(k_*)\\&\leq T_{k_*}(1-T_{k_*})h(k_*)+\big(1-T_{k_*}(1-T_{k_*})\big)H(k_*)\end{aligned}$$

and $x_{i_0}(k_*+1)=x_{i_0}(k_*)$ according to assumption **A4**. This implies

$$\begin{aligned}&\mathbf{P}\Big(x_{i_1}(k_*+1)\leq T_{k_*}(1-T_{k_*})h(k_*)\\&+\big(1-T_{k_*}(1-T_{k_*})\big)H(k_*) \text{ and } x_{i_0}(k_*+1)=x_{i_0}(k_*)\Big)\\&\geq\frac{a_*}{n}\alpha.\end{aligned}$$

Next, according to the weak connectivity assumption **A1**, another node i_2 can be found such that i_2 is connected to $\{i_0,i_1\}$ in $\mathcal{G}_A^*$. There will be two cases.

(i) i_2 is connected to i_0 in $\mathcal{G}_A^*$. Then by a similar analysis we used for bounding $x_{i_1}(k_*+1)$, we obtain

$$\begin{aligned}&\mathbf{P}\Big(x_{i_0}(k_*+2)=x_{i_0}(k_*),\ x_{i_1}(k_*+2)=x_{i_1}(k_*+1),\\&\text{and } x_{i_2}(k_*+2)\leq T_{k_*+1}h(k_*)+(1-T_{k_*+1})H(k_*)\ \Big)\\&\geq\frac{a_*}{n}\alpha.\end{aligned}$$

(ii) i_2 is connected to i_1 in $\mathcal{G}_A^*$. Suppose pair (i_1,i_2) or (i_2,i_1) selected, and $\mathscr{A}_{i_2i_1}$ happens at time k_*+1. Then we have $x_{i_1}(k_*+2)=x_{i_1}(k_*+1)$ and

$$\begin{aligned}&x_{i_2}(k_*+2)=(1-T_{k_*+1})x_{i_2}(k_*+1)+T_{k_*+1}x_{i_1}(k_*)\\&\leq(1-T_{k_*+1})H(k_*+1)+T_{k_*+1}\Big(T_{k_*}(1-T_{k_*})h(k_*)\\&\quad+\big(1-T_{k_*}(1-T_{k_*})\big)H(k_*)\Big)\\&\leq h(k_*)\prod_{k=k_*}^{k_*+1}T_k(1-T_k)+H(k_*)\Big(1-\prod_{k=k_*}^{k_*+1}T_k(1-T_k)\Big)\end{aligned}$$

conditioned that pair (i_0,i_1) or (i_1,i_0) selected, and $\mathscr{A}_{i_1i_0}$ happens at time k_*.

We conclude from either of the two cases that

$$\begin{aligned}&\mathbf{P}\Big(x_\tau(k_*+2)\leq h(k_*)\prod_{k=k_*}^{k_*+1}T_k(1-T_k)\\&+H(k_*)\big(1-\prod_{k=k_*}^{k_*+1}T_k(1-T_k)\big),\ \tau=i_0,i_1,i_2\Big)\geq\Big(\frac{\alpha a_*}{n}\Big)^2.\end{aligned}$$

Continuing we obtain similar bounds for nodes $i_3,\dots,i_{n-1}$, which lead to

$$\begin{aligned}&\mathbf{P}\Big(H(k_*+n-1)\leq h(k_*)\prod_{k=k_*}^{k_*+n-2}T_k(1-T_k)\\&+H(k_*)\Big(1-\prod_{k=k_*}^{k_*+n-2}T_k(1-T_k)\Big)\Big)\geq\Big(\frac{\alpha a_*}{n}\Big)^{n-1}.\end{aligned}\tag{16}$$

We thus obtain

$$\begin{aligned}&\mathbf{P}\Big(\mathcal{H}(k_*+n-1)\leq\Big(1-\prod_{k=k_*}^{k_*+n-2}T_k(1-T_k)\Big)\mathcal{H}(k_*)\Big)\\&\geq\Big(\frac{\alpha a_*}{n}\Big)^{n-1}.\end{aligned}\tag{17}$$

Since assumption **A2** guarantees $\mathcal{H}(k+1)\leq\mathcal{H}(k)$ for all k with probability one, (17) leads to

$$\begin{aligned}&\mathbf{E}\big(\mathcal{H}(k_*+n-1)\big)\\&\leq\Big(1-\Big(\frac{\alpha a_*}{n}\Big)^{n-1}\prod_{k=k_*}^{k_*+n-2}T_k(1-T_k)\Big)\mathbf{E}\big(\mathcal{H}(k_*)\big).\end{aligned}$$

