The Role of Persistent Graphs in the Agreement Seeking of Social Networks

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***Abstract*—This paper investigates the role persistent relations play for a social network to reach a global belief agreement under discrete-time or continuous-time evolution. Each directed arc in the underlying communication graph is assumed to be associated with a time-dependent weight function, which describes the strength of the information flow from one node to another. An arc is said to be persistent if its weight function has infinite $\mathscr{L}_1$ or ℓ_1 norm for continuous or discrete belief evolutions, respectively. The graph that consists of all persistent arcs is called the persistent graph of the underlying network. Three necessary and sufficient conditions on agreement or ϵ-agreement are established. We prove that the persistent graph fully determines the convergence to a common opinion in a social network. It is shown how the convergence rate explicitly depends on the diameter of the persistent graph. For a social networking service like Facebook, our results indicate how permanent friendships need to be and what network topology they should form for the network to be an efficient platform for opinion diffusion.**

***Index Terms*—Consensus, Persistent Graphs, Social Networks, Dynamical Systems, Non-smooth analysis**

I. INTRODUCTION

THE EVOLUTION of *beliefs* or *opinions* is a fundamental problem in the study of social networks. Individuals form opinions on various social events, exchange opinions via interpersonal actions, and revise their opinions from time to time. The underlying interaction, or information exchange network, is naturally formed during this process. The structure of this network essentially determines how beliefs are propagated.

A classical way of modeling the dynamics of opinion evolution over social networks is to simply use a discrete time dynamical system with a row-stochastic state transition matrix, where each entry of the system state corresponds to the opinion of a member of the social network [6], [7], [8], [9]. The central idea here is that information exchange will lead to social aggregation of dispersed beliefs. In [6], the author discussed how the connectivity of the social influence graph affects the common agreement with directed interpersonal actions. Then in [7], [8], agreement convergence conditions were established for more general cases taking advantage of the theories of Markov chains [10], [11]. Following this approach, variations of the model have been considered in [12], [13], [14], [15], [16] for the study of opinion dynamics in social networks, and whether an asymptotic belief agreement can be reached or not has been a central question. In fact, related consensus problems also appear in many different contexts in the study of computer science and engineering, e.g., decentralized and parallel computations [17], [18], [19], coordinations of autonomous agents [20], [21], [22], [23], [24], [25], and sensor networks [26], [27], [28], [29]. Agreement seeking has been extensively studied in the literature for both discrete-time and continuous-time models [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48].

Efforts have also been devoted to the investigation of the role that the interpersonal influence networks play in the formation of individual beliefs [49], [50], [51], [52]. Here researchers studied how the structure of the influence network, i.e., patterns and strengths of the interpersonal influence among the social members, determines the eventual individual opinion and if it corresponds to agreement or not. In [50], structural equation models were studied for the formation of the social opinions. Then in [52], the asymptotic opinion formation was studied under a dynamical recursive system for the opinion evolution. Among the interpersonal influence networks, the influence, or power, of one member of a population over another member is reflected by the corresponding entry of the state transition matrix, which is used to characterize the opinion evolution. Thus, the weight of an arc in the influential communication graph determines the the strength of the interpersonal influence from one social member to another. In most existing works, the arc weights are assumed to either be constants [8], [21], [30], or in a compact set with positive lower and upper bounds [14], [17], [20], [35], [36], [42], [43], [44].

Interpersonal influential power in realistic social networks may vary over a wide range, and may even fade away as time goes by. For example, opinions may be heavily influenced over short time periods by social medias, local leaders, political actors, or random strangers, but over longer time periods a member of the social network may tend to be more influenced by long term relations with families, friends and coworkers. This is to say, the interpersonal influence network is in general time varying, and the pattern of interpersonal influence depends on various social relations in the society. Influential communication can be impulsive, vanishing, persistent, etc. An interesting question arises: are there certain relations that are the ones that actually generate the convergence to a belief agreement, and if so, how do their interconnections influence the convergence rate?

The central aim of the paper is to answer the question. We build a model to classify arcs in the underlying communication graph, and then we give a precise description on how the persistent arcs indeed determine conditions for agreement convergence. We adopt the classical model for belief evolution under general conditions, in which the strength of interper-

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sonal influence can be time varying and unbounded [6], [7], [8], [9]. We define the persistent graph as the graph having links whose weight functions have infinite $\mathscr{L}_1$ or ℓ_1 norm for continuous-time or discrete-time belief dynamics, respectively. Global agreement and ϵ-agreement are defined as whether the maximum state difference converges to zero, and whether the convergence is exponentially fast, respectively. For the discrete-time case, a necessary and sufficient condition is obtained on ϵ-agreement under stochasticity, self-confidence, and arc balance assumptions. Then, for the continuous-time case, two necessary and sufficient conditions are established on global agreement and ϵ-agreement, respectively. In this way, we precisely state how the persistent graph plays a fundamental role in agreement seeking. Additionally, comparisons of our new conditions are given with existing results and the relations between the discrete-time and continuous-time evolutions are highlighted.

From a theoretical point of view, our work generalizes the existing agreement convergence conditions in the sense that we do not impose lower and upper bounds for the arc weights, which is a typical assumption in the literature [6], [8], [9], [20], [21], [35], [36], [42]. We establish necessary and sufficient conditions for ϵ-agreement, which is a problem that have received attention recently [30], [42], [43]. From a practical point of view, our work shows the importance of persistent interpersonal influences in social networks, since the agreement of social opinion is fully determined by the persistent graphs. For real social networks, like Twitter or Facebook, intuitively the persistency of the network can be interpreted in the contact frequency among users. Our results show that the diameter of persistent graph essentially determines the agreement convergence rate. This observation may inspire researchers to explore the structure of persistent graphs for social networks. A future direction of research suggested by the results of our paper is to study the relation between agreement convergence and network topology from social network data.

The rest of the paper is organized as follows. In Section 2, we introduce the network model and define the problem of interest. Then in Sections 3 and 4, the main results and convergence analysis are presented for discrete-time and continuous-time dynamics, respectively. Finally some discussions and concluding remarks are given in Sections 5 and 6.

II. Problem Definition

In this section, we first introduce some basic graph theory [4], and then present the social network model and define the considered problem.

A. Directed Graphs

A (simple) *digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \dots, n\}$ of nodes and an arc set $\mathcal{E}$, where each *arc* $(i, j) \in \mathcal{E}$ is an ordered pair from node $i \in \mathcal{V}$ to another node $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, k$, the sequence is called a (directed) *path* with *length* k. A path from i to j is denoted $i \to j$, and the length of $i \to j$ is denoted $|i \to j|$. A path with no repeated nodes is called a *simple* path. If there exists a path from node i to node j, then node j is said to be reachable from node i. Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. $\mathcal{G}$ is said to be *strongly connected* if it contains path $i \to j$ and $j \to i$ for every pair of nodes i and j; $\mathcal{G}$ is said to be *quasi-strongly connected* if $\mathcal{G}$ has a center [5], [33]; $\mathcal{G}$ is said to be *weakly connected* if $\mathcal{G}$ is connected as an undirected graph ignoring the directions of the arcs (cf. Fig. 1).

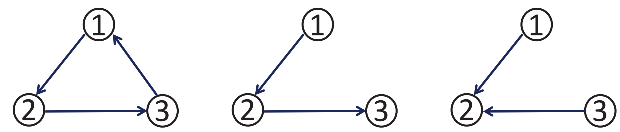

Fig. 1. Strongly, quasi-strongly, and weakly connected digraphs.

The *distance* from i to j, $d(i, j)$, is defined as the length of a shortest (simple) path $i \to j$ when j is reachable from i, and the *diameter* of $\mathcal{G}$ as $d_0 = \max\{d(i, j) | i, j \in \mathcal{V},\ j \text{ is reachable from } i\}$.

B. Social Network Model

In this paper, we consider a social network model with node set $\mathcal{V} = \{1, \dots, n\}$. Let the digraph $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ denote the *underlying* graph of the considered social network. The underlying graph indicates all potential interactions between nodes. Node j is said to be a *neighbor* of i at time t when there is an arc $(j, i) \in \mathcal{E}_*$; each node is supposed to be a neighbor of itself. Let $\mathcal{N}_i = \{i\} \cup \{j : (j, i) \in \mathcal{E}_*\}$ denote the neighbor set of node i.

