Distributed Control of Networked Dynamical Systems: Static Feedback, Integral Action and Consensus

Martin Andreasson, *Student Member, IEEE*, Dimos V. Dimarogonas, *Member, IEEE*,
Henrik Sandberg, *Member, IEEE*, and Karl Henrik Johansson, *Fellow, IEEE*

Abstract—This paper analyzes distributed control protocols for first- and second-order networked dynamical systems. We propose a class of nonlinear consensus controllers where the input of each agent can be written as a product of a nonlinear gain, and a sum of nonlinear interaction functions. By using integral Lyapunov functions, we prove the stability of the proposed control protocols, and explicitly characterize the equilibrium set. We also propose a distributed proportional-integral (PI) controller for networked dynamical systems. The PI controllers successfully attenuate constant disturbances in the network. We prove that agents with single-integrator dynamics are stable for any integral gain, and give an explicit tight upper bound on the integral gain for when the system is stable for agents with double-integrator dynamics. Throughout the paper we highlight some possible applications of the proposed controllers by realistic simulations of autonomous satellites, power systems and building temperature control.

Index Terms—Agents and autonomous systems, cooperative control, electrical power systems, proportional integral (PI) control.

I. Introduction

A. General Motivation

DISTRIBUTED or decentralized control is in many large-scale systems the only feasible control strategy, when sensing and actuation communication is limited. In this paper we distinguish between decentralized control strategies where agents only have access to local measurements, from distributed control strategies where agents have access to local measurements and the measurements from neighboring agents [3], [4]. To attenuate unknown disturbances as well as to stabilize systems in the presence of model imperfections, proportional-integral (PI) controllers are commonly employed in such plants. However, it is still an open problem under which conditions distributed PI-controllers can stabilize a plant in general [19].

One important class of systems that require integral action to attenuate disturbances are electric power systems [18]. Due to sudden load and generation changes as well as model imperfections, a proportional frequency controller cannot reach the desired reference frequency in general. To attenuate such static errors, integrators are commonly employed. Due to the inherent difficulties with distributed PI control, automatic frequency control of power systems is typically carried out at two levels, an inner and an outer level. In the inner control loop, the frequency is controlled with a proportional controller against a dynamic reference frequency. In the outer loop, the reference frequency is controlled with a centralized PI controller to eliminate static errors. While this control architecture works satisfactorily in most of today's power systems, future power system developments might render it unsuitable. For instance, large-scale penetration of renewable power generation increases generation fluctuations, creating a need for fast as well as local disturbance attenuation. Decentralized control of power systems might also provide efficient control under islanding and self-healing features, even when communication between subsystems is limited or even unavailable [25]. Distributed integral action for power system frequency control, as well as numerous other physical systems such as mobile robotic systems, can often be well modeled by consensus-like protocols for double-integrator dynamics. Motivated by this fact, we develop in this paper a more general framework for consensus protocols with integral action for agents with double-integrator dynamics and velocity damping.

Static feedback controllers, in contrast to integrators, have been used in a variety of distributed control problems. For example they are commonly used in the consensus problem. The consensus problem has applications in formation control [12], flocking [17] and rendezvous [7], amongst others. While static linear feedback controllers can solve the basic problems, nonlinear feedback controllers are of great importance when, e.g., dealing with connectivity constraints and collision avoidance [10]. Furthermore, some distributed control problems have inherent nonlinear dynamics, e.g., heat diffusion where heat capacities are nonlinear. Much attention has been devoted to nonlinear consensus protocols in recent years. However, to the best of our knowledge, none of the previous work characterizes the convergence point of nonlinear consensus protocols explicitly. In this paper we introduce a nonlinear consensus protocol for single and double-integrator dynamics. We prove the

asymptotic stability of the dynamics induced by the protocol, and explicitly characterize its equilibrium set.

B. Related Work

Nonlinear interaction functions for consensus problems are well-studied [20], [22], in applications to, e.g., consensus with preserving connectedness [10] and collision avoidance [10]. Sufficient conditions for the convergence of nonlinear protocols for first-order integrator dynamics are given in [2]. Consensus on a general nonlinear function value, referred to as χ-consensus, was studied in [9], by the use of nonlinear gain functions. The literature on χ-consensus has been focused on agents with single-integrator dynamics. However, as we show later, our results can be generalized to hold also for double-integrator dynamics. Consensus protocols where the input of an agent can be separated into a product of a positive function of the agents own state were studied in [6] for single integrator dynamics. In [21], position consensus for agents with double-integrator dynamics under a class of nonlinear interaction functions and nonlinear velocity-damping is studied. In contrast to [21], we study undamped consensus protocols for single- and double-integrator dynamics using integral Lyapunov functions. In [29] double-integrator consensus problems with linear non-homogeneous damping coefficients are considered. We generalize these results to also hold for a class of nonlinear damping coefficients.

Multi-agent systems, as all control processes, are in general sensitive to disturbances. When only relative measurements are available, disturbances may spread through the network. It has for example been shown by [5] that vehicular string formations with only relative measurements cannot maintain coherency under disturbances, as the size of the formation increases. In [30] the robustness of consensus-protocols under disturbances is studied, but limited to the relative states of the agents. However, none of the aforementioned references consider disturbance rejection. In [28] however, the steady-state error for first-order consensus dynamics is minimized by convex optimization over the edge-weights. A similar approach is taken in [16], where the application is vehicle-platooning. In [33], an optimal sensor placement problem for consensus problems is formulated, minimizing the $\mathcal{H}_2$ gain of the system. However, these approaches eliminate output errors if the disturbances are constant only in special cases, as no no integral control is employed.

Consensus with integral action is studied in [13] for agents with single-integrator dynamics. It was shown that the proposed controller attenuates constant disturbances. In [31], the authors take a similar approach to attenuate unknown disturbances. In both references the analysis is limited to agents with single-integrator dynamics. Our proposed PI controller is related to the consensus controllers in [8], [14]. However, these references do not consider the influence of disturbances.

C. Main Contributions

The main contributions of this paper are threefold. Firstly, we propose a class of nonlinear consensus protocols where the input of an agent can be decoupled into a product of a positive gain function of the agents own state, and a sum of interaction functions of its neighbors' relative states. Nonlinear consensus protocols with double-integrator dynamics and state-dependent damping are also considered. The equilibria are characterized for both the first- and second-order nonlinear consensus protocol. Secondly, we propose a distributed PI controller for multi-agent systems with single integrator dynamics and a corresponding controller for agents with damped double-integrator dynamics. Necessary and sufficient conditions for stability are derived, given that the controller gains are uniform. Lastly, we demonstrate some applications of our theoretical results to satellite control, mobile robots, green buildings and frequency control of power systems.

D. Outline

The remainder of this paper is organized as follows. In Section II we introduce four motivating examples, which illustrate the theoretical results developed later in the paper. In Section III we define a mathematical model for a class of distributed control systems, and the considered problem is stated. In Section IV we analyze the problem of distributed control for a class of nonlinear control problems. We explicitly characterize the equilibria by integral equations, and provide sufficient stability criteria by integral Lyapunov functions. In Section V we analyze the problem of distributed control by integral action, where we give necessary and sufficient stability criteria. In Section VI we show the feasibility of the results by applying them to the examples in Section II. The paper ends by concluding remarks in Section VII.

II. MOTIVATING APPLICATIONS

A. Thermal Energy Storage in Smart Buildings

Thermal energy storage has emerged as a possible method for energy-efficient temperature regulation in buildings, as discussed in [32]. By using substances which undergoes a phase transition near the desired maximum temperature in the building, the temperature may be kept below the maximal desired temperature. While the heat capacity of the air in a building is approximately constant, the total heat capacity of the room is highly nonlinear due to the thermal energy storage. The endothermic and exothermic processes of the phase transitions may be modeled by nonlinear heat capacities, which take the form of a continuous approximation of a Dirac delta function at the temperature of the phase transition. The model fits well with a consensus protocol for agents with single-integrator dynamics with nonlinear gain and interaction functions. By a nonlinear extension of Fourier's law [15], the room temperatures are thus well-described by the following nonlinear differential equation:

$$\dot{T}_i = -\gamma_i(T_i) \sum_{j \in \mathcal{N}_i} a_{ij}(T_i - T_j) \tag{1}$$

where T_i is the temperature of room i, $a_{ij}(T_i - T_j)$ is the heat conductivity between room i and j. Here $a_{ij}(\cdot)$ is a nonlinear function $\forall (i,j) \in \mathcal{E}$, and $1/\gamma_i(T_i)$ is the temperature-dependent heat capacity of room i, capturing the dynamics of

the energy storage. It is of interest to determine the asymptotic temperature in the rooms given their initial temperatures.

B. Autonomous Space Satellites

Groups of autonomous space satellites may solve tasks in space that require coordination. E.g., for a solar power plant in space, this could involve formation control of mirrors, reflecting the sunlight to a solar panel. If the agents are far away from any fixed reference frame, it may be assumed that the satellites only have access to their distance and velocity relative to their neighboring satellites. It is however often important to analyze the dynamical behavior of the satellites from a common reference frame, e.g., from the earth. Even if the control laws are linear in the relative velocities in the satellites reference frame, they are in general nonlinear in another reference frame. More specifically, the dynamics of a group of N satellites are assumed to be governed by Newton's second law of motion, resulting in a set of second-order dynamical systems. The control signal is the power applied by each agent's engine, P_i. However, the acceleration in an inertial reference frame is $a_i = P_i/|v_i|$ due to $P_i = \langle F_i, v_i \rangle$, and the force F_i being parallel to v_i, where v_i is agent i's velocity. We assume that the agents only have access to relative measurements. The following dynamics describe the interactions of the satellites:

$$\dot{x}_i = v_i$$
$$\dot{v}_i = -\frac{1}{|v_i| + c} \sum_{j \in \mathcal{N}_i} [a_{ij}(x_i - x_j) + b_{ij}(v_i - v_j)] \quad (2)$$

where $a_{ij}(\cdot)$ and $b_{ij}(\cdot)$ are nonlinear interaction functions, $i = 1, \dots n$ and $\mathcal{N}_i$ denotes the neighbor set of satellite i, and where c is an arbitrarily small constant which ensures the boundedness of $\dot{v}_i$ when $v_i = 0$. Here x_i and v_i denote the position and velocity of satellite i. This motivates the analysis of consensus protocols for agents with double-integrator dynamics and nonlinear gain and interaction functions.

C. Mobile Robot Coordination Under Disturbances

As all control systems, mobile robot systems are susceptible to disturbances. In general, even constant disturbances cause a robot formation with only relative position and velocity measurements to drift. We will consider the particular control objective of reaching position-consensus. To address the issues caused by disturbances to the robots, a distributed PI controller is proposed. We consider robots with the second-order dynamics and the following controller:

$$\dot{x}_i = v_i$$
$$\dot{v}_i = u_i - \gamma v_i + d_i$$
$$u_i = -\sum_{j \in \mathcal{N}_i} \left(b(x_i - x_j) + a \int_0^t (x_i(\tau) - x_j(\tau)) \, d\tau \right) \quad (3)$$

where x_i is the position, v_i is the velocity. The constants $a, b > 0$ are controller parameters, while $\gamma > 0$ is a damping coeffi-

cient, and d_i is a constant disturbance. We will investigate when distributed PI controllers can attenuate static disturbances in mobile-robot networks. Furthermore, given the system-specific damping coefficient γ, we will characterize under which conditions on a and b asymptotic consensus is reached.

