22nd International Symposium on Mathematical Theory of Networks and Systems
July 11-15, 2016. Minneapolis, MN, USA

Products of Doubly Stochastic Matrices*

Weiguo Xia, Ji Liu, Ming Cao, Karl H. Johansson, and Tamer Başar

Abstract— **Doubly stochastic matrices constitute an important class of stochastic matrices, playing a critical role in the study of discrete-time distributed averaging and distributed optimization algorithms. In this extended abstract, we report on several properties of such matrices. Furthermore, we utilize these properties to establish necessary and sufficient conditions for deciding whether a set of doubly stochastic matrices is a consensus set or not.**

Index Terms— **Matrix theory; Control systems; Stability**

I. Introduction

Distributed coordination of multi-agent systems is a topic that has received a great deal of attention in the last two decades [1]–[3]. The consensus problem is one of the benchmark problems in which a group of interacting agents are driven to reach agreement on a variable of interest. In the linear discrete-time consensus process, the update of the agents' states can be characterized by a linear recursion equation with the system matrix being a stochastic matrix [1], [3], [4]. The analysis of the system heavily relies on the property of the infinite product of stochastic matrices.

Properties of stochastic matrices have been studied extensively and a rich set of results on the conditions for the convergence of infinite products of stochastic matrices have been obtained [5]–[7], which can be traced back at least to the work on nonnegative matrices and Markov chains [7]. If we restrict our attention to certain subclasses of stochastic matrices, the study may help simplify those established results for general stochastic matrices. In this extended abstract, we pay special attention to an important class of stochastic matrices, doubly stochastic matrices, which arise often in distributed averaging problems [8] and distributed optimization algorithms [9], [10]. We study their properties, and establish necessary and sufficient conditions to decide whether a set of doubly stochastic matrices is a consensus set, whose definition will be given precisely later. These new results may shed light on better utilization of doubly stochastic matrices.

The remainder of the extended abstract is organized as follows. In Section II, preliminaries on stochastic matrices are introduced. Section III investigates the properties of doubly stochastic matrices and establishes necessary and sufficient conditions for deciding consensus. Conclusions are drawn in Section IV.

II. Preliminaries

We first introduce some preliminaries on stochastic matrices. Let n be a positive integer. A matrix $P = \{p_{ij}\}_{n\times n}$ is said to be *stochastic* if $p_{ij} \geq 0$ for all $i,\ j \in \mathcal{N} \triangleq \{1, 2, \ldots, n\}$, and $\sum_{j=1}^{n} p_{ij} = 1$ for all $i \in \mathcal{N}$. P is said to be *doubly stochastic* if both P and P^T are stochastic. Consider a stochastic matrix P. For a set $\mathcal{A} \subseteq \mathcal{N}$, the set of *one-stage consequent indices* [11] of $\mathcal{A}$ is defined by

$$F_P(\mathcal{A}) = \{j : \ p_{ij} > 0 \text{ for some } i \in \mathcal{A}\}$$

and we call F_P the consequent function of P.

Lemma 1: [12] Let P and Q be two $n \times n$ nonnegative matrices. Then, $F_{PQ}(\mathcal{A}) = F_Q(F_P(\mathcal{A}))$ for all subsets $\mathcal{A} \subseteq \mathcal{N}$.

A stochastic matrix P is indecomposable and aperiodic, and thus called an *SIA* matrix, if $\lim_{m\to\infty} P^m = \mathbf{1}c^T$, where $\mathbf{1}$ is the n-dimensional all-one column vector, and $c = [c_1, \ldots, c_n]^T$ is some column vector satisfying $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. We say that P belongs to the *Sarymsakov class*, or equivalently P is a *Sarymsakov* matrix, if for any two disjoint nonempty sets $\mathcal{A},\ \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset, \tag{1}$$

or

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|, \tag{2}$$

where $|\cdot|$ denotes the cardinality of a set. P is said to be a *scrambling* matrix if for any pair of distinct indices $i, j \in \mathcal{N}$, there exists an index $k \in \mathcal{N}$ such that p_{ik} and p_{jk} are positive.

Stochastic matrices arise often in discrete-time linear consensus processes, which can be modeled by

$$x(k+1) = P(k)x(k), \quad k \geq 1, \tag{3}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ and each $P(k)$ is an $n \times n$ stochastic matrix. The convergence of $x(k)$ to a common value for any initial condition is equivalent to

*Proofs of the assertions in this extended abstract are available from the first author upon request and will appear in an expanded version of this work.

Weiguo Xia is with the School of Control Science and Engineering, Dalian University of Technology, China (wgxiaseu@dlut.edu.cn). Ji Liu and Tamer Başar are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, USA ({jiliu,basar1}@illinois.edu). Ming Cao is with the Faculty of Mathematics and Natural Sciences, ENTEG, University of Groningen, the Netherlands (m.cao@rug.nl). Karl H. Johansson is with ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Sweden (kallej@kth.se).