Note that k_* is chosen arbitrarily in the upper analysis. Particularly, we choose $k_*=K_0(n-1)\geq k_0$ for some integer $K_0\geq 0$, where k_0 is the initial time, we obtain

$$\begin{aligned}&\mathbf{E}\Big(\mathcal{H}\big((s+1)(n-1)\big)\Big)\leq\mathbf{E}\Big(\mathcal{H}\big(K_0(n-1)\big)\Big)\\&\times\prod_{t=K_0}^{s}\Big(1-\Big(\frac{\alpha a_*}{n}\Big)^{n-1}\prod_{k=t(n-1)}^{(t+1)(n-1)-1}T_k(1-T_k)\Big),\end{aligned}$$

which implies

$$\mathbf{E}\Big(\lim_{s\to\infty}\mathcal{H}\big(s(n-1)\big)\Big)\leq\lim_{s\to\infty}\mathbf{E}\Big(\mathcal{H}\big(s(n-1)\big)\Big)=0\tag{18}$$

by Fatou's Lemma and Lemma 1 as long as $\sum_{k=0}^{\infty}\prod_{s=k(n-1)}^{(k+1)(n-1)-1}T_s\big(1-T_s\big)=\infty$. Therefore, observing that $\mathcal{H}(k)$ is non-increasing, (18) leads to

$$\mathbf{P}\Big(\limsup_{k\to\infty}\mathcal{H}(k)=0\Big)=1.\tag{19}$$

The desired conclusion follows. □

We see from Propositions 1 and 2 that it is easier to achieve agreement convergence with symmetric updates, which is consistent with the literature [46].

Again let us consider the case when T_k is monotone. The following lemma holds. We omit the proof since it is based on some simple algebra.

Lemma 2: Let $\{b_k\}_0^\infty$ be a sequence of real numbers with $b_k\in[0,1]$ for all k. Suppose $b_{k+1}\leq b_k$ or $b_{k+1}\geq b_k$ for all k. Then the following statements are equivalent.

a) $\sum_{k=0}^{\infty}\prod_{s=k(n-1)}^{(k+1)(n-1)-1} b_s(1-b_s)=\infty$;

b) $\sum_{s=0}^{\infty}\Big(b_s(1-b_s)\Big)^{n-1}=\infty$.

Combining Proposition 2 and Lemma 2, we obtain the following conclusion.

Theorem 4: Suppose **A1**, **A2** and **A3** hold. Assume that either $T_{k+1}\le T_k$ or $T_{k+1}\ge T_k$ for all k. Then global agreement convergence is achieved a.s. if $\sum_{k=0}^{\infty}\big((1-T_k)T_k\big)^{n-1}=\infty$.

We see from Theorems 3 and 4 that the requirement for the sequence $\{T_k\}_0^{\infty}$ to guarantee a.s. agreement convergence increases from $\sum_{k=0}^{\infty}T_k(1-T_k)=\infty$ to $\sum_{k=0}^{\infty}\big((1-T_k)T_k\big)^{n-1}=\infty$ when the update transits from symmetric to asymmetric. Hence, these results quantify the cost of asymptotic updates versus the strength of attraction.

Remark 1: The convergence conditions established in this section are closely related to the infinite flow graph of random chains discussed in [40], [41]. Note that in our model the (strong or weak) "feedback properties" (cf. [40], [41]) may not necessarily hold since T_k can be arbitrarily close to one.

V. Attraction vs. Repulsion

In this section, we discuss the interplay between the attraction and repulsion updates. Again, we study symmetric and asymmetric updates, respectively.

A. Symmetric Update

Consider the following assumption.

A5. *(Symmetric Update)* The events $\mathscr{A}_{ij}(k)=\mathscr{A}_{ji}(k)$ and $\mathscr{R}_{ij}(k)=\mathscr{R}_{ji}(k)$ a.s. for all (i,j) and k.

Let λ_2^* and λ_n^* be the second smallest and largest eigenvalues of $D-(A+A^T)$ with $D=\mathrm{diag}(d_1\dots d_n)$, $d_i=\sum_{j=1}^{n}(a_{ij}+a_{ji})$, respectively. We have the follow result.

Proposition 3: Suppose **A1** and **A5** hold. Let $D_k\doteq T_k(1-T_k)\alpha-S_k(1+S_k)\gamma$. Then

(i) Global agreement convergence is achieved in the sense that $\lim_{k\to\infty}\mathbf{E}\big(L(k)\big)=0$ if $\prod_{k=0}^{\infty}\big(1-\frac{2}{n}\mathcal{I}_k\big)=0$, where $\mathcal{I}_k=D_k\lambda_2^*$, $D_k\ge 0$ and $\mathcal{I}_k=D_k\lambda_n^*$, $D_k<0$.

(ii) Disagreement convergence is achieved in the sense that $\lim_{k\to\infty}\mathbf{E}\big(L(k)\big)=\infty$, for almost all initial values if $\prod_{k=0}^{\infty}\big(1-\frac{2}{n}\hat{\mathcal{I}}_k\big)=\infty$, where $\hat{\mathcal{I}}_k=D_k\lambda_n^*$, $D_k\ge 0$ and $\hat{\mathcal{I}}_k=D_k\lambda_2^*$, $D_k<0$.