D. Frequency Control of Power Systems

Power systems are among the largest and most complex dynamical systems created by mankind [18]. The interconnectivity of power systems poses many challenges when designing controllers. We model the power system as interconnected second-order systems, known as the swing equation. The swing equation has been used, e.g., in studying transient angle stability of power systems [11] and fault detection in power systems [26], and is given by

$$m_i \ddot{\delta}_i + d_i \dot{\delta}_i = -\sum_{j \in \mathcal{N}_i} k_{ij}(\delta_i - \delta_j) + p_i^m + u_i \quad (4)$$

where δ_i is the phase angle of bus i, m_i and d_i are the inertia and damping coefficient respectively, p_i^m is the electrical power load at bus i and u_i is the mechanical input power. The coefficient $k_{ij} = |V_i||V_j|b_{ij}$, where $|V_i|$ is the absolute value of the voltage of bus i, and b_{ij} is the susceptance of the line (i, j). The frequency of the power system is denoted $\omega_i = \dot{\delta}_i$. Maintaining a steady frequency is one of the major control problems in power systems. If the frequency is not kept close to the nominal operational frequency, generation and utilization equipment may cease to function properly. The frequency is maintained primarily by automatic generation control (AGC), which is carried out at different levels. In the first level, which is carried out locally at each bus, the power generation is controlled by the deviation from a dynamic reference frequency. At the second level, which is carried out by a centralized controller, the reference frequency is controlled based on the deviation of the frequency from a reference frequency at a specific location. When communication is unavailable or limited, a decentralized controller is a possible alternative. A simple decentralized frequency control with integral action would take the form

$$u_i = a \left(\omega^{\text{ref}} - \omega_i(t) \right) + b \int_0^t \left(\omega^{\text{ref}} - \omega_i(t') \right) dt' \quad (5)$$

where ω^{ref} is the reference frequency. The control objective is to ensure that the system frequency reaches the nominal operational frequency, i.e., $\lim_{t \to \infty} \omega_i = \omega^{\text{ref}} \ \forall i \in \mathcal{V}$.

III. PROBLEM FORMULATION

In this section we formalize the previously mentioned problems. We introduce a unified mathematical notation which includes all problems previously described.

A. Notation

Let $\mathcal{G}$ be a graph. Denote by $\mathcal{V} = \{1, \dots, n\}$ the vertex set, and by $\mathcal{E} = \{1, \dots, m\}$ the edge set of $\mathcal{G}$. Let $\mathcal{N}_i$ be the

set of neighboring nodes of node i. Denote by $\mathcal{B}$ the vertex-edge adjacency matrix of $\mathcal{G}$, and let $\mathcal{L}$ be its Laplacian matrix. For undirected graphs it holds that $\mathcal{L} = \mathcal{B}\mathcal{B}^T$. Throughout this paper we will assume that $\mathcal{G}$ is connected, as motivated by the applications considered. We denote by $\mathbb{R}^-/\mathbb{R}^+$ the open negative/positive real axis, and by $\bar{\mathbb{R}}^-/\bar{\mathbb{R}}^+$ its closure. Let $\mathbb{C}^-/\mathbb{C}^+$ denote the open left/right half complex plane, and $\bar{\mathbb{C}}^-/\bar{\mathbb{C}}^+$ its closure. We will denote the scalar position of agent i as x_i, and its velocity as v_i, and collect them into column vectors $x = (x_1, \ldots, x_n)^T$, $v = (v_1, \ldots, v_n)^T$. We denote by $c_{n \times m}$ a vector or matrix of dimension $n \times m$ whose elements are all equal to c. I_n denotes the identity matrix of dimension n. A function $f(\cdot)$ with domain $\mathcal{X}$ is said to be globally Lipschitz (continuous) if there exists $K \in \mathbb{R}^+ : \forall x, y \in \mathcal{X} :$ $\|f(x) - f(y)\| \le K\|x - y\|$.

B. System Model

We consider agents endowed with either single-integrator dynamics

$$\dot{x}_i = d_i + u_i \tag{6}$$

or double-integrator dynamics

$$\dot{x}_i = v_i$$
$$\dot{v}_i = d_i + u_i \tag{7}$$

where d_i is a constant disturbance.

C. Objective

The main objectives of this paper are twofold. Our first objective is to characterize the stability of nonlinear feedback protocols, and to determine under which nonlinear feedback protocols the consensus point of the agents may be determined *a priori*, in the absence of disturbances. The second objective is the design of consensus protocols robust to disturbances. We will focus on constant but unknown disturbances. In both cases, the overall objective will be for all agents to converge to a common state, i.e. $\lim_{t \to \infty} x_i(t) = x^* \forall i \in \mathcal{V}$ for single-integrator dynamics, and $\lim_{t \to \infty} v_i(t) = v^* \forall i \in \mathcal{V}$ for double-integrator dynamics.

IV. DISTRIBUTED CONTROL WITH STATIC NONLINEAR FEEDBACK

Although the consensus problem has received tremendous amounts of attention in the last decade, the vast majority of the studied consensus protocols are linear. In this section we define a class of nonlinear consensus algorithms where the input of each agent can be decoupled into a product of a nonlinear gain function and a nonlinear interaction function. We show that several interesting properties of the system arise due to this coupling. We first study consensus for single-integrator dynamics by nonlinear protocols in Section IV-A. In Section IV-B we consider a nonlinear consensus protocol for agents with double-integrator dynamics. Finally, in Section IV-C a nonlinear con-

sensus protocol with nonlinear, state-dependent damping for agents with double-integrator dynamics is considered.

A. Consensus for Single-Integrator Dynamics

Consider agents with dynamics (6), where $d_i = 0 \ \forall i \in \mathcal{V}$, and where u_i is given by

$$u_i = -\gamma_i(x_i) \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j). \tag{8}$$

Assumption 1: The gain γ_i is continuous and $0 < \underline{\gamma} \le \gamma_i(x) \le \bar{\gamma} \ \forall i \in \mathcal{V}, \ \forall x \in \mathbb{R}$.

Assumption 2: The interaction function $a_{ij}(\cdot)$ is Lipschitz continuous $\forall (i, j) \in \mathcal{E}$, and:
1) $a_{ij}(-y) = -a_{ji}(y) \ \forall (i, j) \in \mathcal{E}, \ \forall y \in \mathbb{R}$
2) $a_{ij}(-y) = -a_{ij}(y) \ \forall (i, j) \in \mathcal{E}, \ \forall y \in \mathbb{R}$
3) $a_{ij}(y) > 0 \ \forall (i, j) \in \mathcal{E}, \ \forall y > 0$,

Remark 1: Assumption 2 guarantees that the agents move in the direction of their neighbors, as well as symmetry in the flow. A consequence of Assumption 2 is that $\alpha_{ij}(0) = 0 \ \forall (i, j) \in \mathcal{E}$, ensuring that any consensus point where $x_i = x_j \ \forall i, j \in \mathcal{V}$ is an equilibrium.

We are now ready to state the main result of this section.

Theorem 1: Given n agents with dynamics (6) with $d_i = 0 \ \forall i \in \mathcal{V}$, and u_i given by (8), where $\gamma_i(\cdot)$ and $a_{ij}(\cdot)$ satisfy Assumptions 1 and 2, respectively, for all $i \in \mathcal{V}$ and for all $(i, j) \in \mathcal{E}$. Then the agents converge asymptotically to an agreement point $\lim_{t \to \infty} x_i(t) = x^* \ \forall i \in \mathcal{V}$, where x^* is uniquely determined by the integral equation for any $x_i(0) = x_i^0, i = 1, \ldots, n$

$$\sum_{i \in \mathcal{V}} \int_0^{x_i^0} \frac{1}{\gamma_i(y)} \mathrm{d}y = \int_0^{x^*} \sum_{i \in \mathcal{V}} \frac{1}{\gamma_i(y)} \mathrm{d}y. \tag{9}$$

Proof: The proof is Lyapunov based, and requires finding an appropriate Lyapunov function, which guarantees convergence to an equilibrium set. This equilibrium set is then characterized by a time-invariant function. Such a function is given by $E(x) = \sum_{i \in \mathcal{V}} \int_0^{x_i} 1/\gamma_i(y) \mathrm{d}y$. Differentiating $E(x)$ with respect to time yields

$$\dot{E}(x(t)) = -\left[\frac{1}{\gamma_1(x_1)}, \ldots, \frac{1}{\gamma_n(x_n)}\right] \Gamma(x)\mathcal{B}a(\mathcal{B}^T x)$$
$$= -1_{1 \times n}\mathcal{B}a(\mathcal{B}^T x) = 0$$

where $\Gamma(x) = \mathrm{diag}([\gamma_1(x_1), \ldots, \gamma_n(x_n)])$, and $a(\cdot)$ is taken component-wise. Hence $E(x)$ is time-invariant and the agreement point x^*, if existing, is given by (9). By Assumption 1, $E(x^* 1_{n \times 1})$ is strictly increasing in x^*, and hence (9) admits a unique solution. Now consider the following candidate Lyapunov function:

$$V(x) = \sum_{i \in \mathcal{V}} \int_{x^*}^{x_i} \frac{y - x^*}{\gamma_i(y)} \mathrm{d}y \tag{10}$$

where x^* is the agreement point given by (9). It can easily be verified that $V(x^* 1_{n \times 1}) = 0$. To show that $V(x) > 0$ for

$x \neq x^* 1_{n \times 1}$, it suffices to show that $\int_{x^*}^{x_i} (y - x^*)/\gamma_i(y) \mathrm{d}y > 0 \; \forall i \in \mathcal{V}, \; \forall x \neq x^* 1_{n \times 1}$. Consider first the case when $x_i > x^*$, where $\int_{x^*}^{x_i} (y - x^*)/\gamma_i(y) \mathrm{d}y = \int_0^{x_i - x^*} z/\gamma_i(z + x^*) \mathrm{d}z > 0$, by the change of variable $z = y - x^*$. The case when $x_i < x^*$ is treated analogously, where $\int_{x^*}^{x_i} (y - x^*)/\gamma_i(y) \mathrm{d}y = \int_0^{x^* - x_i} z/\gamma_i(x^* - z) \mathrm{d}z > 0$, by the change of variable $z = x^* - y$. This also implies that $V(x) = 0 \Rightarrow x = x^* 1_{n \times 1}$. Now consider $\dot{V}(x)$

$$
\begin{aligned}
\dot{V}(x) &= - \sum_{i \in \mathcal{V}} \frac{x_i - x^*}{\gamma_i(x_i)} \gamma_i(x_i) \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) \\
&= - \sum_{i \in \mathcal{V}} x_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) + \sum_{i \in \mathcal{V}} x^* \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j).
\end{aligned}
$$

(11)

Due to the symmetry property in Assumption 2, the first term of (11) may be rewritten as $\sum_{i \in \mathcal{V}} x_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} x_i a_{ij}(x_i - x_j) = (1/2) \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} (x_i - x_j) a_{ij}(x_i - x_j)$. Clearly the second term of (11) satisfies $\sum_{i \in \mathcal{V}} x^* \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) = 0$ due to Assumption 2. Hence, $\dot{V}(x)$ may be rewritten as $\dot{V}(x) = -(1/2) \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} (x_i - x_j) a_{ij}(x_i - x_j) < 0$, unless $x_i = x_j \; \forall i, j \in \mathcal{V}$. Therefore the agents converge to $x_i = x^* \; \forall i \in \mathcal{V}$. $\qquad \square$

Remark 2: The agreement protocol (8) has a physical interpretation. If we consider the smart building problem in Section II-A, and let x_i be the temperature of the rooms, $1/\gamma_i(x_i)$ is the temperature-dependent heat capacity of the nodes. Analogously, $a_{ij}(\cdot)$ is the thermal conductivity of the walls between rooms i and j, being dependent on the temperature gradient between the rooms. The invariant quantity $E(x) = \sum_{i \in \mathcal{V}} \int_0^{x_i} 1/\gamma_i(y) \mathrm{d}y$ is the total stored thermal energy.

Remark 3: The convergence of the dynamics (6), and the control (8) were proven in [27]. However, as opposed to that reference, we characterize here explicitly the equilibrium set. This gives *a priori* knowledge about the point of convergence. Furthermore, in [27], the convergence was studied with a Lyapunov function consisting of the difference of the maximal and minimal state, as opposed to the Lyapunov function consisting of integral equations used in our proof. This new Lyapunov function facilitates the proof by not relying on non-smooth analysis.