Let $x_i(t) \in \mathbb{R}$ be the *belief* of node i at time t. Time is either discrete or continuous. The initial time is $t_0 \geq 0$ in both cases and each node is equipped with an initial belief $x_i(t_0)$. The belief updating rule is in discrete time:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} W_{ij}(t) x_j(t), \quad i = 1, \dots, n \tag{1}$$

and in continuous time:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} W_{ij}(t) \big[x_j(t) - x_i(t)\big], \quad i = 1, \dots, n. \tag{2}$$

Here $W_{ij}(t) : [0, \infty) \to [0, \infty)$ is a nonnegative scalar function which represents the weight of arc (j, i). Clearly $W_{ij}(t)$ describes the strength of the influence of node j on i. Since $W_{ij}(t) = 0$ may happen from time to time, the graph is indeed time-varying.

We define

$$\psi(t) \doteq \min_{i \in \mathcal{V}} \{x_i(t)\}, \quad \Psi(t) \doteq \max_{i \in \mathcal{V}} \{x_i(t)\}$$

as the minimum and maximum state value at time t, respectively. Then

$$\mathcal{H}(t) \doteq \Psi(t) - \psi(t)$$

is a natural agreement measure marking the maximum distances between the individual beliefs. The considered global agreement and ϵ-agreement for both the discrete-time and continuous-time updating rules are defined as follows.

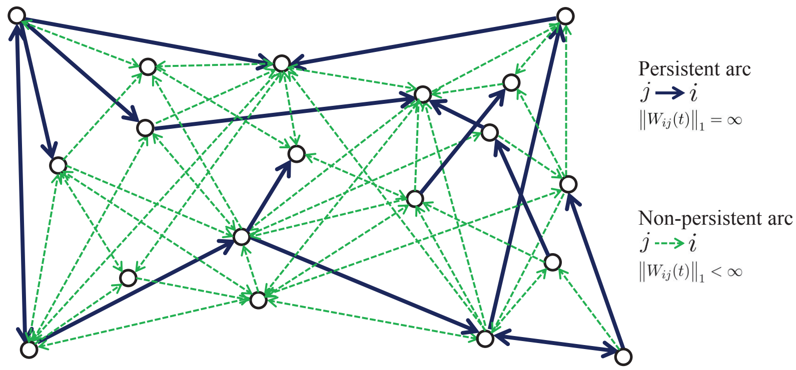

Fig. 2. The complex node interactions consist of persistent (solid) and non-persistent (dashed) arcs. The persistent graph is shown to play a fundamental role for the convergence to an agreement.

Definition 2.1: (a) Global *agreement* is achieved if for any $x(t_0) \doteq (x_1(t_0)\dots x_n(t_0))^T \in \mathbb{R}^n$, we have

$$\lim_{t\to\infty} \mathcal{H}(t) = 0. \tag{3}$$

(b) Global ϵ-agreement is achieved if there exist two constants $0 < \epsilon < 1$ and $T_0 > 0$ such that for any $x(t_0) \in \mathbb{R}^n$ and $t \geq t_0$, we have

$$\mathcal{H}(t + T_0) \leq \epsilon\mathcal{H}(t). \tag{4}$$

Remark 2.1: A global agreement only requires that $\mathcal{H}(t)$ will converge to zero as t tends to infinity. If it is further required that the convergence speed is at least exponentially fast, we use global ϵ-agreement. This definition of ϵ-agreement and other similar concepts have been widely used to characterize the convergence rate of consensus evolutions in the literature, e.g., [30], [42], [43], [44].

C. Persistent Graphs

The goal of this paper is to distinguish the arcs from the underlying graph that are *persistent* over a long time range and how they influence global agreement. To be precise, we impose the following definition for persistent arcs and persistent graphs based on the $\mathscr{L}_1$ or ℓ_1 norms of the weight functions.

Definition 2.2: (a) An arc $(j,i) \in \mathcal{E}_*$ is a persistent arc of the discrete-time updating rule (1) if

$$\sum_{t=0}^{\infty} W_{ij}(t) = \infty,$$

and a persistent arc of the continuous-time updating rule (2) if

$$\int_s^{\infty} W_{ij}(t)dt = \infty \text{ for all } s \geq 0.$$

(b) The graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$ that consists of all persistent arcs is called the persistent graph.

Next, in Sections 3 and 4, we will investigate the discrete-time and continuous-time updating rules, respectively. We will establish sufficient and necessary conditions on global agreement and ϵ-agreement, which illustrate that the notion of persistent graphs is critical to the convergence.

III. Discrete-Time Belief Evolution

In this section, we focus on the discrete-time belief evolution (1). In order to obtain the main result, we need the following assumptions.

A1 *(Stochasticity)* $\sum_{j\in\mathcal{N}_i} W_{ij}(t) = 1$ for all $i \in \mathcal{V}$ and $t \geq 0$.

A2 *(Self-confidence)* There exists $0 < \eta < 1$ such that $W_{ii}(t) \geq \eta$ for $i \in \mathcal{V}$ and $t \geq 0$.

A3 *(Arc Balance)* There exists a constant $A > 1$ such that for any two arcs $(j,i), (m,k) \in \mathcal{E}^p$ and $t \geq 0$, we have

$$A^{-1}W_{ij}(t) \leq W_{km}(t) \leq AW_{ij}(t).$$

The main result for the discrete-time updating rule (1) on global ϵ-agreement is as follows.

Theorem 3.1: Suppose A1, A2 and A3 hold. Global ϵ-agreement is achieved for (1) if and only if

(a) $\mathcal{G}^p$ is quasi-strongly connected;

(b) there exist a constant $a_* > 0$ and an integer $T_* > 0$ such that $\sum_{s=t}^{t+T_*-1} W_{ij}(s) \geq a_*$ for all $t \geq 0$ and $(j,i) \in \mathcal{E}^p$.

In fact, if (a) and (b) hold, then we have

$$\mathcal{H}(t + d_0T_*) \leq \Big(1 - \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\mathcal{H}(t) \tag{5}$$

for all $t \geq t_0$, where d_0 represents the diameter of $\mathcal{G}^p$.

Remark 3.1: Consensus convergence for many variations of (1) has been extensively studied in the literature, e.g., [1], [14], [13], [10], [11], [20], [24], [23], [35]. As for convergence rate, a relatively conservative bound is given in [1], [20], and then generalized in [36], [43]. Recently a sharper bound for convergence rate was obtained in [44]. The self-confidence condition A2 is generally not necessary to ensure a consensus, but the convergence properties may be quite different without A2, especially for the case with time-varying graphs.

Remark 3.2: Most of existing results are based on the assumption that all weight functions $W_{ij}(t)$ in the underlying graph have a positive lower bound whenever they are not zero. Here we just need the self-loop weights, $W_{ii}(t), i = 1,\dots,n$, to have a positive lower bound. As indicated by the proof below, the sufficiency statement of Theorem 3.1 relies on the self-confidence assumption A2, while the arc balance assumption A3 is used in the necessity part.

TABLE I
NOTATIONS

Notation	Definition
$\mathcal{E}_*$	Underlying Arc Set
$\mathcal{E}^p$	Persistent Arc Set
$\theta(t)$	$\sum_{(j,i)\in\mathcal{E}_*\setminus\mathcal{E}^p} W_{ij}(t)$
$\xi^+(t;m)$	$\sum_{j\in\mathcal{N}_m\setminus\{m\}} W_{mj}(t)$
$\xi_0^+(t;m)$	$\sum_{j\in\mathcal{N}_m\setminus\{m\},(j,m)\in\mathcal{E}^p} W_{mj}(t)$

In the following two subsections, we prove the necessity and sufficiency parts of Theorem 3.1, respectively.

Before we state the proofs, we introduce some more notations. For two sets S_1 and S_2, $S_1 \setminus S_2$ is defined as $S_1 \setminus S_2 = \{z : z \in S_1, z \notin S_2\}$. For the underlying graph $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ and the persistent graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$, we denote

$$\theta(t) = \sum_{(j,i)\in\mathcal{E}_*\setminus\mathcal{E}^p} W_{ij}(t) \tag{6}$$

as the overall weights of non-persistent arcs in the underlying graph at time t. The overall weights entering node $m \in \mathcal{V}$ at time t from its neighbors in the underlying and persistent graphs are defined, respectively, as

$$\xi^+(t;m) = \sum_{j\in\mathcal{N}_m\setminus\{m\}} W_{mj}(t),$$

and

$$\xi_0^+(t;m) = \sum_{j\in\mathcal{N}_m\setminus\{m\},(j,m)\in\mathcal{E}^p} W_{mj}(t).$$

We summarize these notations in the following table, which will be used throughout the rest of the paper.

A. Necessity

We need to show that a global ϵ-agreement cannot be achieved without either condition (a) or (b).

The upcoming analysis relies on the following well-known lemmas.

Lemma 3.1: Suppose $0 \le p_k < 1$ for all k. Then $\sum_{k=0}^{\infty} p_k = \infty$ if and only if $\prod_{k=0}^{\infty}(1-p_k) = 0$.