Proof. With assumption **A5**, the considered algorithm can be expressed as $x(k+1)=\Psi(k)x(k)$, where $\Psi(k)$ is a random matrix satisfying

$$\mathbf{P}\Big(\Psi(k)=\Psi^+_{\langle ij\rangle}\doteq I-T_k(e_i-e_j)(e_i-e_j)^T\Big)=\frac{\alpha}{n}(a_{ij}+a_{ji})$$

corresponding to event $\mathscr{A}_{ij}(k)$, and

$$\mathbf{P}\Big(\Psi(k)=\Psi^-_{\langle ij\rangle}\doteq I+S_k(e_i-e_j)(e_i-e_j)^T\Big)=\frac{\gamma}{n}(a_{ij}+a_{ji}).$$

corresponding to event $\mathscr{R}_{ij}(k)$, for all $i\neq j$.

Recall that $L(k)=\sum_{i=1}^{n}|x_i(k)-x_{\mathrm{ave}}|^2$, where $x_{\mathrm{ave}}=\sum_{i=1}^{n}x_i(k_0)/n$ is the initial average. It is crucial to notice that every possible sample of of the random matrix $\Psi(k)$ is symmetric and (generalized) stochastic since its row sum equals one, even though there are negative entries for the matrices $\Psi^-_{\langle ij\rangle}$. Therefore, similar to (11), we have

$$\mathbf{E}\Big(L(k+1)\big|x(k)\Big)=\big(x(k)-x_{\mathrm{ave}}\mathbf{1}\big)^T\mathbf{E}\big(\Psi^2(k)\big)\big(x(k)-x_{\mathrm{ave}}\mathbf{1}\big). \qquad (20)$$

Noticing (12) and

$$\Big(I+S_k(e_i-e_j)(e_i-e_j)^T\Big)^2=I+2S_k(1+S_k)(e_i-e_j)(e_i-e_j)^T$$

we obtain

$$\mathbf{E}\big(\Psi^2(k)\big)=I-2\Big(T_k(1-T_k)\alpha-S_k(1+S_k)\gamma\Big)\frac{1}{n}\Big(D-(A+A^T)\Big). \qquad (21)$$

There are two cases.

(i). Suppose $D_k\ge 0$. Recalling that every eigenvalue λ_i^* of $D-(A+A^T)$ is bounded by $2n$, all the eigenvalues of $\mathbf{E}\big(\Psi^2(k)\big)$ are contained within the unit circle. This implies

$$\mathbf{E}\Big(L(k+1)\big|x(k)\Big)\le\Big(1-\frac{2}{n}D_k\lambda_2^*\Big)L(k). \qquad (22)$$

(ii). Suppose $D_k<0$. Then we have

$$1\le\lambda_i\Big(\mathbf{E}\big(\Psi^2(k)\big)\Big)\le 1-\frac{2}{n}D_k\lambda_n^*$$

for each eigenvalue λ_i of $\mathbf{E}\big(\Psi^2(k)\big)$, which yields

$$\mathbf{E}\Big(L(k+1)\big|x(k)\Big)\le\Big(1-\frac{2}{n}D_k\lambda_n^*\Big)L(k). \qquad (23)$$

Then we see that the first part of the conclusion follows immediately, while the second part follows by verifying the lower bound in (22) and (23). This completes the proof. □

For a.s. disagreement divergence, we present the following result.

Proposition 4: Suppose **A1** and **A5** hold. Disagreement divergence is achieved a.s. for almost all initial conditions if

(i) there exists a constant $S^*>0$ such that $S_k\le S^*$ for all k;

(ii) there exists a constant $0<\varepsilon<1/2$ such that either $T_k\in[0,1/2-\varepsilon]$ or $T_k\in[1/2+\varepsilon,1]$ for all k;

(iii) there exists $0<\tau<1$ such that $\limsup_{m\to\infty}\sum_{k=0}^{m}\mathcal{J}_\tau(k)=O(m)$, where

$$\mathcal{J}_\tau(k)=\log\Big[\big(1+4\tau(S_k^2+S_k)\big)^{p_k}\big(2T_k-1\big)^{2\alpha}\Big]$$

with $p_k=-\frac{\frac{2}{n}\hat{\mathcal{I}}_k+\gamma\left(1+4\tau(S_k^2+S_k)\right)}{4(1-\tau)(S_k^2+S_k)}$, and by definition $b_k=O(c_k)$ means that $\limsup_{k\to\infty}b(k)/c(k)<\infty$ is a nonzero constant.

Proof. We divide the proof into three steps.