B. Consensus for Double-Integrator Dynamics

Consider agents with double-integrator dynamics (7), where $d_i = 0 \; \forall i \in \mathcal{V}$, with u_i given by

$$
u_i = -\gamma_i(v_i) \sum_{j \in \mathcal{N}_i} [a_{ij}(x_i - x_j) + b_{ij}(v_i - v_j)]. \tag{12}
$$

We show that under mild conditions, the controller (12) achieves asymptotic consensus. The following Theorem generalizes the literature on linear second-order consensus as in [24], by extending the analysis of linear gains and interaction functions to nonlinear ones. The nonlinear analysis covers a much richer class of control inputs than the corresponding linear analysis, and we are again able to explicitly characterize

the equilibrium set. By using an integral Lyapunov function, we are thus able to prove that the agents reach consensus for the nonlinear consensus protocol also in the case of double-integrator dynamics.

Theorem 2: Consider agents with dynamics (7), with $d_i = 0 \; \forall i \in \mathcal{V}$, and u_i given by (12), where $a_{ij}(\cdot)$ and $\gamma_i(\cdot)$ satisfy Assumptions 1 and 2, and $b_{ij}(\cdot)$ satisfies Assumption 2, replacing a_{ij} with b_{ij}, for all $i \in \mathcal{V}$ and for all $(i, j) \in \mathcal{E}$, respectively. The agents achieve consensus with respect to x and v, i.e., $|x_i - x_j| \to 0, |v_i - v_j| \to 0 \; \forall i, j \in \mathcal{G}$ as $t \to \infty$ for any initial condition $(x(0), v(0))$. Furthermore, the velocities converge to $\lim_{t \to \infty} v_i(t) = v^* \; \forall i \in \mathcal{V}$, uniquely determined by

$$
\sum_{i \in \mathcal{V}} \int_0^{v_i^0} \frac{1}{\gamma_i(y)} \mathrm{d}y = \int_0^{v^*} \sum_{i \in \mathcal{V}} \frac{1}{\gamma_i(y)} \mathrm{d}y. \tag{13}
$$

Proof: The convergence analysis relies on Lyapunov techniques. For characterizing the equilibrium set, a time-invariant function is introduced. We write (12) in vector form as

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -\Gamma(v) \left[\mathcal{B} a(\bar{x}) + \mathcal{B} b(\mathcal{B}^T v) \right]
\end{aligned}
$$

where $\bar{x} = \mathcal{B}^T x$, and $a(\cdot)$ and $b(\cdot)$ are taken component-wise, and $\Gamma(x) = \mathrm{diag}([\gamma_1(x_1), \dots, \gamma_n(x_n)])$. Consider now the following candidate Lyapunov function, also used in [21], however in another context

$$
V(\bar{x}, v) = \sum_{i \in \mathcal{V}} \left(\int_{v^*}^{v_i} \frac{(y - v^*)}{\gamma_i(y)} \mathrm{d}y \right) + \sum_{(i,j) \in \mathcal{E}} \int_0^{\bar{x}_{ij}} a_{ij}(y) \mathrm{d}y.
$$

Here v^* is the common velocity of the agents at the equilibrium, which we will show is given by (13). It is straightforward to verify that $V([0_{1 \times m}, v^* 1_{1 \times n}]^T) = 0$. By following the proof of the positive semi-definiteness of $V(x)$ in the proof of Theorem 1, replacing x_i and x^* with v_i and v^*, respectively, the positive semi-definiteness of $\sum_{i \in \mathcal{V}} (\int_{v^*}^{v_i} (y - v^*)/\gamma_i(y) \mathrm{d}y)$ follows. For showing the positive semi-definiteness of the second term of $V(\bar{x}, v)$, it suffices to show that $\int_0^{\bar{x}_{ij}} a_{ij}(y) \mathrm{d}y > 0 \; \forall (i, j) \in \mathcal{E}$. For $\bar{x}_{ij} > 0$, this inequality clearly holds. When $\bar{x}_{ij} < 0$ we have $\int_0^{\bar{x}_{ij}} a_{ij}(y) \mathrm{d}y = -\int_{\bar{x}_{ij}}^0 a_{ij}(y) \mathrm{d}y = \int_{\bar{x}_{ij}}^0 a_{ji}(-y) \mathrm{d}y > 0$. We may write $V(\bar{x}, v)$ in vector form as $V(\bar{x}, v) = \int_0^{\bar{x}} 1_{1 \times n} \mathcal{B}^T a(y) \mathrm{d}y + \int_{v^* 1_{n \times 1}}^v \tilde{y}^T \Gamma^{-1}(y) 1_{n \times 1} \mathrm{d}y$, where $\tilde{y} = [y_1 - v^*, \dots, y_n - v^*]^T$. Differentiating $V(\bar{x}, v)$ with respect to time yields

$$
\begin{aligned}
\dot{V} &= a(\bar{x})^T \mathcal{B}^T v - (v - v^* 1_{n \times 1})^T \mathcal{B} \left[a(\bar{x}) + b(\mathcal{B}^T v) \right] \\
&= -v^T \mathcal{B} b(\mathcal{B}^T v) + v^* 1_{1 \times n} \mathcal{B} b(\mathcal{B}^T v) = -v^T \mathcal{B} b(\mathcal{B}^T v) \leq 0
\end{aligned}
$$

due to Assumption 2, with equality if and only if $\mathcal{B}^T v = 0$. We now invoke LaSalle's invariance principle to show that the agreement point satisfies $\dot{v} = 0$. The subspace where $\dot{V}(\bar{x}, v) = 0$ is given by $S_1 = \{(\bar{x}, v) | v = c 1_{n \times 1}\}$. We note that on S_1, $\dot{v} = -\Gamma(v)[\mathcal{B} a(\bar{x}) + \mathcal{B} b(\mathcal{B}^T v)] = -\Gamma(v) \mathcal{B} a(\bar{x}) \neq k(t) 1_{n \times 1}$. To see this, suppose that $\dot{v}(t) = -\Gamma(v) \mathcal{B} a(\bar{x}) =$

$k(t)1_{n\times 1} \Leftrightarrow \mathcal{B}a(\bar{x}) = \Gamma^{-1}(v)k(t)1_{n\times 1}$, where $k(t) \neq 0 \ \forall t$. Premultiplying the above equation with $1_{1\times n}$ yields $0 = 1_{1\times n}\mathcal{B}a(\bar{x}) = k(t)1_{1\times n}\Gamma^{-1}(v)1_{n\times 1} \neq 0$, which is a contradiction since $k(t) \neq 0 \ \forall t$ by assumption. Hence the only trajectories contained in S_1 are those where $v = v^*1_{n\times 1}, \dot{v} = 0$. It can also be shown that no trajectories where $\bar{x} \neq 0$ are contained in S_1. Assume for the sake of contradiction that $\bar{x} \neq 0$ in S_1. Let $i^- = \min_{j\in\mathcal{V}} x_j$ s.t. $\exists k \in \mathcal{N}_{i^-} : x_k > x_{i^-}$. It is clear that such an i^- exists, since otherwise $\bar{x} = 0$. Consequently

$$\dot{v}_{i^-} = -\gamma_{i^-}(v_{i^-}) \sum_{j\in\mathcal{N}_{i^-}} \left[a_{i^-j}(x_{i^-} - x_j) + b_{ij}(v_{i^-} - v_j)\right]$$

$$= -\gamma_{i^-}(v_{i^-}) \sum_{j\in\mathcal{N}_{i^-}} \left[a_{i^-j}(x_{i^-} - x_j)\right]$$

$$> -\gamma_{i^-}(v_{i^-})a_{i^-k}(x_{i^-} - x_k) > 0$$

by the assumption that $x_k > x_{i^-}$. Thus, any trajectory in S_1 where $\bar{x} \neq 0$ cannot remain in S_1, implying that $|x_i - x_j| \to 0, |v_i - v_j| \to 0 \ \forall i, j \in \mathcal{G}$ as $t \to \infty$ and furthermore $\dot{v}(t) = 0_{n\times 1}$. Next we show that $p(v) = \sum_{i\in\mathcal{V}}\int_0^{v_i} 1/\gamma_i(y)dy = \int_0^v 1_{1\times n}\Gamma^{-1}(v)1_{n\times 1}dy$ is invariant under (12). Consider

$$\dot{p}(v(t)) = -1_{1\times n}\Gamma^{-1}(v)\Gamma(v)\left[\mathcal{B}a(\bar{x}) + \mathcal{B}b(\mathcal{B}^T v)\right]$$

$$= -1_{1\times n}\mathcal{B}a(\bar{x}) - 1_{1\times n}\mathcal{B}b(\mathcal{B}^T v) = 0.$$

Thus we conclude that $\lim_{t\to\infty} x(t) = x^*(t)1_{n\times 1}$ and $\lim_{t\to\infty} v(t) = v^*1_{n\times 1}$ with v^* given by (13). The existence and uniqueness of the solution to (13) follows from Assumption 1, and by the proof of Theorem 1, replacing x_i and x^* with v_i and v^*, respectively. □

Remark 4: Theorem 2 has a physical interpretation. If we regard $1/\gamma_i(v_i)$ as the velocity-dependent mass of agent i, e.g., due to the agents' masses scaling with the Lorentz factor $1/\sqrt{1 - v_i^2/v_c^2}$, where v_c is the speed of light, then the invariant quantity $p(v) = \sum_{i\in\mathcal{V}}\int_0^{v_i} 1/\gamma_i(y)dy$ is the total (relativistic) momentum of the mechanical system.

C. Consensus for Double-Integrator Dynamics With State-Dependent Damping

Consider agents with double-integrator dynamics (7), where $d_i = 0 \ \forall i \in \mathcal{V}$, with u_i given by

$$u_i = -\kappa_i(x_i)v_i - \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j). \quad (14)$$

The following theorem generalizes the results in [29] to also include nonlinear state-dependent damping, as well as nonlinear interaction functions. With this framework, we are able to generalize the simple average consensus to a much broader class of consensus functions.

Theorem 3: Consider agents with dynamics (7) and u_i given by (14), where $\kappa_i(\cdot)$ satisfies Assumption 1, replacing γ_i with κ_i, and $a_{ij}(\cdot)$ satisfies Assumption 2 for all $i \in \mathcal{V}$ and for all $(i, j) \in \mathcal{E}$, respectively. Furthermore, the interaction functions $a_{ij(\cdot)}$ satisfy $\lim_{x\to\infty}\int_0^x a_{ij}(y)dy = \infty \ \forall(i, j) \in \mathcal{E}$.[1] Then the agents converge to a common point for any initial position $x_i(0)$. Furthermore, the consensus point is uniquely determined by

$$\sum_{i\in\mathcal{V}}\left(\int_0^{x_i^0}\kappa_i(y)dy + v_i(0)\right) = \int_0^{x^*}\sum_{i\in\mathcal{V}}\kappa_i(y)dy. \quad (15)$$

Proof: The proof of convergence also relies on Lyapunov techniques, and LaSalle's invariance principle. A time-invariant function is introduced to characterize the equilibrium set. This function is given by $E(x, v) = \sum_{i\in\mathcal{V}}(\int_0^{x_i}\kappa_i(y)dy + v_i)$. Differentiating $E(x, v)$ along trajectories of (14) yields $\dot{E}(x, v) = -\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) = 0$, by Assumption 2. We first note that by Assumptions 1 and 2, a unique continuous solution of (7) with u_i given by (14) exists for all $t \geq 0$. Consider the candidate Lyapunov function $V(x, v) = \sum_{i\in\mathcal{V}}[v_i^2/2 + \sum_{j\in\mathcal{N}_i}\int_0^{x_i - x_j} a_{ij}(y)dy] \geq 0$. Differentiating $V(x, v)$ along trajectories of (14) yields

$$\dot{V}(x, v) = \sum_{i\in\mathcal{V}} v_i\left(-\kappa_i(x_i)v_i - \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j)\right)$$

$$+ \sum_{i\in\mathcal{V}}\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j)\right)v_i = -\sum_{i\in\mathcal{V}}\kappa_i(x_i)v_i^2 \leq 0.$$

Since $V(x, v)$ is non-increasing under the dynamics (14), it is clear that for $\Omega = \{[x(t), v(t)] : V(x, v) \leq V(x_0, v_0)\}$, it holds that $[\bar{x}(t), v(t)] \in \Omega \ \forall t \geq 0$, where $\bar{x} = \mathcal{B}^T x$ and $x_0 = x(0), v_0 = v(0)$. Clearly Ω is compact. The following lemma is needed to use LaSalle's invariance principle.