Lemma 3.2: $\log(1-t) \ge -2t$ for all $0 \le t \le 1/2$.

We have the following proposition indicating that $\mathcal{G}^p$ being quasi-strongly connected is not only a necessary condition for (1) to reach global ϵ-agreement, but also necessary for (simple) global agreement, even in the absence of assumptions A2 and A3.

Proposition 3.1: Suppose A1 holds. If global agreement is achieved for (1), then $\mathcal{G}^p$ is quasi-strongly connected.

Proof: Suppose $\mathcal{G}^p$ is not quasi-strongly connected. Then there exist two distinct nodes u and w such that $\mathcal{V}_u \cap \mathcal{V}_w = \emptyset$, where $\mathcal{V}_u = \{\text{nodes from which } u \text{ is reachable in } \mathcal{G}^p\}$ and $\mathcal{V}_w = \{\text{nodes from which } w \text{ is reachable in } \mathcal{G}^p\}$. Moreover, there is no arc entering either $\mathcal{V}_u$ or $\mathcal{V}_w$ in the persistent graph $\mathcal{G}^p$. Let $x_i(t_0) = 0$ for all $i \in \mathcal{V}_u$, and $x_i(t_0) = 1$ for all $i \in \mathcal{V} \setminus \mathcal{V}_u$. Denote $\ell(t) = \max_{i\in\mathcal{V}_u} x_i(t)$ and $\hbar(t) = \min_{i\in\mathcal{V}_w} x_i(t)$. We define $g^+(t;m) = \sum_{j\in\mathcal{N}_m, j\notin\mathcal{V}_u} W_{mj}(t)$ for $m \in \mathcal{V}_u$ and $f^+(t;k) = \sum_{j\in\mathcal{N}_k, j\notin\mathcal{V}_w} W_{kj}(t)$ for $k \in \mathcal{V}_w$. We further denote

$$\zeta_u^+(t) = \sum_{m\in\mathcal{V}_u} g^+(t;m); \quad \zeta_w^+(t) = \sum_{k\in\mathcal{V}_w} f^+(t;k).$$

It is straightforward to see that $\psi(t)$ is non-decreasing and $\Psi(t)$ is non-increasing for (1). It follows that $x_i(t) \in [0,1]$ for all i and $t \ge t_0$. There are two cases.

(i). First, for any $m \in \mathcal{V}_u$, we have

$$\begin{aligned} x_m(t_0+1) &= \sum_{j\in\mathcal{N}_m} W_{mj}(t_0)x_j(t_0) \\ &\le 0\cdot\big(1-g^+(t_0;m)\big) + 1\cdot g^+(t_0;m) \\ &\le \zeta_u^+(t_0), \end{aligned}$$

which yields $\ell(t_0+1) \le \zeta_u^+(t_0)$ immediately. Then, for the next slot we have that for any $m \in \mathcal{V}_u$,

$$\begin{aligned} x_m(t_0+2) &= \sum_{j\in\mathcal{N}_m} W_{mj}(t_0+1)x_j(t_0+1) \\ &\le \sum_{j\in\mathcal{N}_m, j\in\mathcal{V}_u} W_{mj}(t_0+1)\ell(t_0+1) \\ &\quad + \sum_{j\in\mathcal{N}_m, j\notin\mathcal{V}_u} W_{mj}(t_0+1)\cdot 1 \\ &= \zeta_u^+(t_0)\cdot\big(1-g^+(t_0+1;m)\big) \\ &\quad + g^+(t_0+1;m) \\ &\le \zeta_u^+(t_0) + \zeta_u^+(t_0+1), \end{aligned} \tag{7}$$

which leads to $\ell(t_0+1) \le \zeta_u^+(t_0) + \zeta_u^+(t_0+1)$. Continuing we get that for any $s = 1, 2, \dots,$ we have

$$\ell(t_0+s) \le \sum_{t=t_0}^{t_0+s-1} \zeta_u^+(t) \le \sum_{t=t_0}^{\infty} \theta(t) < \infty \tag{8}$$

because there is no arc entering $\mathcal{V}_u$ in the persistent graph $\mathcal{G}^p$.

(ii). Consider now $\mathcal{V}_w$. According to the definition of $\theta(t)$, there exists $T_1 > 0$ such that when $\theta(t) < 1, t \ge T_1$. Let $t_0 \ge T_1$. Then we have $\zeta_w^+(t) \le \theta(t) < 1$ for all $t \ge t_0$ since there is no arc entering $\mathcal{V}_w$ in the persistent graph $\mathcal{G}^p$.
Similarly we obtain $\hbar(t_0+1) \ge 1-\zeta_w^+(t_0)$ since for any $k \in \mathcal{V}_w$, we have

$$\begin{aligned} x_k(t_0+1) &= \sum_{j\in\mathcal{N}_k} W_{kj}(t_0)x_j(t_0) \\ &\ge 0\cdot f^+(t_0;k) + 1\cdot\big(1-f^+(t_0;k)\big) \\ &\ge 1-\zeta_w^+(t_0). \end{aligned}$$

Furthermore, for any $k \in \mathcal{V}_w$, one has

$$\begin{aligned} x_k(t_0+2) &\ge 0\cdot f^+(t_0+1;k) \\ &\quad + \big(1-f^+(t_0;k)\big)\cdot\big(1-\zeta_w^+(t_0)\big) \\ &\ge \big(1-\zeta_w^+(t_0+1)\big)\cdot\big(1-\zeta_w^+(t_0)\big), \end{aligned} \tag{9}$$

and thus $\hbar(t_0+2) \ge \big(1-\zeta_w^+(t_0+1)\big)\cdot\big(1-\zeta_w^+(t_0)\big)$. Proceeding the analysis we know that for any $s = 1, 2, \dots,$

$$\begin{aligned}\hbar(t_0+s) &\geq \prod_{t=t_0}^{t_0+s-1}\big(1-\zeta_w^+(t)\big)\\ &\geq \prod_{t=t_0}^{\infty}\big(1-\theta(t)\big)\\ &\geq \prod_{t=T_1}^{\infty}\big(1-\theta(t)\big)\\ &\doteq \sigma_* > 0, \qquad (10)\end{aligned}$$

where σ_* exists from Lemma 3.1 and the definition of $\theta(t)$.

Because $\sum_{j=0}^{\infty}\theta(t) < \infty$, we can always choose t_0 sufficiently large so that $\sum_{j=t_0}^{\infty}\theta(t) \leq \sigma_*/2$. Therefore, (8) and (10) lead to $\mathcal{H}(t_0+s) \geq \hbar(t_0+s) - \ell(t_0+s) \geq \sigma_*/2 > 0$. A global agreement is thus impossible. This completes the proof. □

We establish a lemma on the upper and lower bounds for some particular nodes.

Lemma 3.3: Suppose A1 holds. Let $x_m(t) = \mu\psi(t) + (1-\mu)\Psi(t)$ with $0 \leq \mu \leq 1$. Then for any integer $T > 0$, we have:

$$\begin{aligned}x_m(t+T) \leq{}& \mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\cdot\psi(t)\\ &+\Big(1-\mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\Big)\cdot\Psi(t). \qquad (11)\end{aligned}$$

and

$$\begin{aligned}x_m(t+T) \geq{}& \mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\cdot\Psi(t)\\ &+\Big(1-\mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\Big)\cdot\psi(t). \qquad (12)\end{aligned}$$

Proof. When $x_m(t) = \mu\psi(t) + (1-\mu)\Psi(t)$, for time $t+1$, we have

$$\begin{aligned}x_m(t+1) &= \sum_{j\in\mathcal{N}_m} W_{mj}(t)x_j(t)\\ &\leq \big(1-\xi^+(t;m)\big)\cdot\big(\mu\psi(t)+(1-\mu)\Psi(t)\big)\\ &\quad + \xi^+(t;m)\Psi(t)\\ &= \mu\big(1-\xi^+(t;m)\big)\cdot\psi(t)\\ &\quad + \Big(1-\mu\big(1-\xi^+(t;m)\big)\Big)\Psi(t). \qquad (13)\end{aligned}$$

For time $t+2$, we obtain

$$\begin{aligned}x_m(t+2) &\leq \big(1-\xi^+(t+1;m)\big)\cdot\Big[\mu\big(1-\xi^+(t;m)\big)\cdot\psi(t)\\ &\quad + \Big(1-\mu\big(1-\xi^+(t;m)\big)\Big)\Psi(t)\Big]\\ &\quad + \xi^+(t+1;m)\Psi(t)\\ &= \mu\prod_{s=t}^{t+1}\big(1-\xi^+(s;m)\big)\cdot\psi(t)\\ &\quad + \Big(1-\mu\prod_{s=t}^{t+1}\big(1-\xi^+(s;m)\big)\Big)\cdot\Psi(t). \qquad (14)\end{aligned}$$

Continuing, we obtain (11).