Step 1. In this step, we show that with probability one and for almost all initial conditions, finite-time agreement convergence cannot be achieved. According to (8), we obtain $\mathbf{P}\Big(\mathcal{H}(k+1)\ge\big(1-2T_k\big)\mathcal{H}(k)\Big)=1$ for all $k\ge 0$ if $T_k\in[0,1/2-\varepsilon]$. Observing that $1-2T_k\ge 2\varepsilon>0$ we see that $\mathcal{H}(k)>0$ for all k with probability one for all initial values satisfying

$\mathcal{H}(k_0) > 0$. This holds also for the other case $T_k \in [1/2+\varepsilon, 1]$ based on a symmetric argument.

Suppose nodes u and v reach the maximum and minimum values at time k, respectively, i.e.,

$$x_u(k) = \max_{i\in\mathcal{V}} x_i(k); \quad x_v(k) = \min_{i\in\mathcal{V}} x_i(k).$$

Then we have

$$\begin{aligned} L(k) &\geq |x_u(k) - x_{\text{ave}}|^2 + |x_v(k) - x_{\text{ave}}|^2 \\ &\geq \frac{1}{2}|x_u(k) - x_v(k)|^2 = \frac{1}{2}\mathcal{H}^2(k), \end{aligned}$$

which implies $L(k) > 0$ with probability one for almost all initial conditions. Therefore, with probability one, we can introduce a sequence of random variables $\{\varpi_k\}_0^\infty$ satisfying

$$L(k+1) = \varpi_k L(k), \ k \geq 0,$$

and

$$\mathbf{E}\big(\varpi_k\big) = \mathbf{E}\big(L(k+1)\big)/\mathbf{E}\big(L(k)\big) \geq 1 - \frac{2}{n}\hat{\mathcal{I}}_k \doteq Z_k. \quad (24)$$

Step 2. We establish a lower bound for $\mathbf{E}\big(\log \varpi_k\big)$ in this step. It is not hard to find that for every possible sample, $\Psi^+_{\langle ij\rangle}$ or $\Psi^-_{\langle ij\rangle}$ of $\Psi(k)$, it holds that

$$\begin{aligned} &\min\Big\{|\lambda_i|: \ \lambda_i \in \sigma(\Psi^+_{\langle ij\rangle}) \cup \sigma(\Psi^-_{\langle ij\rangle})\Big\} \\ &\geq \min\Big\{|\lambda_i|: \ \lambda_i \in \sigma(V_k)\Big\} = 2T_k - 1, \end{aligned} \quad (25)$$

where

$$V_k = \begin{pmatrix} 1-T_k & T_k \\ T_k & 1-T_k \end{pmatrix}. \quad (26)$$

Noticing that

$$L(k+1) \geq \min_{\lambda_i \in \sigma(\Psi(k))} |\lambda_i|^2 L(k),$$

the definition of ϖ_k and (25) yield

$$\mathbf{P}\Big(\varpi_k \geq (2T_k - 1)^2\Big) = \mathbf{P}\Big(\log \varpi_k \geq \log(2T_k - 1)^2\Big) = 1. \quad (27)$$

Similarly, observing that

$$\begin{aligned} &\max\Big\{|\lambda_i|: \ \lambda_i \in \sigma(\Psi^+_{\langle ij\rangle}) \cup \sigma(\Psi^-_{\langle ij\rangle})\Big\} \\ &\leq \max\Big\{|\lambda_i|: \ \lambda_i \in \sigma(\hat{V}_k)\Big\} = 2S_k + 1, \end{aligned} \quad (28)$$

where

$$\hat{V}_k = \begin{pmatrix} 1+S_k & -S_k \\ -S_k & 1+S_k \end{pmatrix},$$

we obtain

$$\mathbf{P}\Big(\varpi_k \leq (2S_k + 1)^2\Big) = \mathbf{P}\Big(\log \varpi_k \leq \log(2S_k + 1)^2\Big) = 1. \quad (29)$$

Noticing (24) and that

$$\mathbf{E}\big(\varpi_k\big) = \int_{\varpi_k \leq 1} \varpi_k + \int_{\varpi_k > 1} \varpi_k \leq 1 + \int_{\varpi_k > 1} \varpi_k,$$

we obtain

$$\int_{\varpi_k > 1} \varpi_k \geq \mathbf{E}\big(\varpi_k\big) - 1 \geq Z_k - 1.$$

Take $0 < \tau < 1$ a constant. The structure of the considered algorithm immediately gives us

$$\begin{aligned} &\mathbf{P}\Big(\varpi_k > 1\Big) \\ &\leq \mathbf{P}\Big(\mathscr{R}_{ij}(k) \text{ happens for some node pair } (i,j)\Big) = \gamma. \end{aligned}$$