Lemma 4: Given the requirements in Theorem 3, $[x(t), v(t)]$ evolve in a compact set, denoted Ω'.

Proof: See Appendix. □

Now let $H = \{(x, y)|\dot{V}(x, v) = 0\} = \{(x, y)|v = 0\}$. Consider any trajectory of (14) with $x \neq x^*(t)1_{n\times 1}$. By (14) and the assumption that $\mathcal{G}$ is connected, $\dot{v}_i \neq 0$ for at least one index i. Thus the largest invariant manifold of E is $\{(x, v)|x = x^*1_{n\times 1}, v = 0\}$. Since Ω' by Lemma 4 is compact and positively invariant, then by LaSalle's invariance principle, the agents converge to a common point $x_i = x^* \ \forall i \in \mathcal{G}$, with $v_i = 0 \ \forall i \in \mathcal{G}$.

It remains to show that the common point to which the agents converge to is the point given by (15), and that the solution is unique. Indeed, consider again the function $E(x, v)$. Since $\dot{E}(x, v) = 0$, and the agents converge to a point x^* with $v_i = 0 \ \forall i \in \mathcal{V}$. It follows that x^* is given by (15). Since $\kappa_i(y) > 0$ by assumption, (15) admits a unique solution. □

The following corollary follows directly from Theorem 3.

Corollary 5: Given n agents starting from rest, i.e., $v_i(0) = 0 \ \forall i \in \mathcal{V}$, with dynamics (7), where u_i is given by (14), the

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[1]We would like to thank the anonymous reviewer for pointing out the necessity of this condition.

agents converge to a common point x^* for any initial position $x_i(0)$, which is uniquely determined by

$$\sum_{i \in \mathcal{V}} \int_0^{x_i^0} \kappa_i(y) \mathrm{d}y = \int_0^{x^*} \sum_{i \in \mathcal{V}} \kappa_i(y) \mathrm{d}y. \qquad (16)$$

Remark 5: In Theorem 1, x^* is given by $\sum_{i \in \mathcal{V}} \int_0^{x_i^0} 1/\gamma_i(y) \mathrm{d}y = \int_0^{x^*} \sum_{i \in \mathcal{V}} 1/\gamma_i(y) \mathrm{d}y$, as opposed to (16) in Corollary 5. The intuition behind this is that in (8), $\gamma_i(x_i)$ acts as a gain of agent i, where an increased $\gamma_i(x_i)$ will increase the speed of agent i. In (14) however, $\kappa_i(x_i)$ acts as damping of agent i, where an increased $\kappa_i(x_i)$ will decrease the speed of agent i.

V. DISTRIBUTED CONTROL WITH INTEGRAL ACTION

Multi-agent systems are, like most control processes, sensitive to disturbances. Generally, static distributed control protocols, such as e.g., P-controllers, cannot reject even constant disturbances. In this section we propose a control protocol for single- and double-integrator dynamics that drives the agents to a common state under static disturbances. By using distributed integral action, we are able to compensate for the disturbances in a distributed setting. Moreover, with the proposed control algorithm, the agents reach the average of their initial positions for arbitrary initial velocities in the absence of disturbances. We study the properties of the control protocols and derive necessary and sufficient conditions under which the multi-agent system is stable.

A. Consensus by Distributed Integral Action for Single-Integrator Dynamics With Damping

Consider agents with single-integrator dynamics (6), with u_i given by

$$u_i = -\sum_{j \in \mathcal{N}_i} \left(b(x_i - x_j) + a \int_0^t (x_i(\tau) - x_j(\tau)) \mathrm{d}\tau \right)$$
$$-\delta(x_i - x_i(0)) \quad (17)$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $\delta \in \bar{\mathbb{R}}^+$ are fixed parameters, and $d_i \in \mathbb{R}$ is an unknown disturbance.

Theorem 6: Under the dynamics (17), with u_i given by (17), the agents converge to a common value x^* for any constant disturbance d_i and any initial condition. If $d_i = 0 \ \forall i \in \mathcal{V}$, the agents converge to $x^* = (1/n) \sum_{i \in \mathcal{V}} x_i(0)$ given $v_i(0) \ \forall i \in \mathcal{V}$. If absolute position measurements are not present, i.e., $\delta = 0$, it still holds that $\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \ \forall i, j \in \mathcal{V}$ for any $d_i \in \mathbb{R}$, $a, b \in \mathbb{R}^+$, while the absolute states diverge, i.e., $\lim_{t \to \infty} |x_i(t)| = \infty \ \forall i \in \mathcal{V}$, unless $1_{1 \times n} d = 0$.

Proof: First consider the case when $\delta = 0$ and $d_i = 0 \ \forall i \in \mathcal{V}$. By introducing the integral states $z = [z_1, \ldots, z_n]^T$ we may rewrite the dynamics (17) in vector form as

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n \times n} & I_n \\ -a\mathcal{L} & -b\mathcal{L} \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ x \end{bmatrix} \qquad (18)$$

together with the initial condition $z(0) = 0_{n \times 1}$. By elementary column operations we note that the characteristic equation of A is given by $0 = \det((a + bs)\mathcal{L} + s^2 I_n)$. By comparing the characteristic polynomial with the characteristic equation of $\mathcal{L}$, being $0 = \det(\mathcal{L} - \kappa I_n)$, with solutions $\kappa = \lambda_i \geq 0$, we obtain the equation $0 = s^2 + \lambda_i b s + \lambda_i a$. This equation has a double root $s = 0$ if $\lambda_i = 0$, and the remaining solutions $s \in \mathbb{C}^-$ if $\lambda_i > 0$. Since the above equation has exactly two solutions for every λ_i, it follows that the algebraic multiplicity of the eigenvalue 0 must be equal to two. It is well-known that for connected graphs $\mathcal{G}$, λ_1 is the only zero-eigenvalue of the Laplacian $\mathcal{L}$. By straightforward calculations we obtain that $e_1 = [1_{1 \times n}, 0_{1 \times n}]^T$ is an eigenvector and $e_2 = [0_{1 \times n}, 1_{1 \times n}]^T$ is a generalized eigenvector of A corresponding to the eigenvalue 0. It can also be verified that $v_1 = (1/n)[1_{1 \times n}, 0_{1 \times n}]$ and $v_2 = (1/n)[0_{1 \times n}, 1_{1 \times n}]$ are a left eigenvector and a generalized left eigenvector of A, respectively, corresponding to the eigenvalue 0. It is easily verified that $v_1 e_1 = 1, v_2 e_2 = 1$ and $v_2 e_1 = 0, v_1 e_2 = 0$. If we let P be a matrix consisting of the normalized eigenvectors of A, we can chose the first columns of P to be e_1 and e_2, and the first rows of P^{-1} to be v_1 and v_2, respectively. Since all remaining eigenvalues of A have strictly negative real part we obtain

$$\lim_{t \to \infty} e^{At} = \lim_{t \to \infty} P e^{Jt} P^{-1}$$

$$= P \lim_{t \to \infty} \begin{bmatrix} 1 & t & 0_{1 \times (2n-2)} \\ 0 & 1 & 0_{1 \times (2n-2)} \\ 0_{(2n-2) \times 1} & 0_{(2n-2) \times 1} & e^{J't} \end{bmatrix} P^{-1}$$

$$= \lim_{t \to \infty} \frac{1}{n} \begin{bmatrix} 1_{n \times n} & t 1_{n \times n} \\ 0_{n \times n} & 1_{n \times n} \end{bmatrix}$$

where J is a Jordan matrix. Thus, given an initial position $x(0) = x_0$, we obtain $\lim_{t \to \infty} x_i(t) = (1/n) \sum_{i \in \mathcal{V}} x_{0,i} \ \forall i \in \mathcal{V}$, i.e., the agents converge to the average of their initial positions.

We now consider the case where $\delta = 0$ and $d_i \neq 0$ for at least one $i \in \mathcal{V}$. Define the output

$$[y_x] = [0_{m \times n} \ \mathcal{B}^T] \begin{bmatrix} z \\ x \end{bmatrix}$$

and consider the linear coordinate change

$$x = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n \times 1} \ S \end{bmatrix} u \quad u = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1 \times n} \\ S^T \end{bmatrix} x$$

$$z = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n \times 1} \ S \end{bmatrix} w \quad w = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1 \times n} \\ S^T \end{bmatrix} z \qquad (19)$$

where S is a matrix such that $[(1/\sqrt{n})1_{n \times 1}, S]$ is an orthonormal matrix. In the new coordinates, the system dynamics become

$$\dot{w} = u$$
$$\dot{u} = \begin{bmatrix} 0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & -a S^T \mathcal{L} S \end{bmatrix} w + \begin{bmatrix} 0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & -b S^T \mathcal{L} S \end{bmatrix} u$$
$$+ \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1 \times n} \\ S^T \end{bmatrix} d. \qquad (20)$$

We note that the states u_1 and w_1 are both unobservable and uncontrollable. We thus omit these states to obtain a minimal realization by defining the new coordinates $u' = [u_2, \ldots, u_n]^T$ and $w' = [w_2, \ldots, w_n]^T$, thus obtaining the system dynamics

$$\begin{bmatrix} \dot{w}' \\ \dot{u}' \end{bmatrix} = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{(n-1)} \\ -aS^T\mathcal{L}S & -bS^T\mathcal{L}S \end{bmatrix} \begin{bmatrix} w' \\ u' \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ S^T d \end{bmatrix}.$$

Clearly $x^T S^T \mathcal{L} S x \geq 0$, with equality only if $Sx = k1_{n\times 1}$. However, since $[(1/\sqrt{n})1_{n\times 1}, S]$ is orthonormal, $1_{1\times n}Sx = 0_{1\times n}x = 0 = k1_{1\times n}1_{n\times 1} = kn$, which implies $k = 0$. Hence $S^T \mathcal{L} S$ is positive definite and thus invertible, and we may define

$$\begin{bmatrix} w'' \\ u'' \end{bmatrix} = \begin{bmatrix} w' \\ u' \end{bmatrix} - \begin{bmatrix} \frac{1}{a}(S^T\mathcal{L}S)^{-1}S^T d \\ 0_{(n-1)\times 1} \end{bmatrix}.$$

It is easily verified that the origin is the only equilibrium of the system dynamics, which in the new coordinates are given by

$$\begin{bmatrix} \dot{w}'' \\ \dot{u}'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{(n-1)} \\ -aS^T\mathcal{L}S & -bS^T\mathcal{L}S \end{bmatrix}}_{\triangleq A} \begin{bmatrix} w'' \\ u'' \end{bmatrix}.$$

By elementary column operations, the characteristic polynomial in κ of A is given by $\det(\kappa^2 I_{(n-1)} + (b\kappa + a)S^T\mathcal{L}S)$. By comparing this polynomial with the characteristic polynomial $\det(sI + S^T\mathcal{L}S)$, which since $S^T\mathcal{L}S$ is positive definite has solutions $-s_i < 0$, we know that the eigenvalues of A must satisfy $\kappa^2 + s_i b\kappa + s_i a = 0$, with solutions $\kappa \in \mathbb{C}^-$. Thus A is Hurwitz. From the dynamics (20), it is clear that $\dot{u}_1 = (1/\sqrt{n})1_{1\times n}d$. Hence $\lim_{t\to\infty} u_1(t) = \pm\infty$ unless $1_{1\times n}d = 0$. Since u' is bounded, by the coordinate change (19), x is bounded iff $1_{1\times n}d = 0$.

