# Generalized Sarymsakov Matrices

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*Abstract*—**Within the set of stochastic, indecomposable, aperiodic (SIA) matrices, the class of Sarymsakov matrices is the *largest* known subset that is closed under matrix multiplication, and more critically whose compact subsets are all consensus sets. This paper shows that a *larger* subset with these two properties can be obtained by generalizing the standard definition for Sarymsakov matrices. The generalization is achieved by introducing the notion of the SIA index of a stochastic matrix, whose value is 1 for Sarymsakov matrices, and then exploring those stochastic matrices with larger SIA indices. In addition to constructing the larger set, this paper introduces another class of generalized Sarymsakov matrices, which contains matrices that are not SIA, and studies their products. Sufficient conditions are provided for an infinite product of matrices from this class, converging to a rank-one matrix. Finally, as an application of the results just described and to confirm their usefulness, a necessary and sufficient combinatorial condition, the "avoiding set condition," for deciding whether or not a compact set of stochastic matrices is a consensus set is revisited. In addition, a necessary and sufficient combinatorial condition is established for deciding whether or not a compact set of doubly stochastic matrices is a consensus set.**

*Index Terms*—**Cooperative control, doubly stochastic matrices, multi-agent systems, products of stochastic matrices, Sarymsakov matrices.**

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## I. INTRODUCTION

**O**VER the last decade, there has been considerable interest in consensus problems that are concerned with a network of agents trying to agree on a specific value of some variable [2]–[13]. Similar research problems have arisen decades ago in statistics [14] and computer science [15]. While different aspects of consensus processes, such as convergence rates [16]–[18], measurement delays [16], stability [6], [19], controllability [20], and robustness [21], have been investigated, and many variants of consensus problems, such as average consensus [22], asynchronous consensus [16], quantized consensus [23]–[26], group consensus [27], [28], constrained consensus [29], and modulus consensus [30]–[34], have been proposed and studied, some fundamental issues regarding linear discrete-time consensus processes still remain open, one of which can be stipulated in precise terms as follows.

A linear discrete-time consensus process is typically modeled by a linear recursion equation of the form

$$x(k + 1) = P(k)x(k), \quad k \geq 1 \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ and each $P(k)$ is an $n \times n$ stochastic matrix. It is well known that reaching a consensus for any initial state in this model is equivalent to the convergence of the product $P(k) \cdots P(2)P(1)$ to a rank-one matrix as $k$ goes to infinity. Sufficient conditions for such an infinite product of stochastic matrices converging to a rank-one matrix have been widely studied in the literature; see, for example, [2], [4], [6], [7], [10], [11], and [13].

In this context, one fundamental issue that comes up is that, given a set of $n \times n$ stochastic matrices $\mathcal{P}$, what the conditions on $\mathcal{P}$ are such that for any infinite sequence of matrices $P(1), P(2), P(3), \ldots$ from $\mathcal{P}$, the sequence of left-products $P(1), P(2)P(1), P(3)P(2)P(1), \ldots$ converges to a rank-one matrix. We will call $\mathcal{P}$ satisfying this property a *consensus set* (the formal definition will be given in the next section). The existing literature on characterizing a consensus set can be traced back to at least the work of Wolfowitz [35] in which stochastic, indecomposable, aperiodic (SIA) matrices have been introduced. Recently, it has been shown in [36] that the problem of deciding whether $\mathcal{P}$ is a consensus set or not is NP-hard; a combinatorial necessary and sufficient condition for such a decision has also been provided there as well. Even in the light of these classical as well as recent findings, the following fundamental question remains: What is the largest subset of the class of $n \times n$ stochastic matrices whose compact subsets are all consensus sets? In [37], this question is answered under the assumption that each stochastic matrix has positive diagonal en-

tries. For general stochastic matrices, however, the question has remained open. This paper aims at addressing this challenging question by studying some well-known classes of SIA matrices.