Now we conclude that

$$\begin{aligned} Z_k - 1 \leq \int_{\varpi_k > 1} \varpi_k &\leq \hat{p}_k(2S_k + 1)^2 \\ &+ \big(1 - \tau + \tau(2S_k + 1)^2\big)(\gamma - \hat{p}_k), \end{aligned} \quad (30)$$

where by definition

$$\hat{p}_k \doteq \mathbf{P}\big(1 - \tau + \tau(2S_k + 1)^2 \leq \varpi_k \leq (2S_k + 1)^2\big).$$

After some simple algebra we see from (30) that

$$\begin{aligned} \hat{p}_k &\geq \frac{Z_k - 1 - \gamma\big(1 - \tau + \tau(2S_k + 1)^2\big)}{4(1-\tau)(S_k^2 + S_k)} \\ &= -\frac{\frac{2}{n}\hat{\mathcal{I}}_k + \gamma\big(1 + 4\tau(S_k^2 + S_k)\big)}{4(1-\tau)(S_k^2 + S_k)} \doteq p_k. \end{aligned} \quad (31)$$

Combining (27), (29) and (31), we eventually arrive at the following lower bound of $\mathbf{E}\log\varpi_k$:

$$\begin{aligned} \mathbf{E}\log\varpi_k &\geq \hat{p}_k \log\Big(1 - \tau + \tau(2S_k + 1)^2\Big) + \alpha\log(2T_k - 1)^2 \\ &\geq \mathcal{J}_\tau(k). \end{aligned} \quad (32)$$

Step 3. In this step, we complete the final piece of the proof by a contradiction argument. Suppose there exist two constants $M_0 \geq 0$ and $0 < p < 1$ such that

$$\mathbf{P}\Big(\limsup_{k\to\infty} \mathcal{H}(k) \leq M_0\Big) = p. \quad (33)$$

Noticing that $L(k) = \sum_{i=1}^n |x_i(k) - x_{\text{ave}}|^2 \leq n\mathcal{H}^2(k)$, we further conclude

$$\mathbf{P}\Big(\limsup_{k\to\infty} L(k) \leq nM_0^2\Big) \geq p,$$

which yields

$$\mathbf{P}\Big(\limsup_{m\to\infty} \log L(m+1) \leq \log\big(nM_0^2\big)\Big) \geq p.$$

This leads to

$$\mathbf{P}\Big(\lim_{m\to\infty} \frac{\sum_{k=0}^m \log\varpi_k}{m} \leq 0\Big) \geq p. \quad (34)$$

On the other hand, noting that the node pair selection process is independent of time and node state, and that $\mathbf{V}(\log\varpi_k)$ is bounded according to (27) and (29), we can apply the strong law of large numbers and conclude from (32) that

$$\begin{aligned} &\mathbf{P}\Big(\lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^m \big(\log\varpi_k - \mathcal{J}_\tau(k)\big) \geq 0\Big) \\ &\geq \mathbf{P}\Big(\lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^m \big(\log\varpi_k - \mathbf{E}\log\varpi_k\big) = 0\Big) = 1, \end{aligned}$$

which contradicts (34) if $\limsup_{m\to\infty}\sum_{k=0}^m \mathcal{J}_\tau(k) = O(m)$.

The desired conclusion thus follows and this completes the proof. $\square$

We conclude this subsection by the following conclusion under the condition when T_k and S_k are time-invariant.

Theorem 5: Suppose **A1** and **A5** hold. Let $T_\star \in [0,1]$ and $S_\star > 0$ be two given constants. Assume that $T_k \equiv T_\star$ and $S_k \equiv S_\star$. Then

$$D_* = S_\star(1+S_\star)\gamma - T_\star(1-T_\star)\alpha$$

is a critical convergence measure regarding the state convergence of the considered network. To be precise, we have

(i) Global agreement convergence is achieved both in the sense that $\lim_{k\to\infty} \mathbf{E}\big(L(k)\big) = 0$ and almost surely if $D_* < 0$;

(ii) State oscillation is achieved in expectation, i.e., $\mathbf{E}\big(L(k)\big) = L(k_0)$ for all $k \geq k_0$ if $D_* = 0$;

(iii) Disagreement divergence is achieved in the sense that $\lim_{k\to\infty} \mathbf{E}\big(L(k)\big) = \infty$ for almost all initial values if $D_* > 0$;

(iv) Disagreement divergence is achieved a.s. for almost all initial conditions if $T_\star \neq 1/2$ and D_* is sufficiently large, i.e., there exists $0 < \tau < 1$ such that

$$\big(1+4\tau(S_\star^2+S_\star)\big)^{p^*}\big(2T_\star-1\big)^{2\alpha} > 1,$$

where

$$p^* = \frac{2D_*\lambda_2^* - n\gamma\big(1+4\tau(S_\star^2+S_\star)\big)}{4n(1-\tau)(S_\star^2+S_\star)}.$$

Proof. We just need to verify the a.s. convergence claim in (i) since all the other conclusions follow straightforwardly from Propositions 3 and 4.