Now consider the case where $\delta > 0$ and $d_i = 0 \; \forall i \in \mathcal{V}$. The dynamics can be written as:

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -a\mathcal{L} & -b\mathcal{L} - \delta I \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ \delta x(0) \end{bmatrix}. \quad (21)$$

Define the output of the system

$$[y_x] = [0_{m\times n} \; \mathcal{B}^T] \begin{bmatrix} z \\ x \end{bmatrix}$$

and consider the linear coordinate change

$$z = \begin{bmatrix} \frac{1}{\sqrt{n}}1_{n\times 1} \; S \end{bmatrix} w \quad w = \begin{bmatrix} \frac{1}{\sqrt{n}}1_{1\times n} \\ S^T \end{bmatrix} z \quad (22)$$

where S is a matrix such that $[(1/\sqrt{n})1_{n\times 1}, S]$ is an orthonormal matrix. In the new coordinates, the system dynamics (21) become

$$\begin{bmatrix} \dot{w} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & \begin{bmatrix} \frac{1}{\sqrt{n}}1_{1\times n} \\ S^T \end{bmatrix} \\ [0_{n\times 1} \; -a\mathcal{L}S] & -b\mathcal{L} - \delta I \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ \delta x(0) \end{bmatrix}.$$

We note that the state w_1 is unobservable and uncontrollable. We thus omit this state to obtain a minimal realization by defin-

ing the new coordinates $w' = [w_2, \ldots, w_n]^T$, thus obtaining the system dynamics

$$\begin{bmatrix} \dot{w}' \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)\times(n-1)} & S^T \\ -a\mathcal{L}S & -b\mathcal{L} - \delta I_n \end{bmatrix}}_{\triangleq A} \begin{bmatrix} w' \\ x \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ \delta x(0) \end{bmatrix}. \quad (23)$$

By elementary column operations, the characteristic polynomial of A may be written as $0 = \det(s^2 I_n + s(b\mathcal{L} + \delta I_n) + a\mathcal{L}SS^T) \triangleq \det(Q(s))$. For a given $s \in \mathbb{C}$, the previous equation has a solution only if $x^T Q(s)x = 0$ for some x satisfying $x^T x = 1$. This equation becomes

$$0 = s^2 \underbrace{x^T x}_{a_2} + s \underbrace{x^T(b\mathcal{L} + \delta I_n)x}_{a_1} + \underbrace{ax^T\mathcal{L}SS^T x}_{a_0}.$$

By the Routh-Hurwitz stability criterion, the above equation has all its solutions $s \in \mathbb{C}^-$ if and only if $a_i > 0$ for $i = 0, 1, 2$. Clearly $a_2 = 1$ by definition, while $a_1 > 0$ by the positive definiteness of $\mathcal{L} + \delta I_n$. It is easily shown that $SS^T = I_n - 1/n1_{n\times n}$, by the orthonormality of $[(1/\sqrt{n})1_{n\times 1}, S]$. This implies that $\mathcal{L}SS^T = \mathcal{L}$, implying that $a_0 \geq 0$. If $a_0 > 0$, all solutions are stable, whereas if $a_0 = 0$, $s = 0$ is also a solution. Thus, any eigenvalue of A must be either zero, or have negative real part. However, it is easily verified that A is full rank. Thus 0 cannot be an eigenvalue of A, implying that A is Hurwitz. The first $n - 1$ rows of the equilibrium of (23) imply $S^T x = 0_{(n-1)\times 1}$, which implies $x = x^* 1_{n\times 1}$. Finally, the last n rows of (23) imply $1_{1\times n}(-a\mathcal{L}Sw' - b\mathcal{L}x^* 1_{n\times 1} - \delta x^* 1_{n\times 1} + \delta x(0)) = 0$, so $nx^* = \sum_{i\in\mathcal{V}} x_i(0)$

The case when $\delta > 0$ and $d_i \neq 0$ for at least one $i \in \mathcal{V}$ is analogous to the case when $\delta = 0$ and $d_i \neq 0$, and the proof is thus omitted. □

B. Consensus by Distributed Integral Action for Double-Integrator Dynamics With Damping

Consider agents with double-integrator dynamics (7), with input given by the velocity-damped PI-controller

$$u_i = -\sum_{j\in\mathcal{N}_i} \left(b(x_i - x_j) + a\int_0^t (x_i(\tau) - x_j(\tau))\,\mathrm{d}\tau \right) - \gamma v_i - \delta(x_i - x_i^0) \quad (24)$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $\gamma \in \mathbb{R}^+$, $\delta \in \mathbb{R}^+$ and $d_i \in \mathbb{R}$ is an unknown scalar disturbance. The above protocol does not require communication of the integral state between the agents, as it suffices for each agent to measure its neighbors' states and integrate the relative differences.

Theorem 7: Under the dynamics (7) with u_i given by (24), the agents converge to a common value x^* for any constant disturbance d_i, provided that $a < b\gamma$. If $d_i = 0 \; \forall i \in \mathcal{V}$, the agents converge to $x^* = (1/n)\sum_{i\in\mathcal{V}} x_i(0)$ for arbitrary $v_i(0)$. If absolute position measurements are not present, i.e., $\delta = 0$, it still holds that $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \; \forall i, j \in \mathcal{V}$ for any set of disturbances d_i. However the absolute states are in general unbounded, i.e., $\lim_{t\to\infty} |x_i(t)| = \infty \; \forall i \in \mathcal{V}$, unless $1_{1\times n}d = 0$. Also, in this case the system is stable if and only if $a < b\gamma$.

Proof: The proof follows the same principle ideas as the proof of Theorem 6. However, as second-order dynamics are considered, the problem is inherently different from first-order dynamics. First consider the case where $\delta = 0$. Let also $d_i = 0 \; \forall i \in \mathcal{V}$. By introducing the state vector $z = [z_1, \ldots, z_n]^T$ we may rewrite the dynamics as

$$
\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n \times n} & I_n & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & I_n \\ -a\mathcal{L} & -b\mathcal{L} & -\gamma I_n \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ x \\ v \end{bmatrix}
$$

together with the additional initial condition $z(0) = 0_{n \times 1}$. By elementary column operations it is easily shown that the characteristic polynomial of A can be written as $0 = \det((a + bs)\mathcal{L} + s^2(s + \gamma)I)$, where I is the identity matrix of appropriate dimensions. Comparing the above equation with the characteristic polynomial of $\mathcal{L}$, we get that $0 = s^3 + \gamma s^2 + \lambda_i bs + \lambda_i a$, where λ_i is an eigenvalue of $\mathcal{L}$. If $\lambda_i > 0$, the above equation has all its solutions $s \in \mathbb{C}^-$ if and only if $a < b\gamma$, and $a, b, \gamma > 0$ by the Routh-Hurwitz stability criterion. Since $\mathcal{G}$ by assumption is connected, $\lambda_1 = 0$ and $\lambda_i > 0 \; \forall i = 2, \ldots, n$. For $\lambda_1 = 0$, the above equation has the solutions $s = 0$, $s = -\gamma$. By straightforward calculations it can be shown that $e_1^1 = [1_{1 \times n}, \; 0_{1 \times n}, \; 0_{1 \times n}]^T$ and $e_1^2 = [0_{1 \times n}, \; 1_{1 \times n}, \; 0_{1 \times n}]^T$ are an eigenvector and a generalized eigenvector of A, respectively, corresponding to the eigenvalue 0. Furthermore $v_1 = 1/(\gamma^2 n)[\gamma^2 1_{1 \times n}, \; 0_{1 \times n}, \; -1_{1 \times n}]$ and $v_2 = 1/(\gamma n)[0_{1 \times n}, \; \gamma 1_{1 \times n}, \; 1_{1 \times n}]$ are a generalized left eigenvector and a left eigenvector of A corresponding to the eigenvalue 0. Furthermore $v_1 e_1^1 = 1$, $v_2 e_1^2 = 1$ and $v_2 e_1^1 = 0$, $v_1 e_1^2 = 0$. Hence the first columns of P can be chosen as e_1^1 and e_1^2, and the first rows of P^{-1} can be chosen to be v_1 and v_2. Since all other eigenvalues of A have strictly negative real part we obtain

$$
\begin{aligned}
&\lim_{t \to \infty} e^{At} \\
&= \lim_{t \to \infty} P e^{Jt} P^{-1} \\
&= P \lim_{t \to \infty} \begin{bmatrix} 1 & t & 0_{1 \times (3n-2)} \\ 0 & 1 & 0_{1 \times (3n-2)} \\ 0_{(3n-2) \times 1} & 0_{(3n-2) \times 1} & e^{J't} \end{bmatrix} P^{-1} \\
&= \lim_{t \to \infty} P \begin{bmatrix} 1 & t & 0_{1 \times (3n-2)} \\ 0 & 1 & 0_{1 \times (3n-2)} \\ 0_{(3n-2) \times 1} & 0_{(3n-2) \times 1} & 0_{(3n-2) \times (3n-2)} \end{bmatrix} P^{-1} \\
&= \lim_{t \to \infty} \frac{1}{n} \begin{bmatrix} 1_{n \times n} & t 1_{n \times n} & \frac{t\gamma-1}{\gamma^2} 1_{n \times n} \\ 0_{n \times n} & 1_{n \times n} & \frac{1}{\gamma} 1_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}
\end{aligned}
$$

where J is a Jordan matrix. Given any initial position $x(0) = x_0$, $z(0) = 0_{n \times 1}$, $v(0) = v_0$, we obtain $\lim_{t \to \infty} x_i(t) = (1/n) \sum_{i \in \mathcal{V}} x_{0,i} + (1/\gamma n) \sum_{i \in \mathcal{V}} v_{0,i} \; \forall i \in \mathcal{V}$.

Now let us turn our attention to the case where $\delta > 0$ and $d_i \neq 0$ for at least one $i \in \mathcal{V}$. We define the output of the system as

$$
\begin{bmatrix} y_x \\ y_v \end{bmatrix} = \begin{bmatrix} 0_{m \times n} & \mathcal{B}^T & 0_{m \times n} \\ 0_{m \times n} & 0_{m \times n} & \mathcal{B}^T \end{bmatrix} \begin{bmatrix} z \\ x \\ v \end{bmatrix}
$$

and consider the same linear coordinate change of z, x and v as applied to z and x in the proof of Theorem 6. In the new coordinates the system dynamics are

$$
\begin{aligned}
\dot{z}' &= x' \\
\dot{x}' &= v' \\
\dot{v}' &= \begin{bmatrix} 0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & -aS^T \mathcal{L} S \end{bmatrix} z' + \begin{bmatrix} 0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & -bS^T \mathcal{L} S \end{bmatrix} x' \\
&\quad - \gamma v' + \begin{bmatrix} \frac{1}{n} 1_{1 \times (n)} \\ S^T \end{bmatrix} d.
\end{aligned} \tag{25}
$$

We note that the states z_1, x_1' and v_1' are unobservable and uncontrollable. We thus omit these states to obtain a minimal realization by defining the new coordinates $z'' = [z_2', \ldots, z_n']^T$, $x'' = [x_2', \ldots, x_n']^T$ and $v'' = [v_2', \ldots, v_n']^T$ we obtain the system dynamics

$$
\begin{bmatrix} \dot{z}'' \\ \dot{x}'' \\ \dot{v}'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)^2} & I_{(n-1)^2} & 0_{(n-1)^2} \\ 0_{(n-1)^2} & 0_{(n-1)^2} & I_{(n-1)^2} \\ -aS^T \mathcal{L} S & -bS^T \mathcal{L} S & -\gamma I_{(n-1)^2} \end{bmatrix}}_{\triangleq A'} \begin{bmatrix} z'' \\ x'' \\ v'' \end{bmatrix}
$$
$$
+ \begin{bmatrix} 0_{(n-1) \times 1} \\ 0_{(n-1) \times 1} \\ S^T d \end{bmatrix}.
$$

We now shift the state space by defining

$$
\begin{bmatrix} z^{(3)} \\ x^{(3)} \\ v^{(3)} \end{bmatrix} = \begin{bmatrix} z'' \\ x'' \\ v'' \end{bmatrix} - \begin{bmatrix} \frac{1}{a}(S^T \mathcal{L} S)^{-1} S^T d \\ 0_{(n-1) \times 1} \\ 0_{(n-1) \times 1} \end{bmatrix}.
$$

It is easily verified that the origin is the only equilibrium of the system dynamics, and that the stability in the new coordinates is characterized by the matrix A'. By a similar argument used when showing that A has eigenvalues with non-positive real part, we may show that A' has eigenvalues with non-positive real part. But since $S^T \mathcal{L} S$ is full-rank, A' must also be full-rank, and hence A' is Hurwitz. Thus $\lim_{t \to \infty} x^{(3)} = \lim_{t \to \infty} x'' = 0_{(n-1) \times 1}$, which implies that $\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \; \forall i, j \in \mathcal{V}$, even in the presence of disturbances d_i. It is also clear that whenever $a \geq b\gamma$, at least one eigenvalue will have non-negative real part, and that its (generalized) eigenvector will be distinct from e_1 and e_2. From the dynamics (25), it is clear that $\dot{x}_1' = (1/n) 1_{1 \times n} d$. Hence $\lim_{t \to \infty} x_1'(t) = \pm\infty$ unless $1_{1 \times n} d = 0$. Since x'' is bounded, by the coordinate change (19), x is bounded if and only if $1_{1 \times n} d = 0$.