In equality (12) can be easily obtained using a symmetric analysis as for (11). □

We are now in a place to present the following conclusion, which shows the necessity of condition (b) in Theorem 3.1.

Proposition 3.2: Suppose A1 and A3 hold. If global ϵ-agreement is achieved for (1), then there exist a constant $a_* > 0$ and an integer $T_* > 0$ such that $\sum_{s=t}^{t+T_*} W_{ij}(s) \geq a_*$ for all $t \geq 0$ and $(j,i) \in \mathcal{G}^p$.

Proof: We prove the conclusion by contradiction. Suppose the condition does not hold. Then there exists a persistent arc $(j_*, i_*) \in \mathcal{E}^p$ such that for any real number $\delta > 0$ and integer $T > 0$, there exists an integer $t_*(T,\delta)$ satisfying

$$\sum_{s=t_*}^{t_*+T-1} W_{i_*j_*}(s) < \delta \qquad (15)$$

Since $(j_*, i_*) \in \mathcal{G}^p$, it is straightforward to see that $t_*(T,\epsilon) \to \infty$ as $T \to \infty$ for any fixed ϵ. Thus, we can assume that (15) also holds for the arcs in $\mathcal{E}_* \setminus \mathcal{E}^p$. Moreover, without loss of generality, we can also assume that $\xi^+(s;i) \leq 1/2$ for all i and $t_* \leq s \leq t_* + T - 1$. With arc balance assumption A3 and Lemma 3.2, for any $0 < \epsilon < 1$ we take $\delta = \frac{1}{2}A^{-1}(n-1)^{-1}\cdot\log\big(\frac{1+\epsilon}{2}\big)^{-1}$, and then (15) implies

$$\begin{aligned}\prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;i)\big) &= e^{\sum_{s=t_*}^{t_*+T-1}\log\big(1-\xi^+(s;i)\big)}\\ &\geq e^{-2\sum_{s=t_*}^{t_*+T-1}\xi^+(s;i)}\\ &\geq e^{-2A(n-1)\delta}\\ &= e^{-\log\big(\frac{1+\epsilon}{2}\big)^{-1}}\\ &= \frac{1+\epsilon}{2} \qquad (16)\end{aligned}$$

for all $i \in \mathcal{V}$.

Moreover, taking $x_m(t_*) = \psi(t_*)$ and $x_k(t_*) = \Psi(t_*)$, we know from Lemma 3.3 that

$$\begin{aligned}x_m(t_*+T) \leq{}& \prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;m)\big)\cdot\psi(t_*)\\ &+\Big(1-\prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;m)\big)\Big)\cdot\Psi(t_*) \qquad (17)\end{aligned}$$

and

$$\begin{aligned}x_k(t_*+T) \geq{}& \prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;k)\big)\cdot\Psi(t_*)\\ &+\Big(1-\prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;k)\big)\Big)\cdot\psi(t_*). \qquad (18)\end{aligned}$$

Therefore, with (16), (17) and (18), we eventually obtain

$$\begin{aligned}\mathcal{H}(t_*+T) &\geq x_k(t_*+T) - x_m(t_*+T)\\ &\geq \Big[\prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;k)\big)\\ &\quad + \prod_{s=t_*}^{t_*+T-1}\big(1-\xi^+(s;m)\big) - 1\Big]\cdot\mathcal{H}(t_*)\\ &> \big(2\cdot\frac{1+\epsilon}{2} - 1\big)\mathcal{H}(t_*)\\ &= \epsilon\mathcal{H}(t_*). \qquad (19)\end{aligned}$$

The desired conclusion thus follows. □

The necessity claim in Theorem 3.1 follows from Propositions 3.1 and 3.2.

B. Sufficiency

We now present the sufficiency proof of Theorem 3.1. In fact, we are going to prove a stronger statement which does not rely on the arc balance assumption A3.

Proposition 3.3: Suppose A1 and A2 hold. Global ϵ-agreement is achieved for (1) if $\mathcal{G}^p$ is quasi-strongly connected and there exist a constant $a_*>0$ and an integer $T_*>0$ such that $\sum_{s=t}^{t+T_*-1} W_{ij}(s) \geq a_*$ for all $t\geq 0$ and $(j,i)\in\mathcal{G}^p$.

Proof: Let $i_0\in\mathcal{V}$ be a center of $\mathcal{G}^p$. Take $t_0\geq 0$. Assume first that

$$x_{i_0}(t_0)\leq \frac{1}{2}\psi(t_0)+\frac{1}{2}\Psi(t_0). \tag{20}$$

Then from Lemma 3.3, one has

$$\begin{aligned} x_{i_0}(t_0+T) &\leq \frac{1}{2}\prod_{s=t_0}^{t_0+T-1}\big(1-\xi^+(s;i_0)\big)\cdot\psi(t_0)\\ &\quad+\Big(1-\frac{1}{2}\prod_{s=t_0}^{t_0+T-1}\big(1-\xi^+(s;i_0)\big)\Big)\cdot\Psi(t_0)\\ &\leq \frac{\eta^T}{2}\psi(t_0)+\Big(1-\frac{\eta^T}{2}\Big)\Psi(t_0) \end{aligned}\tag{21}$$

for all $T=0,1,\dots$.

Denote $\mathcal{V}_1$ as the node set consisting of all the nodes of which i_0 is a neighbor in $\mathcal{G}^p$, i.e., $\mathcal{V}_1=\{j:(i_0,j)\in\mathcal{E}^p\}$. Note that $\mathcal{V}_1$ is nonempty because i_0 is a center. For any $i_1\in\mathcal{V}_1$, there exists an instance $\bar{t}_1\in[t_0,t_0+T_*-1]$ such that $W_{i_1i_0}(\bar{t}_1)\geq a_*/T_*$ because $\sum_{t=t_0}^{t_0+T_*-1}W_{i_1i_0}(t)\geq a_*$. Suppose $\bar{t}_1=t_0+\varrho_1$ with $\varrho_1\in[0,T_*-1]$. Then with (21), we have

$$\begin{aligned} x_{i_1}(\bar{t}_1+1) &= x_{i_1}(t_0+\varrho_1+1)\\ &\leq W_{i_1i_0}(t_0+\varrho_1)x_{i_0}(t_0+\varrho_1)\\ &\quad+\big(1-W_{i_1i_0}(t_0+\varrho_1)\big)\Psi(t_0)\\ &\leq \frac{a_*}{T_*}\cdot\Big[\frac{\eta^{\varrho_1}}{2}\psi(t_0)+\big(1-\frac{\eta^{\varrho_1}}{2}\big)\Psi(t_0)\Big]\\ &\quad+\Big(1-\frac{a_*}{T_*}\Big)\Psi(t_0)\\ &= \eta^{\varrho_1}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0)+\Big(1-\eta^{\varrho_1}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0). \end{aligned}\tag{22}$$

Based on Lemma 3.3, we can further conclude

$$\begin{aligned} x_{i_1}(t_0+\varrho_1+T) &\leq \eta^{\varrho_1+T-1}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0)\\ &\quad+\Big(1-\eta^{\varrho_1+T-1}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0) \end{aligned}\tag{23}$$

for all $T=1,2,\dots$, which implies

$$\begin{aligned} x_{i_1}(t_0+T_*+K) &\leq \eta^{T_*+K}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0)\\ &\quad+\Big(1-\eta^{T_*+K}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0) \end{aligned}\tag{24}$$

for all $K=0,1,\dots$.

Next, since $\mathcal{G}^p$ is quasi-strongly connected, we can denote $\mathcal{V}_2$ as the node set consisting of all the nodes each of which has a neighbor in $\{i_0\}\cup\mathcal{V}_1$ within $\mathcal{G}^p$. For any $i_2\in\mathcal{V}_2$, there exist a node $i_*\in\{i_0\}\cup\mathcal{V}_1$ and an instance $\bar{t}_2=t_0+T_*+\varrho_2$ with $\varrho_2\in[0,T_*-1]$ such that $W_{i_2i_*}(\bar{t}_1)\geq a_*/T_*$. Similarly we have

$$\begin{aligned} x_{i_2}(\bar{t}_2+1) &\leq W_{i_2i_*}(t_0+T_*+\varrho_2)x_{i_*}(t_0+T_*+\varrho_2)\\ &\quad+\big(1-W_{i_2i_*}(t_0+T_*+\varrho_2)\big)\Psi(t_0)\\ &\leq \frac{a_*}{T_*}\cdot\Big[\eta^{T_*+\varrho_2}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0)\\ &\quad+\Big(1-\eta^{T_*+\varrho_2}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0)\Big]+\Big(1-\frac{a_*}{T_*}\Big)\Psi(t_0)\\ &= \frac{\eta^{T_*+\varrho_2}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\cdot\psi(t_0)\\ &\quad+\Big(1-\frac{\eta^{T_*+\varrho_2}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\Big)\Psi(t_0), \end{aligned}\tag{25}$$

and therefore

$$\begin{aligned} x_{i_2}(t_0+2T_*+K) &\leq \frac{\eta^{2T_*+K}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\psi(t_0)\\ &\quad+\Big(1-\frac{\eta^{2T_*+K}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\Big)\Psi(t_0) \end{aligned}$$

for all $K=0,1,\dots$.