We invoke the following supermartingale convergence theorem to illustrate the almost sure convergence for the case $D_* < 0$.

Lemma 3: [2] Let $\xi_k, k \geq 0$ be a sequence of nonnegative random variables with $\mathbf{E}V_0 < \infty$. If

$$\mathbf{E}\big(\xi_{k+1}\big|\xi_0,\dots,\xi_k\big) \leq (1-c_k)\xi_k$$

with $c_k \in [0,1]$ and $\sum_{k=0}^\infty c_k = \infty$, then $\lim_{k\to\infty}\xi_k = 0$ almost surely.

When $D_* < 0$, from (22) we have

$$\mathbf{E}\Big(L(k+1)\big|x(k)\Big) \leq \Big(1 - \frac{2}{n}D_*\lambda_2^*\Big)L(k). \tag{35}$$

Then based on Lemma 3, $L(k)$ tends to zero a.s., which is equivalent to a.s. agreement convergence. □

Remark 2: It is surprising that the convergence measure D_* in Theorem 5 does not rely on the network topology. This is to say, if all the nodes may misbehave with equal probability as the proposed algorithm, then there is no particular topology which can be viewed as "better" than others in terms of agreement convergence.

B. Asymmetric Update

In this subsection, we discuss asymmetric node updates. We introduce the following assumption.

A6. *(Asymmetric Update)* Both $\mathbf{P}(\mathscr{A}_{ij}(k)\bigcap\mathscr{A}_{ji}(k)) = 0$ and $\mathbf{P}(\mathscr{R}_{ij}(k)\bigcap\mathscr{R}_{ji}(k)) = 0$ for all (i,j) and k.

The main result on a.s. agreement convergence under asymmetric update is as follows.

Proposition 5: Suppose **A1** and **A6** hold. Global agreement convergence is achieved a.s. if

(i) $0 \leq \big(\frac{\alpha a_*}{n}\big)^{n-1}\hat{T}_k - \big(1-\big(1-\gamma\big)^{n-1}\big)\big(\hat{S}_k - 1\big) \leq 1$ for all $k \geq 0$, where $\hat{T}_k = \prod_{m=k(n-1)}^{(k+1)(n-1)-1} T_m(1-T_m)$ and $\hat{S}_k = \prod_{m=k(n-1)}^{(k+1)(n-1)-1}\big(S_m+1\big)$;

(ii) $\sum_{k=0}^\infty \big(\frac{\alpha a_*}{n}\big)^{n-1}\hat{T}_k - \big(1-\big(1-\gamma\big)^{n-1}\big)\big(\hat{S}_k-1\big) = \infty$.

Proof. Following the considered algorithm we have

$$\mathbf{P}\Big(\mathcal{H}(k_*+n-1) \leq \Big(\prod_{k=k_*}^{k_*+n-2}\big(S_k+1\big)\Big)\mathcal{H}(k_*)\Big) = 1 \tag{36}$$

and

$$\mathbf{P}\Big(\mathcal{H}(k_*+n-1) > \mathcal{H}(k_*)\Big) \leq 1 - \big(1-\gamma\big)^{n-1} \tag{37}$$

since $\mathcal{H}(k_*+n-1) > \mathcal{H}(k_*)$ implies that repulsion happens at least one time during $[k_*, k_*+n-1)$.

We conclude from (17), (36), and (37) that

$$\begin{aligned}
&\mathbf{E}\Big(\mathcal{H}(k_*+n-1)\Big|x(k_*)\Big)\\
&\leq \Bigg[1 - \Big(\frac{\alpha a_*}{n}\Big)^{n-1}\prod_{k=k_*}^{k_*+n-2}T_k(1-T_k)\\
&\quad + \Big(1-\big(1-\gamma\big)^{n-1}\Big)\Big(\prod_{k=k_*}^{k_*+n-2}\big(S_k+1\big)-1\Big)\Bigg]\mathcal{H}(k_*)
\end{aligned}$$

for all $k_* > 0$. This implies

$$\mathbf{P}\Big(\lim_{m\to\infty}\mathcal{H}\big(m(n-1)\big) = 0\Big) = 1 \tag{38}$$

with conditions (i) and (ii), again from Lemma 3. Moreover, condition (i) leads to that S_k is upper bounded. The desired conclusion thus follows. □

Next, we study a.s. disagreement divergence. The following conclusion holds.

Proposition 6: Suppose **A1** and **A6** hold. Disagreement divergence is achieved a.s. for almost all initial values if

(i) there exist two constants $S^* > 0$ and $0 < T^* < 1$ such that $S_k \leq S^*$ and $T_k \leq T^*$ for all k.