The stability analysis of the case when $\delta > 0$ is analogous to the corresponding part of the proof of Theorem 6, and hence omitted. If $d_i = 0 \; \forall i \in \mathcal{V}$, stationarity of $v(t)$ implies: $\lim_{t \to \infty} 1_{1 \times n}(-a\mathcal{L}z - b\mathcal{L}x - \delta x + \delta x(0) - \gamma v) = 0$, so $nx^* = \sum_{i \in \mathcal{V}} x_i(0)$. $\quad\square$

VI. Motivating Applications Revisited

A. Thermal Energy Storage in Smart Buildings

We here return to the example of thermal energy storage in smart buildings, introduced in Section II-A. Recall that the temperatures dynamics in the rooms can be described by (1).

The heat conductivity a is assumed to be constant and uniform, implying $a_{ij}(x) = a \; \forall (i, j) \in \mathcal{E}$, where $a = 0.5 \text{ W}/°\text{C}$.
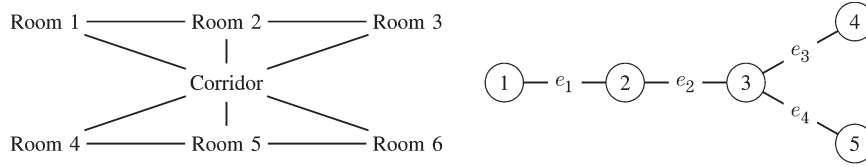
Fig. 1. The leftmost figure illustrates the floor topology. The rightmost figure shows the communication topology of the space satellites.
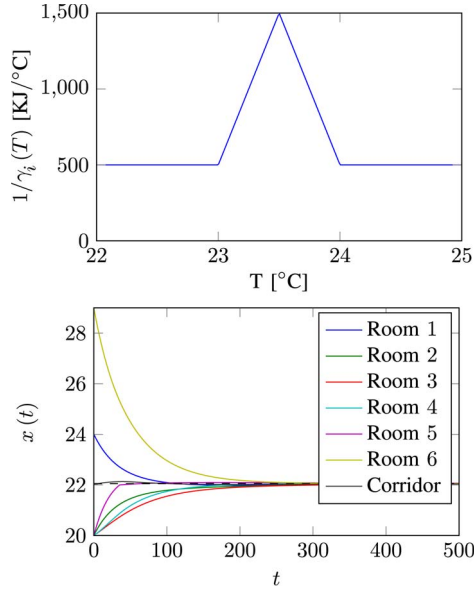

Fig. 2. The top figure shows the heat capacities of Room 2 and 5. The bottom figure shows the temperatures in the building floor.

Consider the floor topology in Fig. 1. Let the desired maximum temperature be given by $t_b = 23°C$. The heat capacity is assumed to be given by Fig. 2 for $i \in \{\text{Room 2}, \text{Room 5}\}$ due to thermal energy storage installations, and $1/\gamma_i(T) = 50 \text{ kJ}/°C$ for $i \in \{\text{Room 1}, \text{Room 3}, \text{Room 4}, \text{Room 6}, \text{Corridor}\}$ where no thermal energy storage is installed. The initial temperatures was assumed to be 29°C for room 6, 24°C for room 1, 22°C for the corridor and 20°C for the other rooms. The temperatures as a function of time are shown in Fig. 2 for given initial temperatures. We note that the temperature in room 2 and 5 never exceeds the desired maximum temperature $t_b = 23°C$ due to the thermal energy storage, and that the temperatures converge to a temperature below t_b in all rooms, in accordance with Theorem 1.

B. Autonomous Space Satellites

Consider a group of autonomous space satellites with unitary masses. The satellites are denoted as 1, ..., 5, and their communication topology is given by the undirected graph in Fig. 1. The objective is to reach consensus in one dimension by a distributed control law using only relative position and velocity measurements. The raw control signal is the power applied to the agent's engine, P_i. However, the acceleration in an inertial reference frame is $a_i = P_i/|v_i|$ due to $P_i = \langle F_i, v_i \rangle$ and P_i being parallel to the agent's velocity v_i. We assume that the agents only have access to relative measurements, and hence are unaware of their absolute positions. Assuming

that $a_i = P_i/(|v_i| + c)$, $c > 0$, to ensure that the accelerations remain bounded, this scenario can be modelled by (12). This is clearly a special case of the dynamics (7) with u_i given by (12). The interaction functions in this example are chosen to be $a_{ij}(y) = 2b_{ij}(y) = 20(e^{|y|} - 1)\text{sgn}(y) \ \forall (i,j) \in \mathcal{E}$, which satisfies Assumption 2. It is clear that this situation cannot be modelled by any previously proposed linear consensus protocols. Fig. 3 shows the state trajectories for different initial conditions. As predicted by Theorem 2, consensus is reached, and the final consensus velocity, as seen from an observer, can be calculated by (15).

C. Mobile Robot Coordination Under Disturbances

In this section we revisit the example of mobile robots from Section II-C. The dynamics of the robots are given by (3). Let the damping coefficient be given by $\gamma = 3$, and the static gain $b = 5$. We consider the system with a constant disturbance $d = [1, 0, 0, 0, 0]^T$, and for the different integral gains $a = 0$, $a = 1$, and $a = 15$. The initial conditions are given by $x(0) = [5, -6, 8, 4, 5]^T$, $v(0) = [0, 0, 0, 0, 0]^T$. The setup we will consider consists of a string of 5 mobile robots, whose communication topology is a string graph.

By Theorem 7 stability is guaranteed if and only if $a < b\gamma$. In Fig. 4, the state trajectories are shown for different choices of a. We observe that asymptotic consensus amongst the mobile robots is only reached for $a = 1$. For $a = 0$, consensus is not reached due to the presence of a static disturbance. When $a = 1$, the disturbance is attenuated by the integrators, and asymptotic consensus is reached. However, as a is increased to $15 = b\gamma$, the system becomes marginally stable, i.e., stable but not asymptotically stable. By increasing a further to 20, the system becomes unstable, in accordance with Theorem 7.

D. Frequency Control of Power Systems

In this section we demonstrate that a similar protocol to the one proposed in Section V-B, can be employed for frequency control of power systems. Let us consider a power system, whose topology modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each node, here referred to as a bus, is assumed to obey the linearized swing (4). By defining $\delta = [\delta_1 \ldots, \delta_n]^T$, we may rewrite (4) as

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -M\mathcal{L}_k & -MD \end{bmatrix} \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ Mp^m \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ Mu \end{bmatrix} \quad (26)$$

where $M = \text{diag}(1/m_1, \ldots, 1/m_n)$, $D = \text{diag}(d_1, \ldots, d_n)$, $\mathcal{L}_k$ is the weighted Laplacian with edge weights k_{ij}, $p^m = [p_1^m, \ldots, p_n^m]^T$ and $u = [u_1, \ldots, u_n]^T$.
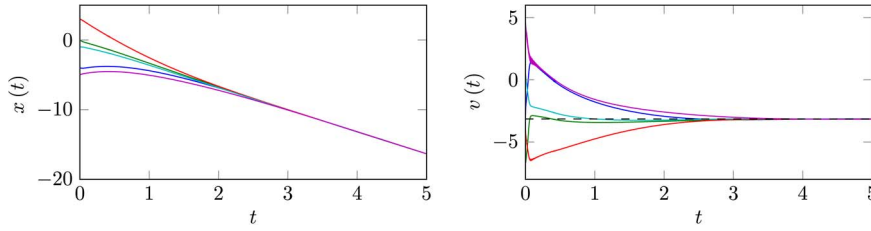
Fig. 3. The leftmost figure shows the positions of the satellites for the initial conditions $x(0) = [-4, 0, 3, -1, -5]^T$, $v(0) = [-3, -7, 3, -1, 0]^T$, while their velocities.
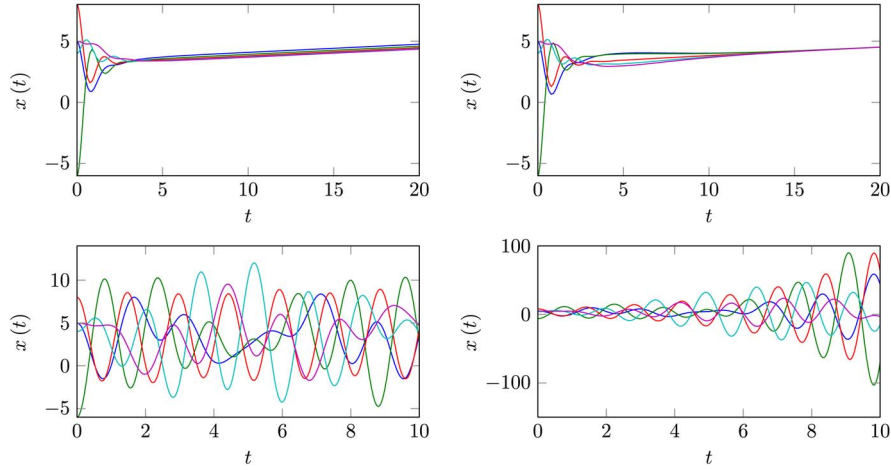

Fig. 4. The upper left figure shows the state trajectories of (3) when $a = 0$, the upper right figure shows the state trajectories when $a = 1$, the lower left figure shows the state trajectories when $a = 15$, and the lower right figure shows the state trajectories when $a = 20$.
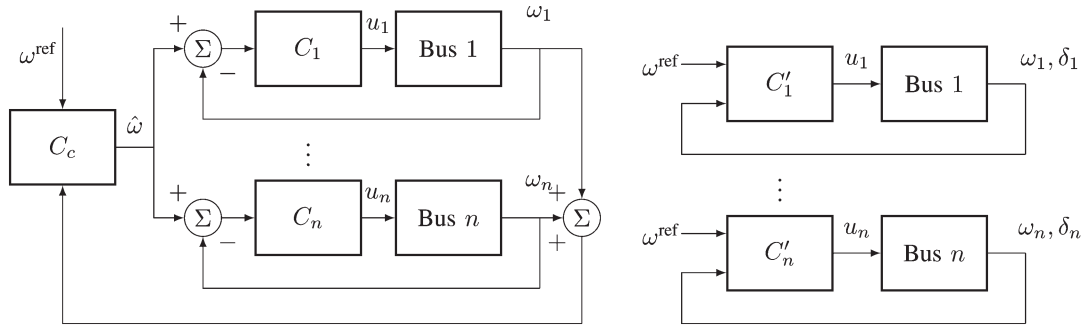

Fig. 5. The left figure illustrates the centralized controller, while the right figure illustrates the decentralized controller.