The work of Xia was supported in part by the Fundamental Research Funds for the Central Universities. The work of Liu and Başar was supported in part by the U.S. Air Force Office of Scientific Research (AFOSR) MURI grant FA9550-10-1-0573. The work of Cao was supported in part by the European Research Council (ERC-StG-307207). The work of Johansson was supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council.

the convergence of the matrix product $P(k)\cdots P(2)P(1)$ to a matrix with all rows equal as k goes to infinity. When the sequence of matrices $P(1), P(2), \dots$ is picked from a set of stochastic matrices $\mathcal{P}$, an important question to answer is as follows: What are the conditions on $\mathcal{P}$ so that every infinite left-product $\cdots P(k)\cdots P(2)P(1)$ converges to a rank-one matrix? A set having this property is called a consensus set.

Definition 1: (*Consensus set*) Let $\mathcal{P}$ be a set of $n \times n$ stochastic matrices. $\mathcal{P}$ is a consensus set if for each sequence of matrices $P(1), P(2), P(3), \dots$ from $\mathcal{P}$, $P(k)\cdots P(2)P(1)$ converges to a rank-one matrix $\mathbf{1}c^T$ as $k \to \infty$, where $c_i \geq 0$ and $\sum_{i=1}^n c_i = 1$.

Necessary and sufficient conditions for $\mathcal{P}$ being a consensus set have been established in [5], [7], [13]–[15].

Theorem 1: [15] Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices. The following conditions are equivalent:

1) $\mathcal{P}$ is a consensus set.
2) For each integer $k \geq 1$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ is SIA.
3) There is an integer $\nu \geq 1$ such that for each $k \geq \nu$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ is scrambling.
4) There is an integer $\alpha \geq 1$ such that for each $k \geq \alpha$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(k)\cdots P(1)$ belongs to the Sarymsakov class.

A combinatorial necessary and sufficient condition for deciding whether a compact set of stochastic matrices is a consensus set or not has been established in [14].

Theorem 2: [14] A compact set $\mathcal{P}$ of $n \times n$ stochastic matrices is not a consensus if, and only if, there exist two sequences of nonempty subsets of $\mathcal{N}$, $S_1, S_2, \dots, S_l$, and $S_1', S_2', \dots, S_l'$, of length $l \leq 3^n - 2^{n+1} + 1$ and a sequence of matrices $P(1), P(2), \dots, P(l)$ from $\mathcal{P}$ such that

$$S_i \cap S_i' = \emptyset \text{ for all } i = 1, \dots, l,$$

and for all $i = 1, \dots, l-1$,

$$F_{P(i)}(S_i) \subseteq S_{i+1},\ F_{P(l)}(S_l) \subseteq S_1,$$
$$F_{P(i)}(S_i') \subseteq S_{i+1}',\ F_{P(l)}(S_l') \subseteq S_1'.$$

The above results hold for all compact sets of general stochastic matrices. We pay special attention to doubly s-tochastic matrices here, explore their properties, and establish necessary and sufficient conditions for determining whether a compact set of doubly stochastic matrices is a consensus set or not.

III. Doubly Stochastic Matrices

We first present several properties of doubly stochastic matrices that may not hold for general stochastic matrices.

Lemma 2: [16] Let P be a doubly stochastic matrix. For any nonempty set $\mathcal{A} \subseteq \mathcal{N}$, $|F_P(\mathcal{A})| \geq |\mathcal{A}|$.

Lemma 3: Let P be a doubly stochastic matrix. For any two disjoint nonempty subsets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, if $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$, then $|F_P(\mathcal{A})| > |\mathcal{A}|$ and $|F_P(\tilde{\mathcal{A}})| > |\tilde{\mathcal{A}}|$.

The following lemma reveals when a doubly stochastic matrix is a Sarymsakov matrix.

Proposition 1: [16] Let P be a doubly stochastic matrix. P is a Sarymsakov matrix if, and only if, for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, $|F_P(\mathcal{A})| > |\mathcal{A}|$.

A necessary and sufficient condition for a stochastic matrix to be SIA is as follows.

Theorem 3: [15] A stochastic matrix P is SIA if, and only if, for any two disjoint nonempty subsets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an integer $k \geq 1$ such that either

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset, \tag{4}$$

or

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P^k(\mathcal{A}) \cup F_P^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{5}$$

For doubly stochastic matrices, we have a simplified condition.

Proposition 2: Let P be a doubly stochastic matrix. P is an SIA matrix if, and only if, for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, there exists a positive integer k such that $|F_P^k(\mathcal{A})| > |\mathcal{A}|$.

For doubly stochastic matrices satisfying the condition

$$p_{ij} > 0 \text{ if and only if } p_{ji} > 0 \text{ for all } i \neq j, \tag{6}$$

we have the following result.

Proposition 3: [16] Let P be a doubly stochastic matrix satisfying condition (6). If P is SIA, then P is a Sarymsakov matrix.

For doubly stochastic matrices, the necessary and sufficient condition for deciding consensus can be obtained using Proposition 2. We first state the following result.