(ii) there exists an integer $Z \geq 0$ such that $\sum_{k=0}^m \mathcal{J}_Z(k) = O(m)$, where

$$\begin{aligned}
\mathcal{J}_Z(k) &= \Big(\frac{\gamma a_*}{n}\Big)^{Z+1}\log\Big(\frac{1}{n-1}\prod_{\varsigma=k(Z+1)}^{(k+1)(Z+1)-1}\big(1+S_\varsigma\big)\Big)\\
&\quad + \big(1-(1-\alpha)^{Z+1}\big)\log\prod_{\varsigma=k(Z+1)}^{(k+1)(Z+1)-1}\big(1-T_\varsigma\big).
\end{aligned} \tag{39}$$

Proof. Suppose node pair (i,j) is selected at time k. According to the definition of the considered randomized algorithm, we obtain

$$\big|x_i(k+1)-x_j(k+1)\big| = \begin{cases} \big|x_i(k)-x_j(k)\big|, & \text{if } \mathscr{N}_{ij}(k) \text{ happens;}\\ (1-T_k)\big|x_i(k)-x_j(k)\big|, & \text{if } \mathscr{A}_{ij}(k) \text{ happens;}\\ (1+S_k)\big|x_i(k)-x_j(k)\big|, & \text{if } \mathscr{R}_{ij}(k) \text{ happens.}\end{cases} \tag{40}$$

Therefore, with assumption **A6**, we obtain

$$\mathbf{P}\Big(\mathcal{H}(k+1) \geq \big(1-T^*\big)\mathcal{H}(k)\Big) = 1$$

for all $k \geq 0$. This implies for all initial values satisfying $\mathcal{H}(k_0) > 0$, agreement convergence is achieved only in infinite time with probability one. As a result, we can well define a sequence of random variable, $\{\hat{\varpi}_k\}_0^\infty$, such that $\mathcal{H}(k+1) = \hat{\varpi}_k \mathcal{H}(k),\ k \geq 0$.

Now with (40), it is straightforward to conclude that

$$\mathbf{P}\Big(\hat{\varpi}_k \geq 1 - T_k\Big) = 1 \quad (41)$$

and

$$\mathbf{P}\Big(\hat{\varpi}_k < 1\Big) \leq \alpha. \quad (42)$$

Moreover, based on the weak connectivity assumption **A1**, for any $k \geq 0$, there always exist two nodes i_0 and j_0 such that either $a_{i_0 j_0} > 0$ or $a_{j_0 i_0} > 0$, and $\big|x_{i_0}(k) - x_{j_0}(k)\big| \geq \frac{1}{n-1}\mathcal{H}(k)$. Note that if $a_{i_0 j_0} > 0$ or $a_{j_0 i_0} > 0$, and $\big|x_{i_0}(k) - x_{j_0}(k)\big| \geq \mu\mathcal{H}(k)$ for some $\mu > 0$, we have $\big|x_{i_0}(k+1) - x_{j_0}(k+1)\big| \geq (1+S_k)\mu\mathcal{H}(k)$ with probability $\gamma a_*/n$.

Thus, the case with $\mathscr{R}_{ij}(k)$ happening in (40) leads to

$$\mathbf{P}\Big(\hat{\varpi}_k \geq \frac{1+S_k}{n-1}\Big) \geq \frac{\gamma a_*}{n}, \quad (43)$$

and

$$\mathbf{P}\Big(\hat{\varpi}_{k+s}\cdots\hat{\varpi}_k \geq \frac{1}{n-1}\prod_{\varsigma=k}^{k+s}(1+S_\varsigma)\Big) \geq \Big(\frac{\gamma a_*}{n}\Big)^{s+1} \quad (44)$$

for all $s \geq 0$, recalling that $a_* = \min\{a_{ij} :\ a_{ij} > 0\}$ is the lower bound of the nonzero entries of A.

Therefore, letting $Z \geq 0$ be an integer, we can eventually conclude from (41), (42) and (44) that

$$\sum_{k=k_*}^{k_*+Z} \mathbf{E}\log\hat{\varpi}_k \geq \Big(\frac{\gamma a_*}{n}\Big)^{Z+1}\log\Big(\frac{1}{n-1}\prod_{k=k_*}^{k_*+Z}(1+S_k)\Big) + \Big(1-(1-\alpha)^{Z+1}\Big)\log\prod_{k=k_*}^{k_*+Z}(1-T_k).$$

The desired conclusion follows from the same argument as the proof of Proposition 4 based on the strong law of large numbers, again due to the fact that the pair selection process is independent of the node states. This completes the proof. □

We also end the discussion by a theorem for the case when T_k and S_k are time-invariant. Applying the same analysis methods of proving Propositions 5 and 6, we obtain the following result.