1) Centralized PI Control: We will here present a a centralized frequency control protocol for power systems and analyze its stability properties. Traditionally, frequency control of a power systems is carried out at two levels, see e.g., [18]. In the first level, the frequency is controlled with a proportional controller against a dynamic reference frequency. At the second level, the dynamic reference frequency is controlled with a proportional controller to eliminate static errors. We model the first level, proportional controller of an arbitrary bus i as

$$u_i = a(\hat{\omega} - \omega_i). \tag{27}$$

The second level proportional controller, regulating $\hat{\omega}$ is assumed to be given by

$$\dot{\hat{\omega}} = b\left(\omega^{\text{ref}} - \frac{1}{n}\sum_{i\in\mathcal{V}}\omega_i\right) \tag{28}$$

where we have assumed that the average frequency of the buses is measured by the central controller.[2] Note that the second level controller integrates $\hat{\omega}$, thus acting as an integral controller. The centralized controller architecture is illustrated in Fig. 5.

Proposition 1: The power system described by (26) where u_i is given by (27), (28) satisfies $\lim_{t\to\infty}\omega_i(t) = \omega^{\text{ref}}$ for any set of initial conditions, given that $a, b > 0$.

Proof: We may write (26) with u given by (27), (28) as

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 0_{1\times n} & -\frac{b}{n}1_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} & I_n \\ aM1_{n\times 1} & -M\mathcal{L}_k & -MD - aM \end{bmatrix} \begin{bmatrix} \hat{\omega} \\ \delta \\ \omega \end{bmatrix} + \begin{bmatrix} b\omega^{\text{ref}} \\ 0_{n\times 1} \\ Mp^m \end{bmatrix}. \tag{29}$$

---

[2] In reality the frequency is often measured at a specific bus. This will typically lead to longer delays, since disturbances need to propagate through the system before control action can be taken.

Define the output of the system as

$$[\,y_\omega\,] = [\,0_{n\times 1} \quad 0_{n\times n} \quad I_n\,] \begin{bmatrix} \hat{\omega} \\ \delta \\ \omega \end{bmatrix}$$

and consider the linear coordinate change

$$\delta = \begin{bmatrix} \frac{1}{\sqrt{n}}1_{n\times 1} \, S \end{bmatrix} \delta' \quad \delta' = \begin{bmatrix} \frac{1}{\sqrt{n}}1_{1\times n} \\ S^T \end{bmatrix} \delta \qquad (30)$$

where S is a matrix such that $[(1/\sqrt{n})1_{n\times 1}, S]$ is an orthonormal matrix. In the new coordinates, the system dynamics (29) become

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}' \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 0_{1\times n} & -\frac{b}{n}1_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} & \begin{bmatrix} \frac{1}{\sqrt{n}}1_{1\times n} \\ S^T \end{bmatrix} \\ aM1_{n\times 1} \, [0_{n\times 1} \, -M\mathcal{L}_k S] & -MD - aM \end{bmatrix} \begin{bmatrix} \hat{\omega} \\ \delta' \\ \omega \end{bmatrix}$$
$$+ \begin{bmatrix} b\omega^{\mathrm{ref}} \\ 0_{n\times 1} \\ Mp^m \end{bmatrix}.$$

We note that δ_1' is unobservable, and hence omit this state by defining $\delta'' = [\delta_2', \ldots, \delta_n']^T$. In these coordinates the system dynamics (29) become

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}'' \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0_{1\times(n-1)} & -\frac{b}{n}1_{1\times n} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & S^T \\ aM1_{n\times 1} & -M\mathcal{L}_k S & -MD - aM \end{bmatrix}}_{\triangleq A} \begin{bmatrix} \hat{\omega} \\ \delta' \\ \omega \end{bmatrix}$$
$$+ \begin{bmatrix} b\omega^{\mathrm{ref}} \\ 0_{n\times 1} \\ Mp^m \end{bmatrix}. \qquad (31)$$

By elementary row and column operations, we may rewrite the characteristic equation of A as $\det(M\mathcal{L}_k SS^T + (ab/n)M1_{n\times n} + s(MD + aM) + s^2 I_n)= 0$. This equation is equivalent to

$$\det\left(\mathcal{L}_k SS^T + \frac{ab}{n}1_{n\times n} + s(D + aI_n) + s^2 M^{-1}\right) = 0. \quad (32)$$

For a given s, this equation has one solution only if $x^T(\mathcal{L}_k SS^T + (ab/n)1_{n\times n} + s(D + aI_n) + s^2 M^{-1})x = 0$ has a solution for some $\|x\| = 1$. Hence, if the previous equation has all its solutions in $\mathbb{C}^-$ for all $\|x\| = 1$, then (32) has all its solutions in $\mathbb{C}^-$. By similar arguments used in the proof of Theorem 6, we can show that $\mathcal{L}_k SS^T = \mathcal{L}_k$. Thus, if the equation

$$\underbrace{x^T\left(\mathcal{L}_k + \frac{ab}{n}1_{n\times n}\right)x}_{a_0} + s\underbrace{x^T(D + aI_n)x}_{a_1} + s^2\underbrace{x^T M^{-1}x}_{a_2} = 0$$

has all its solutions in $\mathbb{C}^-$, then A has only one zero eigenvalue, and all other eigenvalues in $\mathbb{C}^-$. By the Routh-Hurwitz stability criterion, the above equation has all its solutions in $\mathbb{C}^-$ iff $a_i > 0$, $i = 0, 1, 2$. Clearly $a_0 > 0$ since $x^T\mathcal{L}x > 0$ for $x \neq c1_{n\times 1}$, where $c \in \mathbb{R}$, and $1_{n\times n}1_{n\times 1} \neq 0_{n\times 1}$. Also $a_1, a_2 > 0$ since $D + aI_n$ and M^{-1} are diagonal with positive elements. We conclude thus that A is Hurwitz. Stationarity of (31) implies $\omega = \omega^{\mathrm{ref}}1_{n\times 1}$. □

Decentralized PI Control: In this section we analyze a decentralized frequency controller, where each bus controls its own frequency based only on local phase and frequency measurements. Thus, there is no need to send control signals or reference values to the buses. This controller architecture might be favorable due to security concerns when sending unencrypted data over large areas. Another benefit is improved performance when the tripping of one or several power lines causes the network to be split up into two or more sub-networks, so called islanding. The controller of node i is assumed to be given by (5), here written as

$$\dot{z}_i = \omega_i - \omega^{\mathrm{ref}} \qquad (33)$$
$$u_i = a(\omega^{\mathrm{ref}} - \omega_i) - bz_i. \qquad (34)$$

The controller architecture is illustrated in Fig. 5. The decentralized controller (33), (34) is typically not practically feasible with only frequency measurements available at the generation buses. Even the slightest measurement error will be integrated and cause instability, see, e.g., [18]. However, with recent advances in phasor measurement unit (PMU) technology however, phase measurements are becoming increasingly available [23]. By integrating (33) we obtain $z_i = \omega^{\mathrm{ref}}t - \delta_i$. This implies that in order to accurately estimate the integral state z_i, each generator bus needs access only to time and phase measurements, both provided by PMU's with high accuracy.

Proposition 2: The power system described by (26) where u_i is given by (33), (34) satisfies $\lim_{t\to\infty} \omega_i(t) = \omega^{\mathrm{ref}}$ for any set of initial conditions, given that $a, b > 0$.

Proof: If we consider $[\mathcal{B}^T\delta, \omega]$ to be the output, the dynamics of (26) may be modified as long as the dynamics of $[\mathcal{B}^T\delta, \omega]$ are left unchanged. We thus may rewrite (26) with u given by (33), (34) as

$$\begin{bmatrix} \dot{z} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n \\ -M\mathcal{L}_k - bM & -MD - aM \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ \omega \end{bmatrix}$$
$$+ \begin{bmatrix} -\omega^{\mathrm{ref}}1_{n\times 1} \\ M(p^m + a\omega^{\mathrm{ref}}1_{n\times 1}) \end{bmatrix}$$

since $\dot{\delta} - \dot{z} = \omega^{\mathrm{ref}}1_{n\times 1}$, implying that $\delta - z = \delta_0 - t\omega^{\mathrm{ref}}1_{n\times 1}$. Since $\mathcal{L}_k 1_{n\times 1} = 0_{n\times 1}$, the output dynamics of the above equation is equivalent to that of (26) with respect to the output $[\mathcal{B}^T\delta, \omega]$. By elementary column operations, we may rewrite the characteristic equation of A as

$$\det(s^2 I_n + sMD + M\mathcal{L} + bI_n) = 0 \Leftrightarrow$$
$$\det(s^2 M^{-1} + sD + \mathcal{L} + bM^{-1}) = 0. \qquad (35)$$

For a given s, the above equation has a solution only if $x^T(s^2 M^{-1} + sD + \mathcal{L} + bM^{-1})x = 0$ has a solution. Hence, if the previous equation has all its solutions in $\mathbb{C}^-$ for all $\|x\| = 1$, then (35) has all its solutions in $\mathbb{C}^-$. Thus, if the equation

$$\underbrace{x^T(\mathcal{L} + bM^{-1})x}_{a_0} + s\underbrace{x^T Dx}_{a_1} + s^2\underbrace{x^T M^{-1}x}_{a_2} = 0$$
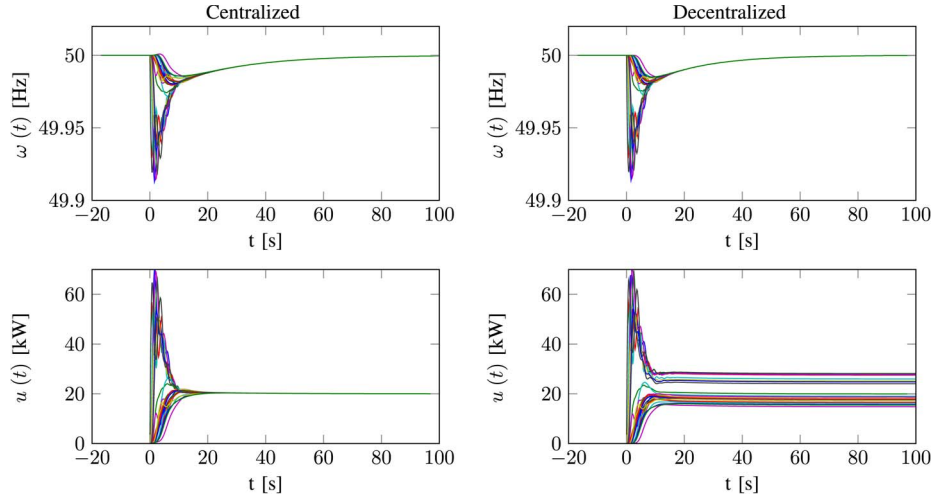
Fig. 6.   The upper left figure shows the bus frequencies with centralized frequency control, while the lower left figure shows the control signals at all buses. The upper right figure shows the bus frequencies with decentralized frequency control, while the lower right figure shows the control signals at all buses. The controller parameters were $a = 0.8$, $b = 0.04$, for both the centralized and the decentralized controller.
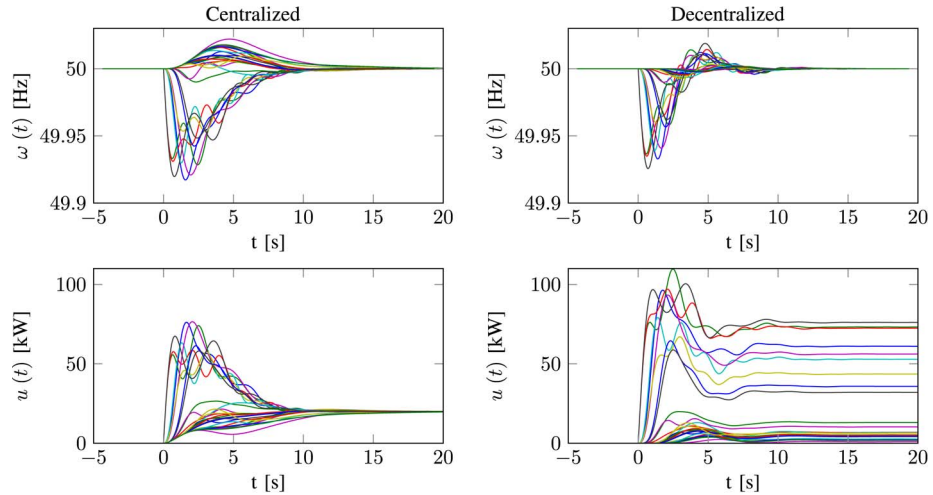