Proceeding the estimate, $\mathcal{V}_3,\dots,\mathcal{V}_k$ can be similarly defined until $\big(\cup_{i=1}^k\mathcal{V}_i\big)\cup\{i_0\}=\mathcal{V}$. Moreover, it is not hard to see that i_0 can be selected so that $k=d_0$, where d_0 is the diameter of $\mathcal{G}^p$, and thus

$$\begin{aligned} x_i(t_0+d_0T_*) &\leq \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\cdot\psi(t_0)\\ &\quad+\Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\Psi(t_0),\quad i=1,\dots,n \end{aligned}\tag{26}$$

which yields

$$\begin{aligned} \Psi(t_0+d_0T_*) &\leq \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\cdot\psi(t_0)\\ &\quad+\Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\Psi(t_0). \end{aligned}\tag{27}$$

With (27), we eventually have

$$\begin{aligned} \mathcal{H}(t_0+d_0T_*) &\leq \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\cdot\psi(t_0)\\ &\quad+\Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\Psi(t_0)-\psi(t_0)\\ &= \Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\mathcal{H}(t_0). \end{aligned}\tag{28}$$

For the opposite case of (20) with

$$x_{i_0}(t_0)>\frac{1}{2}\psi(t_0)+\frac{1}{2}\Psi(t_0), \tag{29}$$

(28) is obtained using a symmetric argument by bounding $\psi(t_0+d_0T_*)$ from below.

Therefore, the desired conclusion follows with $\epsilon=1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2$ and $T_0=d_0T_*$ since (28) holds independent with the choice of t_0. □

IV. Continuous-time Belief Evolution

In this section, we turn to the continuous-time updating rule. We need an assumption on the continuity of each weight function $W_{ij}(t)$ for the existence of trajectories of (2).

A4 *(Continuity)* Each $W_{ij}(t),\ (j,i)\in\mathcal{E}_*$ is continuous except for a set with measure zero.

With assumption A4, the set of discontinuity points for the right-hand side of equation (2) has measure zero. Therefore, the Caratheodory solutions of (2) exist for arbitrary initial conditions, and they are absolutely continuous functions that satisfy (2) for almost all t on the maximum interval of existence [3], [54]. In the following, each solution of (2) is considered in the sense of Caratheodory without explicit mention.

Let us first study the feasibility of the solutions of (2). Consider (2) with initial condition $x(t_0)=(x_1(t_0),\dots,x_n(t_0))^T=x^0\in\mathbb{R}^n, t_0\geq 0$.

The upper Dini derivative of a function $h:(a,b)\to\mathbb{R}$ at t is defined as

$$D^+h(t)=\limsup_{s\to 0^+}\frac{h(t+s)-h(t)}{s}$$

The next result is useful for the calculation of Dini derivatives [53], [33].

Lemma 4.1: Let $V_i(t,x):\mathbb{R}\times\mathbb{R}^m\to\mathbb{R},\ i=1,\dots,n$, be C^1 and $V(t,x)=\max_{i=1,\dots,n}V_i(t,x)$. If $\mathcal{I}(t)=\big\{i\in\{1,\dots,n\}:V(t,x(t))=V_i(t,x(t))\big\}$ is the set of indices where the maximum is reached at t, then $D^+V(t,x(t))=\max_{i\in\mathcal{I}(t)}\dot V_i(t,x(t))$.

The following lemma establishes the monotonicity of $\Psi(t)$ and $\psi(t)$.

Lemma 4.2: For all $t\geq t_0\geq 0$, we have $D^+\Psi(t)\leq 0$ and $D^+\psi(t)\geq 0$.

Proof. We prove $D^+\Psi(t)\leq 0$. The other part can be proved similarly.

Let $\mathcal{I}_0(t)$ represent the set containing all the agents that reach the maximum in the definition of $\Psi(t)$ at time t, i.e., $\mathcal{I}(t)=\{i\in\mathcal{V}|\ x_i(t)=\Psi(t)\}$. Then according to Lemma 4.1, we obtain

$$\begin{aligned}D^+\Psi(t)&=\max_{i\in\mathcal{I}_0(t)}\dot x_i(t)\\&=\max_{i\in\mathcal{I}_0(t)}\Big[\sum_{j\in N_i}W_{ij}(t)(x_j(t)-x_i(t))\Big]\\&\leq 0,\end{aligned}\tag{30}$$

which completes the proof. □

Lemma 4.2 implies, $\mathcal{H}(t)$ is non-increasing for all $t\geq t_0$, and therefore each (Caratheodory) trajectory of (2) is bounded within the initial states of the nodes. As a result, the trajectories exist in $[t_0,\infty)$ for any initial condition.

The main result on global consensus and ϵ-consensus is stated in the following two theorems.

Theorem 4.1: Suppose A3 and A4 hold. Global agreement is achieved for (2) if and only if $\mathcal{G}^p$ is quasi-strongly connected.

Theorem 4.2: Suppose A3 and A4 hold. Global ϵ-agreement is achieved for (2) if and only if

(a) $\mathcal{G}^p$ is quasi-strongly connected;

(b) there exists two constants $a_*,\ \tau_0>0$ such that $\int_t^{t+\tau_0}W_{ij}(s)ds\geq a_*$ for all $t\geq 0$ and $(j,i)\in\mathcal{G}^p$.

Moreover, if (a) and (b) hold, then we have

$$\mathcal{H}\Big(t+\tau_0\cdot\Big\lceil\frac{d_0\log 2}{a_*}\Big\rceil\Big)\leq\Big(1-\frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t),\tag{31}$$

where $m_0=\big(\frac{\omega_0}{2}\big)^2\frac{1}{(n-1)A}$ with $\omega_0=e^{-\int_0^\infty\theta(t)dt}$, d_0 is the diameter of $\mathcal{G}^p$, and $\lceil z\rceil$ represents the smallest integer which is no smaller than z.

Theorem 4.1 implies that the connectivity of the persistent graph $\mathcal{G}^p$ totally determines whether an agreement can be achieved globally. Furthermore, Theorem 4.2 implies that $\int_0^T W_{ij}(t)dt=O(T)$ is a critical condition to ensure a global ϵ-consensus.

Remark 4.1: Consensus for (2) was first studied in [21], where the convergence rate was shown to be determined by the second largest eigenvalue of the Laplacian of the communication graph. Further discussions can be found in [24], [33], [38].

Remark 4.2: Theorems 4.1 and 4.2 still hold if assumption A3 is replaced by the following integral version.

A5. (Integral Arc Balance) There exists a constant $A>1$ such that for any two arcs $(j,i),(m,k)\in\mathcal{E}^p$, we have

$$A^{-1}\int_a^b W_{ij}(t)dt\leq\int_a^b W_{km}(t)dt\leq A\int_a^b W_{ij}(t)dt$$

for all $0\leq a<b$.

Remark 4.3: If we have $\int_{t=t_0}^T W_{ij}(t)dt=\infty,(j,i)\in\mathcal{G}^p$ for some finite T, it follows from the proof of Theorem 4.1 below that (2) will reach a global agreement in finite time when t tends to T.

A. Preliminaries

In this subsection, we establish two lemmas which describe the boundaries of how much each individual arc affects the nodes' dynamics. Then the proof of Theorems 4.1 and 4.2 will be proposed in the next two subsections.

Lemma 4.3: Suppose $x_m(s)\leq\mu\psi(s)+(1-\mu)\Psi(s)$ for some $s\geq t_0$ and $m\in\mathcal{V}$ with $0\leq\mu\leq 1$ a giving constant. Then we have

$$x_m(t)\leq\mu e^{-\int_s^t\xi^+(\tau;m)d\tau}\psi(s)+\big[1-\mu e^{-\int_s^t\xi^+(\tau;m)d\tau}\big]\Psi(s)\tag{32}$$

for all $t\geq s$.