Theorem 6: Suppose **A1** and **A6** hold. Let $T_\star \in [0,1]$ and $S_\star > 0$ be two given constants. Assume that $T_k \equiv T_\star$ and $S_k \equiv S_\star$. Then we have

(i) Global agreement convergence is achieved a.s. if

$$\Big(1-(1-\gamma)^{n-1}\Big)\Big((S_\star+1)^{n-1}-1\Big) < \Big(\frac{\alpha a_*}{n}\Big)^{n-1}\Big(\max\{T_\star, 1-T_\star\}\Big)^{n-1};$$

(ii) Disagreement divergence is achieved a.s. for almost all initial conditions if there exists an integer $Z \geq 0$ such that

$$\Big(\frac{\gamma a_*}{n}\Big)^{Z+1}\log\frac{(1+S_\star)^{Z+1}}{n-1} + \Big(1-(1-\alpha)^{Z+1}\Big)(Z+1)\log\Big(1-T_\star\Big) > 0. \quad (45)$$

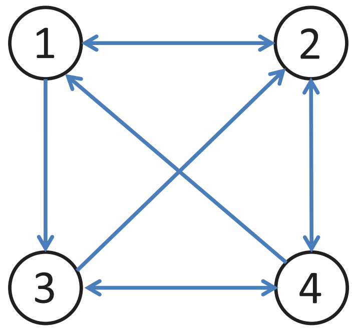

Fig. 1. The underlying communication graph.

C. Numerical Example

We present a numerical example in order to illustrate the critical measure established in Theorem 5. Consider four nodes $1, \ldots, 4$. The node meeting probability matrix is given by

$$A = [a_{ij}] = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{pmatrix}.$$

The induced graph $\mathcal{G}_A$ from A is shown in the Fig. 1. The initial values are taken as $x_i(0) = i, i = 1, \ldots, 4$. For the algorithm considered in Section 2, we take $\alpha = \beta = \gamma = 1/3$ and let $T_k \equiv T_\star = 1/4$ and $S_k \equiv S_\star$.

We study three cases, corresponding to $S_\star = (\sqrt{7}-2)/4$, $(\sqrt{7}-2)/4 - 0.05$, $(\sqrt{7}-2)/4 + 0.05$, respectively. The corresponding values of $D_* = S_\star(1+S_\star)\gamma - T_\star(1-T_\star)\alpha$ are then given by 0, -0.0212, and 0.0229. We run the considered randomized algorithm for 10^5 times, and then take the average value of the consensus measure $L(k) = \sum_{i=1}^4 \big(x_i(k) - x_{\text{ave}}\big)^2$ at every time step as the empirical estimate of the expected value of $L(k)$. The transition of $\mathbf{E}(L(k))$ for these three cases of D_* is shown in Fig. 2. From the numerical result we see that $\mathbf{E}(L(k))$ diverges when $D_* = 0.0229 > 0$, converges when $D_* = -0.0212 < 0$, and keeps constant when $D_* = 0$. The numerical result is consistent with the conclusion in Theorem 5.1.

VI. Conclusions

This paper proposed a model for investigating node misbehavior in distributed information processing over random networks. At each instance, two nodes were selected for a meeting with a given probability. When nodes meet, there were three events for the node update: attraction, neglect, or repulsion. Attraction event follows the standard averaging algorithm targeting a consensus; neglect event means the selected node will stick to its current state; repulsion event represents the case when nodes are against the consensus convergence. Each node was assumed to follow one of these three update rules at random. Both symmetric and asymmetric node updates were studied. After obtaining two general impossibility theorems, a series of necessary and/or sufficient conditions were established for the network to reach a.s. agreement convergence, or a.s. disagreement divergence. To the best of our knowledge, the obtained results for the first

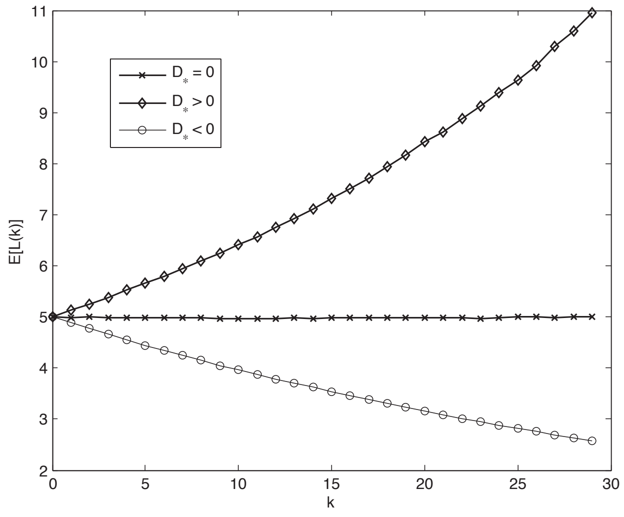

Fig. 2. The expected value of $L(k)$ for different D_*.

time in the literature gave a clear description on the possible disagreement divergence for agreement protocols due to node misbehavior. More challenges lie in the optimal policy for the nodes to take bad action from a tradeoff between the risk of being discovered and the result it generates, and the case when bad action only takes place for some particular neighboring relations.

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