Fig. 7.   The upper left figure shows the bus frequencies with centralized frequency control, while the lower left figure shows the control signals at all buses. The upper right figure shows the bus frequencies with decentralized frequency control, while the lower right figure shows the control signals at all buses. The controller parameters were $a = 0.8$, $b = 0.8$, for both the centralized and the decentralized controller.

has all its solutions in $\mathbb{C}^-$, then A is stable. By the Routh-Hurwitz stability criterion, the above equation has all its solutions in $\mathbb{C}^-$ iff $a_i > 0$, $i = 0$, 1, 2. Clearly $a_0 > 0$ since $x^T \mathcal{L} x > 0$ for $x \neq c1_{n \times 1}$ for any $c \in \mathbb{R}$, and $M^{-1}1_{n \times 1} \neq 0_{n \times 1}$. Also $a_1, a_2 > 0$ since D and M^{-1} are diagonal with positive elements. We conclude thus that A is Hurwitz. Now consider the coordinate shift

$$\begin{bmatrix} z' \\ \omega' \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} - \begin{bmatrix} z_o \\ \omega_o \end{bmatrix}$$

where

$$z_0 = (bI_n + \mathcal{L}_k)^{-1}(D\omega^{\mathrm{ref}}1_{n \times 1} - p^m), \quad \omega_0 = \omega^{\mathrm{ref}}1_{n \times 1}.$$

In the translated coordinates, the origin is the only equilibrium of the system. Hence $\lim_{t \to \infty} \omega_i(t) = \omega^{\mathrm{ref}} \; \forall i \in \mathcal{V}$.   □

2) Simulations: The centralized and decentralized frequency controllers were tested on the IEEE 30 bus test system [1]. The line admittances were extracted from [1] and the

voltages were assumed to be 132 kV for all buses. The values of M and D were assumed to be given by $m_i = 10^5$ kgm^2 and $d_i = 1 \; s^{-1}$, respectively, $\forall i \in \mathcal{V}$. The power system was assumed to be initially in an operational equilibrium, until the power load is increased by a step of 200 kW in the buses 2, 3 and 7. This will immediately result in decreased frequencies at these buses. The frequency controllers at the buses will then control the frequencies towards the reference frequency of $\omega^{\mathrm{ref}} = 50$ Hz. The controller parameters were set to $a = 0.8$, $b = 0.04$, for both the centralized and the decentralized controller. The step responses of the frequencies are plotted in Fig. 6. We note that for the centralized PI controller, the generation is increased uniformly among the generators. If however the integral action is distributed amongst the generators, some generators will increase their generation more than others. Fig. 7 shows the step response under the significantly larger integral action, $\beta = 0.8$ for both the centralized and the decentralized controller. We notice that the step response of the decentralized controller shows better performance compared to

the centralized controller. This is due to that the centralized controller can only measure the average frequency in the power system, as opposed to the individual frequencies which the decentralized controller can measure.

Note that the controller parameters are assumed to be identical for both the centralized and the decentralized controller. This might not be restrictive when the generators are homogeneous. However, when generators are more heterogeneous, this might be restrictive.

VII. CONCLUSION

In this paper we have studied a class of nonlinear consensus protocols for single and double-integrator dynamics. Necessary and sufficient conditions for consensus were derived for static communication topologies under single and double-integrator dynamics. In all cases, expressions for the convergence points were given. We have also studied consensus controllers with integral action for agents with single integrator dynamics and agents with damped double-integrator dynamics. We proved that with the proposed consensus controllers, the agents reach asymptotic consensus even in the presence of constant disturbances. If we allow for absolute position measurements the agents in addition converge asymptotically to a common state. In the absence of disturbances, the proposed consensus protocols asymptotically solve the initial average consensus problem. We have demonstrated by simulations that the proposed controllers have applications in controlling autonomous satellites in space, control of mobile robots, building temperature control and frequency control of electrical power systems.

APPENDIX
PROOF OF LEMMA 4

Since Ω is compact, the relative states $\bar{x}$ are bounded. Then clearly x is bounded if and only if the average $x' = (1/n) \sum_{i \in \mathcal{V}} x_i$ is also bounded, as seen by the following inequalities:

$$\|x\|_\infty < n\|\bar{x}\|_\infty + |x'|$$

$$|x'| = \left| \frac{1}{n} \sum_{i \in \mathcal{V}} x_i \right| < \frac{1}{n}\|x\|_\infty.$$

Let $E_0 = E(x_0, v_0) = \sum_{i \in \mathcal{V}} (\int_0^{x_i} \kappa_i(y) \mathrm{d}y + v_i)$, where $[x_0, v_0]$ denotes the initial condition. Since $[\bar{x}(t), v(t)]$ evolve in the compact set Ω, $v_i(t)$ is also bounded. Hence $\forall i \in \mathcal{V} \exists M \in \mathbb{R}^+ : |v_i(t)| \leq M \quad \forall t \geq 0, \quad \forall i \in \mathcal{V}$. By Assumption 1 $\kappa_i(x) \geq \underline{\kappa} > 0 \, \forall i \in \mathcal{V}$, $\forall x \in \mathbb{R}$. Using these inequalities we obtain

$$\left| \sum_{i \in \mathcal{V}} \int_0^{x_i} \kappa_i(y) \mathrm{d}y \right| \leq nM + |E_0|. \tag{36}$$

Assume for the sake of contradiction that $x'(t)$ is unbounded. Let us consider the case when $x'(t) \to +\infty$. Since $\bar{x}$ is bounded by say $M' > 0$ in the ∞-norm, and $\mathcal{G}$ is connected, $|x_i(t) - x_j(t)|$ is bounded by $(n-1)M' \; \forall i, j \in \mathcal{V}$.

Thus $x_i(t) > 0 \; \forall i \in \mathcal{V}$ whenever $x'(t) > (n-1)M'$. Thus, if $x'(t) > (n-1)M'$, we obtain that $\sum_{i \in \mathcal{V}} \int_0^{x_i} \kappa_i(y) \mathrm{d}y \geq \sum_{i \in \mathcal{V}} \underline{\kappa} x_i$. By assumption, $x'(t)$ is unbounded, implying that also $\sum_{i \in \mathcal{V}} x_i(t)$ is unbounded. Thus $\exists t_1 : \sum_{i \in \mathcal{V}} x_i(t_1) > \max\{1/\underline{\kappa}(nM + |E_0|), M'\}$. But this contradicts (36). Hence $x'(t)$ must be bounded. The cases when $x'(t) \to -\infty$ as well as the case when no limit of $x'(t)$ exists are treated analogously. We conclude that x must be bounded, and thus Ω' is compact by the Heine-Borel Theorem. $\square$

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Henrik Sandberg (M'04) received the M.Sc. degree in engineering physics and the Ph.D. degree in automatic control from Lund University, Lund, Sweden, in 1999 and 2004, respectively.

He is an Associate Professor with the Automatic Control Laboratory, KTH Royal Institute of Technology, Stockholm, Sweden. From 2005 to 2007, he was a Post-Doctoral Scholar with the California Institute of Technology, Pasadena, USA. In 2013, he was a Visiting Scholar at the Laboratory for Information and Decision Systems (LIDS), MIT, Cambridge, USA. He has also held visiting appointments with the Australian National University and the University of Melbourne, Australia. His current research interests include secure networked control, power systems, model reduction, and fundamental limitations in control.

Dr. Sandberg received the Best Student Paper Award from the IEEE Conference on Decision and Control in 2004 and an Ingvar Carlsson Award from the Swedish Foundation for Strategic Research in 2007. He is currently an Associate Editor of the IFAC Journal *Automatica*.

Martin Andreasson (S'12) received the M.Sc. degree in engineering physics from the KTH Royal Institute of Technology, Stockholm, Sweden, in 2011 where he is currently pursuing the Ph.D. degree.

His research interests include distributed control of multi-agent systems, and control of power systems.

Dimos V. Dimarogonas (M'10) was born in Athens, Greece, in 1978. He received the Diploma in electrical and computer engineering and the Ph.D. in mechanical engineering from the National Technical University of Athens (NTUA), Greece, in 2001 and 2006, respectively.

Between May 2007 and February 2009, he was a Postdoctoral Researcher at the Automatic Control Laboratory, School of Electrical Engineering, ACCESS Linnaeus Center, Royal Institute of Technology (KTH), Stockholm, Sweden. Between February 2009 and March 2010, he was a Postdoctoral Associate at the Laboratory for Information and Decision Systems (LIDS) at the Massachusetts Institute of Technology (MIT), Boston. He is currently an Assistant Professor at the Automatic Control Laboratory, ACCESS Linnaeus Center, Royal Institute of Technology (KTH), Stockholm. His current research interests include multi-agent systems, hybrid systems and control, robot navigation and networked control.

Dr. Dimarogonas received a Docent in Automatic Control from KTH in 2012. He serves in the Editorial Board of Automatica and the IET Control Theory and Applications and is a member of the Technical Chamber of Greece.

Karl Henrik Johansson (F'13) received the M.Sc. and Ph.D. degrees in electrical engineering from Lund University, Lund, Sweden.

He is Director of the KTH ACCESS Linnaeus Centre and Professor at the School of Electrical Engineering, Royal Institute of Technology, Sweden. He is a Wallenberg Scholar and has held a six-year Senior Researcher Position with the Swedish Research Council. He is Director of the Stockholm Strategic Research Area ICT The Next Generation.He has held visiting positions at UC Berkeley (1998–2000) and California Institute of Technology (2006–2007). His research interests are in networked control systems, hybrid and embedded system, and applications in transportation, energy, and automation systems.

Dr. Johansson received the Best Paper Award of the IEEE International Conference on Mobile Ad-hoc and Sensor Systems. In 2009, he was also awarded Wallenberg Scholar, as one of the first ten scholars from all sciences by the Knut and Alice Wallenberg Foundation. He was awarded an Individual Grant for the Advancement of Research Leaders from the Swedish Foundation for Strategic Research in 2005. He received the triennial Young Author Prize from IFAC in 1996 and the Peccei Award from the International Institute of System Analysis, Austria, in 1993. He received Young Researcher Awards from Scania in 1996 and from Ericsson in 1998 and 1999. He has been a member of the IEEE Control Systems Society Board of Governors and the Chair of the IFAC Technical Committee on Networked Systems. He has been on the Editorial Boards of several journals, including *Automatica*, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and *IET Control Theory and Applications*. He is currently on the Editorial Board of IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS and the *European Journal of Control*. He has been Guest Editor for special issues, including the one on "Wireless Sensor and Actuator Networks" of IEEE TRANSACTIONS ON AUTOMATIC CONTROL 2011. He was the General Chair of the ACM/IEEE Cyber-Physical Systems Week 2010 in Stockholm and IPC Chair of many conferences. He has served on the Executive Committees of several European research projects in the area of networked embedded systems.