Proof: Based on Lemma 4.2, we see that

$$\begin{aligned}\dot x_m(t)&=\sum_{j\in\mathcal{N}_m}W_{mj}(t)\big[x_j(t)-x_m(t)\big]\\&\leq\sum_{j\in\mathcal{N}_m}W_{mj}(t)\big[\Psi(s)-x_m(t)\big]\\&=-\xi^+(t;m)\big[x_m(t)-\Psi(s)\big],\quad t\geq s.\end{aligned}\tag{33}$$

This implies

$$\begin{aligned}x_m(t)&\leq e^{-\int_s^t\xi^+(\tau;m)d\tau}x_m(s)+\big[1-e^{-\int_s^t\xi^+(\tau;m)d\tau}\big]\Psi(s)\\&\leq\mu e^{-\int_s^t\xi^+(\tau;m)d\tau}\psi(s)+\big[1-\mu e^{-\int_s^t\xi^+(\tau;m)d\tau}\big]\Psi(s)\end{aligned}\tag{34}$$

by Grönwall's inequality. The proof is completed. □

We give a lemma investigating the dynamic evolution between two connected nodes.

Lemma 4.4: Suppose $(l, m) \in \mathcal{E}_*$ and there exists a constant $0 < \mu < 1$ such that

$$x_l(t) \le \mu\psi(s_0) + (1-\mu)\Psi(s_0), \quad t \in [s_0, s].$$

Then we have

$$x_m(t) \le \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du \cdot \psi(s_0) + \Big[1 - \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du\Big]\Psi(s_0) \tag{35}$$

for all $t \in [s_0, s]$.

Proof: Similar to (33), for any $t \in [s_0, s]$, we have

$$\dot{x}_m(t) \le \big[\xi^+(t;m) - W_{ml}(t)\big] \cdot \big[\Psi(s_0) - x_m(t)\big] + W_{ml}(t)\big[\mu\psi(s_0) + (1-\mu)\Psi(s_0) - x_m(t)\big].$$

Therefore, noting the fact that

$$\int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau}\xi^+(u;m)du = \int_{s_0}^{t} \frac{d}{du}\Big[e^{-\int_u^t \xi^+(\tau;m)d\tau}\Big] = 1 - e^{-\int_{s_0}^t \xi^+(\tau;m)d\tau},$$

we obtain

$$\begin{aligned}
x_m(t) &\le e^{-\int_{s_0}^t \xi^+(\tau;m)d\tau} x_m(s_0) \\
&\quad + \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau}\big[\xi^+(u;m) - W_{ml}(u)\big]du \cdot \Psi(s_0) \\
&\quad + \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du \\
&\quad \cdot \Big[\mu\psi(s_0) + (1-\mu)\Psi(s_0)\Big] \\
&\le e^{-\int_{s_0}^t \xi^+(\tau;m)d\tau}\Psi(s_0) \\
&\quad + \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau}\big[\xi^+(u;m) - W_{ml}(u)\big]du \cdot \Psi(s_0) \\
&\quad + \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du \\
&\quad \cdot \Big[\mu\psi(s_0) + (1-\mu)\Psi(s_0)\Big] \\
&= \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du \cdot \psi(s_0) \\
&\quad + \Big[1 - \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du\Big]\Psi(s_0),
\end{aligned} \tag{36}$$

for all $t \in [s_0, s]$ by Grönwall's inequality and some simple manipulations. This completes the proof. □

B. Proof of Theorem 4.1

Sufficiency

Let $i_0 \in \mathcal{V}$ be a center of $\mathcal{G}^p$. Assume first that

$$x_{i_0}(t_0) \le \frac{1}{2}\psi(t_0) + \frac{1}{2}\Psi(t_0). \tag{37}$$

Denote $\omega_0 = e^{-\int_0^\infty \theta(t)dt}$. Then we have $0 < \omega_0 \le 1$. Thus, based on Lemma 4.3 and noting the fact that $\psi(t_0) \le \Psi(t_0)$, we have

$$\begin{aligned}
x_{i_0}(t) &\le \frac{1}{2} e^{-\int_{t_0}^t \xi^+(\tau;i_0)d\tau}\psi(t_0) \\
&\quad + \big[1 - \frac{1}{2} e^{-\int_{t_0}^t \xi^+(\tau;i_0)d\tau}\big]\Psi(t_0) \\
&\le \frac{\omega_0}{2} e^{-\int_{t_0}^t \xi_0^+(\tau;i_0)d\tau}\psi(t_0) \\
&\quad + \Big[1 - \frac{\omega_0}{2} e^{-\int_{t_0}^t \xi_0^+(\tau;i_0)d\tau}\Big]\Psi(t_0).
\end{aligned}$$

Define

$$\hat{t}_1 = \inf\Big\{t \ge t_0 :\ e^{-\int_{t_0}^t \xi_0^+(\tau;i_0)d\tau} = \frac{1}{2}\Big\}. \tag{38}$$

We see that $\hat{t}_1$ is finite from the definition of $\mathcal{E}^p$. As a result, we obtain

$$x_{i_0}(t) \le \frac{\omega_0}{4}\psi(t_0) + \big[1 - \frac{\omega_0}{4}\big]\Psi(t_0), \quad t \in [t_0, \hat{t}_1]. \tag{39}$$

Next, we denote the node set consisting of all the nodes of which i_0 is a neighbor in $\mathcal{G}^p$ as $\mathcal{V}_1$, i.e., $\mathcal{V}_1 = \{j : (i_0, j) \in \mathcal{E}^p\}$. Note that $\mathcal{V}_1$ is nonempty because i_0 is a center. Then for any $i_1 \in \mathcal{V}_1$, we see from Lemma 4.4 that

$$\begin{aligned}
x_{i_1}(\hat{t}_1) &\le \frac{\omega_0}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \cdot \psi(t_0) \\
&\quad + \Big[1 - \frac{\omega_0}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du\Big]\Psi(s_0) \\
&\le \frac{\omega_0^2}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \cdot \psi(t_0) \\
&\quad + \Big[1 - \frac{\omega_0^2}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du\Big]\Psi(s_0).
\end{aligned} \tag{40}$$

The arc balance assumption A3 implies that

$$\int_u^{\hat{t}_1} \xi_0^+(t;i_1)dt \le \int_u^{\hat{t}_1} (n-1)AW_{i_1 i_0}(t)dt,$$

which yields

$$\begin{aligned}
&\int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \\
&\ge \int_{t_0}^{\hat{t}_1} e^{-(n-1)A\int_u^{\hat{t}_1} W_{i_1 i_0}(\tau)d\tau} W_{i_1 i_0}(u)du \\
&= \frac{1}{(n-1)A} \int_{t_0}^{\hat{t}_1} \frac{d}{du} e^{-(n-1)A\int_u^{\hat{t}_1} W_{i_1 i_0}(\tau)d\tau} \\
&= \frac{1}{(n-1)A} \cdot \Big[1 - e^{-(n-1)A\int_{t_0}^{\hat{t}_1} W_{i_1 i_0}(\tau)d\tau}\Big].
\end{aligned} \tag{41}$$

On the other hand, we also have

$$\int_{t_0}^{\hat{t}_1} \xi_0^+(t;i_0)dt \le \int_{t_0}^{\hat{t}_1} (n-1)AW_{i_1 i_0}(t)dt.$$

Thus, we know from (41) and the definition of $\hat{t}_1$ that

$$\begin{aligned}\int_{t_0}^{\hat{t}_1} & e^{-\int_u^{\hat{t}_1}\xi_0^+(\tau;i_1)d\tau}W_{i_1i_0}(u)du\\ &\geq \frac{1}{(n-1)A}\cdot\Big[1-e^{-(n-1)A\int_{t_0}^{\hat{t}_1}W_{i_1i_0}(\tau)d\tau}\Big]\\ &\geq \frac{1}{(n-1)A}\cdot\Big[1-e^{-\int_{t_0}^{\hat{t}_1}\xi_0^+(\tau;i_0)d\tau}\Big]\\ &=\frac{1}{2(n-1)A}. \end{aligned} \tag{42}$$

Equations (40) and (42) result in

$$x_{i_1}(\hat{t}_1)\leq \frac{m_0}{2}\psi(t_0)+(1-\frac{m_0}{2})\Psi(t_0) \tag{43}$$

for all $i_1\in\mathcal{V}_1$, where $m_0=\big(\frac{\omega_0}{2}\big)^2\frac{1}{(n-1)A}$.