Theorem 4: Let $\mathcal{P}$ be a compact set of $n \times n$ doubly stochastic matrices, and let

$$b(n) \triangleq \binom{n}{\lfloor \frac{n-1}{2} \rfloor},$$

where $\lfloor \frac{n-1}{2} \rfloor$ is the greatest integer that is no larger than $\frac{n-1}{2}$ and

$$\binom{n}{\lfloor \frac{n-1}{2} \rfloor} = \frac{n!}{\lfloor \frac{n-1}{2} \rfloor! \left(n - \lfloor \frac{n-1}{2} \rfloor\right)!}.$$

$\mathcal{P}$ is a consensus set if, and only if, for each $k \geq b(n)$ and any $P(i) \in \mathcal{P},\ 1 \leq i \leq k$, the matrix $P(1)\cdots P(k-1)P(k)$ belongs to the Sarymsakov class.

Remark 1: The next result, Theorem 5, says that "α" in condition (4) in Theorem 1 can be taken as $b(n)$ when all the matrices in $\mathcal{P}$ are doubly stochastic matrices, instead of $\frac{1}{2}(3^n - 2^{n+1} + 1)$ (see Theorem 4.7 in [6]) for general stochastic matrices. □

Theorem 5: Let $\mathcal{P}$ be a compact set of $n \times n$ doubly stochastic matrices. Then, $\mathcal{P}$ is not a consensus set if, and only if, there exist a sequence of nonempty subsets of $\mathcal{N}$, $S_1, S_2, \dots, S_l$ of length $l \leq b(n)$, and a sequence of matrices $P(1), P(2), \dots, P(l)$ from $\mathcal{P}$ such that for all $i \in \{1, 2, \dots, l-1\}$,

$$F_{P(i)}(S_i) \subseteq S_{i+1}, \quad F_{P(l)}(S_l) \subseteq S_1. \tag{7}$$

Remark 2: It has been shown in [14] that the process of deciding whether a finite set of stochastic matrices is a consensus set or not is NP-hard. Theorem 5 can be used to decide whether a finite set of doubly stochastic matrices is a consensus set or not, and can be helpful for checking the complexity. □

For a finite set of doubly stochastic matrices satisfying (6), deciding whether this set is a consensus set or not can be accomplished in polynomial time. In fact, this is true for several classes of stochastic matrices. We identify below some of such classes of stochastic matrices.

Definition 2: A stochastic matrix P is said to belong to the *class* $\mathcal{M}$ if either P is a Sarymsakov matrix, or P is not an SIA matrix.

The class $\mathcal{M}$ is actually composed of all the Sarymakov matrices and the non-SIA matrices.

Theorem 6: Let $\mathcal{P}$ be a finite set of $n \times n$ stochastic matrices. If $\mathcal{P} \subseteq \mathcal{M}$, then whether $\mathcal{P}$ is a consensus set or not can be decided in $O(n^4)$ time.

Remark 3: There are several classes of stochastic matrices that belong to the class $\mathcal{M}$. For example, stochastic matrices with positive diagonal entries [4], [17], symmetric stochastic matrices [14], and doubly stochastic matrices satisfying (6) (Proposition 3). Theorem 6 shows that for a finite set of matrices from these classes of matrices, a polynomial time algorithm can be derived to decide whether the set is a consensus set or not. □

IV. Conclusions

We have reported on several properties of doubly s-tochastic matrices, and presented a necessary and sufficient condition for a doubly stochastic matrix to be SIA. A combinatorial necessary and sufficient condition has been derived for deciding whether a compact set of doubly stochastic matrices is a consensus set or not.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):985–1001, 2003.

[2] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[3] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[4] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47(2):575–600, 2008.

[5] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 14(5):733–737, 1963.

[6] A. Paz. *Introduction to Probabilistic Automata*. Academic Press, 1971.

[7] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer-Verlag, New York, 2006.

[8] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[9] A. Nedić and A. Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.

[10] A. Nedić, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–937, 2010.

[11] E. Seneta. Coefficients of ergodicity: Structure and applications. *Advances in Applied Probability*, 11(3):576–590, 1979.

[12] D. J. Hartfiel. *Nonhomogeneous Matrix Products*. World Scientific, Singapore, 2002.

[13] J. M. Anthonisse and H. Tijms. Exponential convergence of products of stochastic matrices. *Journal of Mathematical Analysis and Applications*, 59(2):360–364, 1977.

[14] V. D. Blondel and A. Olshevsky. How to decide consensus: A combinatorial necessary and sufficient condition and a proof that consensus is decidable but NP-hard. *SIAM Journal on Control and Optimization*, 52(5):2707–2726, 2014.

[15] W. Xia and M. Cao. Sarymsakov matrices and asynchronous implementation of distributed coordination algorithms. *IEEE Transcations on Automatic Control*, 59(8):2228–2233, 2014.

[16] W. Xia, J. Liu, M. Cao, K. H. Johansson, and T. Başar. Products of generalized stochastic Sarymsakov matrices. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pages 3621–3626, 2015.

[17] J. Liu, S. Mou, A. S. Morse, B. D. O. Anderson, and C. Yu. Deterministic gossiping. *Proceedings of the IEEE*, 99(9):1505–1524, 2011.