We continue to estimate the upper bound of nodes in $\{i_0\}\cup\mathcal{V}_1$ when $t\geq\hat{t}_1$. Define

$$\mathcal{Y}(t)=\max_{i\in\{i_0\}\cup\mathcal{V}_1}x_i(t).$$

Then $\mathcal{Y}(\hat{t}_1)\leq\frac{m_0}{2}\psi(t_0)+(1-\frac{m_0}{2})\Psi(t_0)$. Similar to Lemma 4.3, we find that

$$D^+\mathcal{Y}(t)\leq-\beta(t)[\mathcal{Y}(t)-\Psi(\hat{t}_1)],\quad t\geq\hat{t}_1,$$

where $\beta(t)=\sum_{i\in\{i_0\}\cup\mathcal{V}_1,j\notin\{i_0\}\cup\mathcal{V}_1}W_{ij}(t)$. This implies

$$\begin{aligned}\mathcal{Y}(t)&\leq e^{-\int_{\hat{t}_1}^t\beta(\tau)d\tau}\mathcal{Y}(\hat{t}_1)+\Big[1-e^{-\int_{\hat{t}_1}^t\beta(\tau)d\tau}\Big]\Psi(\hat{t}_1)\\ &\leq e^{-\int_{\hat{t}_1}^t\beta(\tau)d\tau}\Big[\frac{m_0}{2}\psi(t_0)\\ &\quad+(1-\frac{m_0}{2})\Psi(t_0)\Big]+\Big[1-e^{-\int_{\hat{t}_1}^t\beta(\tau)d\tau}\Big]\Psi(t_0)\\ &\leq\frac{m_0}{2}\cdot\omega_0e^{-\int_{\hat{t}_1}^t\hat{\beta}(\tau)d\tau}\psi(t_0)\\ &\quad+\Big[1-\frac{m_0}{2}\cdot\omega_0e^{-\int_{\hat{t}_1}^t\hat{\beta}(\tau)d\tau}\Big]\Psi(t_0), \end{aligned}\tag{44}$$

where $\hat{\beta}(t)=\sum_{i\in\{i_0\}\cup\mathcal{V}_1,j\notin\{i_0\}\cup\mathcal{V}_1,(j,i)\in\mathcal{E}^p}W_{ij}(t)$. We can then define

$$\mathcal{V}_2=\Big\{j\notin\{i_0\}\cup\mathcal{V}_1:\exists i\in\{i_0\}\cup\mathcal{V}_1\text{ s.t. }(i,j)\in\mathcal{E}^p\Big\}$$

and

$$\hat{t}_2=\inf\Big\{t\geq\hat{t}_1:\ e^{-\int_{\hat{t}_1}^t\hat{\beta}(\tau)d\tau}=\frac{1}{2}\Big\}$$

and similar analysis with (43) gives a bound to any node $i_2\in\mathcal{V}_2$ as

$$x_{i_2}(\hat{t}_2)\leq\frac{m_0^2}{2}\psi(t_0)+\big(1-\frac{m_0^2}{2}\big)\Psi(t_0). \tag{45}$$

Moreover, (45) also holds for nodes in $\{i_0\}\cup\mathcal{V}_1$.

Since $\mathcal{G}^p$ has a center, we can proceed the estimation to nodes in $\mathcal{V}_2,\dots,\mathcal{V}_k$ until $\big(\cup_{j=1}^k\mathcal{V}_j\big)\cup\{i_0\}=\mathcal{V}$ with $\hat{t}_2,\dots,\hat{t}_k$ such that

$$x_i(\hat{t}_k)\leq\frac{m_0^k}{2}\psi(t_0)+(1-\frac{m_0^k}{2})\Psi(t_0) \tag{46}$$

for all $i\in\mathcal{V}$, which leads to

$$\Psi(\hat{t}_k)\leq\frac{m_0^k}{2}\psi(t_0)+(1-\frac{m_0^k}{2})\Psi(t_0). \tag{47}$$

We see that i_0 can be chosen so that $k\leq d_0$ always holds, where d_0 is the diameter of $\mathcal{G}^p$. Denoting $t_1=\hat{t}_k$, we eventually arrive at

$$\begin{aligned}\mathcal{H}(t_1)&=\Psi(t_1)-\psi(t_1)\\ &\leq\frac{m_0^{d_0}}{2}\psi(t_0)+\big(1-\frac{m_0^{d_0}}{2}\big)\Psi(t_0)-\psi(t_0)\\ &=\Big(1-\frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t_0). \end{aligned}\tag{48}$$

Although the analysis up to now is based on assumption (37), we see that (48) also holds for the other case with $x_{i_0}(t_0)>\frac{1}{2}\psi(t_0)+\frac{1}{2}\Psi(t_0)$ using a symmetric argument by investigating the lower bound of $\psi(t_1)$.

Similar estimate can be carried out for $t_k,k=2,3,\dots$, which leads to

$$\mathcal{H}(t_{k+1})\leq\Big(1-\frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t_k) \tag{49}$$

for all $t_k,k=1,2,\dots$, which yields

$$\mathcal{H}(t_k)\leq\Big(1-\frac{m_0^{d_0}}{2}\Big)^k\mathcal{H}(t_0). \tag{50}$$

Therefore, we can now conclude that $\lim_{t\to\infty}\mathcal{H}(t)=0$ because $\mathcal{H}(t)$ is non-increasing and $0<m_0<1$. The sufficiency statement of Theorem 4.1 is thus proved.

Necessity

We follow the same line as the proof of Proposition 3.1. Suppose $\mathcal{G}^p$ is not quasi-strongly connected. Let $\mathcal{V}_u$, $\mathcal{V}_w$, $\ell(t)$ and $\hbar(t)$ follow the definitions in the proof of Proposition 3.1. Also let $x_i(t_0)=0$ for all $i\in\mathcal{V}_u$, and $x_i(t_0)=1$ for all $i\in\mathcal{V}\setminus\mathcal{V}_u$. According to Lemma 4.2, we have $x_i(t)\in[0,1]$.

Based on Lemma 4.1, we have

$$\begin{aligned}D^+\ell(t)&=\max_{i\in\mathcal{I}_1(t)}\Big[\sum_{j\in\mathcal{N}_i}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &\leq\max_{i\in\mathcal{I}_1(t)}\Big[\sum_{j\in\mathcal{N}_i\setminus\mathcal{V}_u}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &=\max_{i\in\mathcal{I}_1(t)}\Big[\sum_{j\in\mathcal{N}_i,\ (j,i)\in\mathcal{E}_*\setminus\mathcal{E}^p}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &\leq\theta(t)\cdot\big(1-\ell(t)\big) \end{aligned}\tag{51}$$

where $\mathcal{I}_1(t)$ is the index set that contains the nodes where the maximum is reached and $\theta(t)$ is defined in (6).

Similarly we have

$$\begin{aligned}D^+\hbar(t)&=\min_{i\in\mathcal{I}_2(t)}\Big[\sum_{j\in\mathcal{N}_i}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &\geq\min_{i\in\mathcal{I}_2(t)}\Big[\sum_{j\in\mathcal{N}_i\setminus\mathcal{V}_w}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &=\min_{i\in\mathcal{I}_2(t)}\Big[\sum_{j\in\mathcal{N}_i,\ (j,i)\in\mathcal{E}_*\setminus\mathcal{E}^p}W_{ij}(t)\big(x_j(t)-x_i(t)\big)\Big]\\ &\geq-\theta(t)\cdot\hbar(t) \end{aligned}\tag{52}$$

where $\mathcal{I}_2(t)$ is the index set that contains the nodes where the minimum is reached.

With (51) and (52), denoting $L(t)=\hbar(t)-\ell(t)$, we obtain

$$D^+L(t)\geq-\theta(t)\cdot(\hbar(t)-\ell(t)+1)=-\theta(t)\cdot(L(t)+1),$$

which is equivalent to

$$D^+\Big[e^{\int_{t_0}^t \theta(\tau)d\tau}(L(t)+1)\Big] \geq 0. \tag{53}$$

Therefore, we have

$$L(t) \geq 2e^{-\int_{t_0}^t \theta(\tau)d\tau} - 1. \tag{54}$$

Since $\int_0^\infty \theta(t)dt < \infty$, we can choose t_0 sufficiently large to ensure $e^{-\int_{t_0}^t \theta(\tau)d\tau} \geq \frac{2}{3}$ for all $t \geq t_0$. This leads to $\mathcal{H}(t) \geq L(t) \geq 1/3,\ t \geq t_0$. The necessity part of Theorem 4.1 thus follows.

C. Proof of Theorem 4.2

We first prove the necessity statement. Based on Theorem 4.1, we only need to prove that condition (b) in Theorem 4.2 is necessary. Suppose (b) in Theorem 4.2 does not hold. Then similar to the proof of Proposition 3.2, we have that $\forall 0 < \epsilon < 1, T > 0, \exists t_*(T,\epsilon) \geq 0$ and $(j_0, i_0) \in \mathcal{E}^p$ such that

$$\int_{t_*}^{t_*+T} W_{i_0 j_0}(\tau)d\tau < \frac{1}{2}A^{-1}|\mathcal{E}_*|^{-1}\log\epsilon^{-1}, \tag{55}$$

where $|\mathcal{E}_*|$ represents the number of elements in $\mathcal{E}_*$. Assumption A3 further implies

$$\int_{t_*}^{t_*+T} W_{ij}(\tau)d\tau < \frac{1}{2}|\mathcal{E}_*|^{-1}\log\epsilon^{-1} \tag{56}$$

for all $(j,i) \in \mathcal{E}^p$. It is straightforward to see that $t_*(T,\epsilon) \to \infty$ as $T \to \infty$ for any fixed ϵ according to the definition of persistent arcs. Thus, without loss of generality, we can assume that the upper bound in (55) also holds for the arcs in $\mathcal{E}_* \setminus \mathcal{E}^p$.

From similar argument we used to obtain (53),

$$D^+\mathcal{H}(t) \geq -2\Big[\sum_{(j,i)\in\mathcal{E}_*} W_{ij}(t)\Big]\mathcal{H}(t), \quad t \geq t_0. \tag{57}$$

Therefore, letting the system initial time be $t_0 = t_*$ with $\mathcal{H}(t_*) > 0$, where t_* is defined in (55), we see from (55) and (56) that

$$\begin{aligned} 2\sum_{(j,i)\in\mathcal{E}_*}\int_{t_*}^{t_*+T} W_{ij}(\tau)d\tau &< 2|\mathcal{E}_*|\cdot\big(\frac{1}{2}|\mathcal{E}_*|^{-1}\log\epsilon^{-1}\big) \\ &= \log\epsilon^{-1}. \end{aligned} \tag{58}$$

Consequently, (57) and (58) lead to

$$\mathcal{H}(t_*+T) \geq e^{-2\sum_{(j,i)\in\mathcal{E}_*}\int_{t_*}^{t_*+T} W_{ij}(\tau)d\tau}\mathcal{H}(t_*) > \epsilon\mathcal{H}(t_*). \tag{59}$$

Then the necessity part of Theorem 4.2 holds because ϵ and T are arbitrarily chosen in (59).

Next, we prove the sufficiency part of Theorem 4.2 based on the convergence analysis in the proof of Theorem 4.1.

When there exist two constants $a_*,\ \tau_0 > 0$ such that $\int_t^{t+\tau_0} W_{ij}(\tau)d\tau \geq a_*$ for all $t \geq 0$ and $(j,i) \in \mathcal{G}^p$, we have

$$\int_t^{t+\tau_0} b_0(\tau)d\tau \geq a_*,\ t \geq 0, \tag{60}$$

where $b_0(t) = \min_{(j,i)\in\mathcal{G}^p} W_{ij}(t)$.

Let us revisit the proof of Theorem 4.1. The definition of $\hat{t}_1$ in (38) satisfies

$$\begin{aligned} \hat{t}_1 &= \inf\Big\{t \geq t_0:\ e^{-\int_{t_0}^t \xi_0^+(\tau;i_0)d\tau} = \frac{1}{2}\Big\} \\ &\leq \inf\Big\{t \geq t_0:\ \ e^{-\int_{t_0}^t b_0(\tau)d\tau} = \frac{1}{2}\Big\}. \end{aligned} \tag{61}$$

Similarly, for $\hat{t}_j, j = 2,\dots,k$ with $k \leq d_0$, we have

$$\hat{t}_j \leq \inf\Big\{t \geq \hat{t}_j:\ \ e^{-\int_{\hat{t}_{j-1}}^t b_0(\tau)d\tau} = \frac{1}{2}\Big\}. \tag{62}$$

Thus, for $t_1 = \hat{t}_k$ in (48), it holds that

$$\begin{aligned} t_1 &\leq \inf\Big\{t \geq t_0:\ \ e^{-\int_{t_0}^t b_0(\tau)d\tau} = \Big(\frac{1}{2}\Big)^{d_0}\Big\} \\ &= \inf\Big\{t \geq t_0:\ \ \int_{t_0}^t b_0(\tau)d\tau = d_0\log 2\Big\}. \end{aligned} \tag{63}$$

Based on (60), we have

$$\Big\lfloor\frac{t-t_0}{\tau_0}\Big\rfloor a_* \leq \int_{t_0}^t b_0(\tau)d\tau,$$

where $\lfloor z \rfloor$ represents the largest integer which is no larger than z. This immediately implies

$$t_1 \leq t_0 + \tau_0\cdot\Big\lceil\frac{d_0\log 2}{a_*}\Big\rceil, \tag{64}$$

where $\lceil z \rceil$ represents the smallest integer which is no smaller than z.

Therefore, it can be concluded from Lemma 4.2 and (48) that

$$\mathcal{H}\Big(t_0 + \tau_0\cdot\Big\lceil\frac{d_0\log 2}{a_*}\Big\rceil\Big) \leq \Big(1 - \frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t_0). \tag{65}$$

The desired conclusion follows since (65) holds independent with the choice of t_0. Thus, we have now completed the proof of Theorem 4.2.

V. Discussions

In this section, we present some comparisons between our results with existing work, and comparisons between the discrete-time and continuous-time belief evolutions.

A. Relation to Cut-balanced Graphs

In [46], a cut-balance condition is introduced in the sense that there exists a constant $K \geq 1$ such that for all t and any nonempty subset $S \subseteq \mathcal{V}$, it holds that

$$K^{-1}\sum_{i\in S, j\notin S} W_{ji}(t) \leq \sum_{i\in S, j\notin S} W_{ij}(t) \leq K\sum_{i\in S, j\notin S} W_{ji}(t). \tag{66}$$

If the persistent graph $\mathcal{G}^p$ is strongly connected, the arc balance assumption A3 implies condition (66) over $\mathcal{G}^p$. Therefore, in this particular case, assumption A3 is a special case of the cut-balance condition in [46], though assumption (66) in [46] is over the underlying graph $\mathcal{G}_*$. Except for this slight difference, the convergence statements in Theorem 3.1 and Theorem 4.1 are consistent with the results given in [46] for strongly connected graphs.

On the other hand, when $\mathcal{G}^p$ is quasi-strongly connected, the cut-balance condition never holds even under assumption

A3, because there may be no arc pointing to the center node. Hence, in general, the results given in this paper provides conditions for node agreement independent of the conditions in [46].

B. Discrete-time vs. Continuous-time

Theorems 3.1 and 4.2 share quite similar structure and statement. However, there are some fundamental differences between them.

(a) The discrete-time result in Theorem 3.1 highly relies on the self-confidence condition A2. Without A2, oscillations among the nodes may become inevitable and periodic solutions of (1) may arise for almost all initial condition even under A1 and A3. Note that the arc balance condition A3 is only useful for the necessity part of Theorem 3.1.

(b) For the continuous-time result in Theorem 4.2, each self weight $W_{ii}(t)$ does not even show up in the model (2). The arc balance condition A3 is essential for the dynamics. Without A3, oscillations may occur if the arc weights of the persistent graph alternatively become large.

Therefore, we can conclude that under directed communication graphs, the self-confidence condition is critical for discrete-time belief agreement, as is the arc balance condition for continuous-time case. In fact, for agreement seeking over directed graphs, certain self-confidence or arc balance conditions are essentially inevitable (cf. [46], [48]). One may also assume bidirectional communication or bounded arc weights in order for obtaining desired agreement, but these conditions are indeed equivalent, or related to certain self-confidence and arc balance conditions.

An interesting question is whether a similar conclusion can be made for the discrete-time model (1) as the statement in Theorem 4.1. This question is open for general directed graphs and needs additional explorations. More discussions on this problem can be found in [47] and [48] on the ergodicity of random and deterministic stochastic chains, where the definition of infinite flow corresponds to the persistent graph considered in this paper.

VI. Conclusions

Individuals are equipped with beliefs in social activities. The evolution of the beliefs can be modeled as dynamical systems over graphs using for instance the widely studied consensus algorithms. This paper studied persistent graphs under discrete-time and continuous-time consensus algorithms. Sufficient and necessary conditions were established on the persistent graph for the network to reach global agreement or ϵ-agreement. It was shown that the persistent graph essentially determines both the convergence and convergence rate to an agreement.

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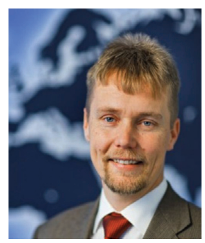

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