It is known that the set of Sarymsakov matrices, first introduced by Sarymsakov [38] and redefined in [39], forms a semigroup [40] and is the *largest* known subset of the class of stochastic matrices whose compact subsets are all consensus sets; in particular, the set is closed under matrix multiplication, and any infinitely long left-product of the elements from any of its compact subsets converges to a rank-one matrix [41]. In this paper, we construct a *larger* set of stochastic matrices whose compact subsets are all consensus sets. The key idea is to generalize the definition of the Sarymsakov matrices so that the original set of Sarymsakov matrices is contained as a proper subset.

In this paper, we introduce two approaches to generalize the definition, and thus study two classes of generalized Sarymsakov matrices and their products. The first class of generalized Sarymsakov matrices, called Type-I generalized Sarymsakov matrices, makes use of the concept of the SIA index of a stochastic matrix (the formal definition will be given in Section III). We show that the set of $n \times n$ stochastic matrices with SIA indices no larger than $k$ is closed under matrix multiplication only when $k = 1$, which turns out to be the original Sarymsakov class. This result reveals why exploring a set larger than the set of Sarymsakov matrices whose compact subsets are all consensus sets is a challenging problem. We construct a set that consists of all Sarymsakov matrices plus one specific pattern of SIA matrices, which is thus slightly larger than the Sarymsakov class, and show that it is closed under matrix multiplication and each of its compact subsets is a consensus set. The other class of generalized Sarymsakov matrices, called Type-II generalized Sarymsakov matrices, contains matrices that may not be SIA. For this class, we provide sufficient conditions for the convergence of the product of an infinite sequence of matrices from this class to a rank-one matrix. A special case in which all the generalized Sarymsakov matrices are doubly stochastic is also discussed. To elucidate the importance of Sarymsakov matrices, we provide an alternative proof for the necessary and sufficient combinatorial condition given in [36] for deciding whether a compact set of stochastic matrices is a consensus set using the property of Sarymsakov matrices, and establish a necessary and sufficient condition for deciding whether a compact set of doubly stochastic matrices is a consensus set.

Consensus and distributed averaging (a particular type of consensus process, which aims to compute the average of all agents' initial values [42]) problems have found applications in a wide range of fields including sensor networks [43], robotic teams [44], social networks [45], and electric power grids [46]. Extending the existing conditions for reaching a consensus or seeking conditions for more general scenarios will facilitate the implementation of a consensus process in those applications. This paper makes contributions toward this direction in the following three ways. First, a key difference between this paper and the existing literature is that the stochastic matrices considered in this paper are not required to have positive diagonal entries.

This relaxation is important in the sense that when each agent in a network updates its own variable, it can completely ignore the current value of its own variable, which provides more freedom in the design of each agent's local update rule. Second, this paper constructs a larger set of stochastic matrices whose compact subsets are all consensus sets. Naturally the larger such a set becomes, the more choices for its subsets one will have and thus more freedom to construct consensus sets. Third, this paper establishes sufficient conditions for the convergence of the product of an infinite sequence of stochastic matrices (or doubly stochastic matrices) to a rank-one matrix by considering the generalized Sarymsakov matrices, which are novel in view of the existing results, and thus useful in the design of consensus (or distributed averaging) processes.

The common theme that runs throughout this paper is the following. Considering the fact that the set of Sarymsakov matrices is the largest known subset of the class of stochastic matrices whose compact subsets are all consensus sets, this paper studies two types of generalized Sarymsakov matrices in order to construct a larger such set and establishes novel conditions for reaching a consensus. Type-I generalized Sarymsakov matrices generalize the "one-stage consequent indices" in the definition of Sarymsakov matrices to "$k$-stage consequent indices" for any integer $k \geq 1$ (see Definition 2). By investigating the properties of this type of generalized Sarymsakov matrices for different values of $k$, we reveal why constructing a set larger than the set of Sarymsakov matrices whose compact subsets are all consensus sets is a challenging problem (see Theorem 4), and explore a possible way to construct such a set (see Theorem 5). Type-II generalized Sarymsakov matrices allow one inequality in the definition of Sarymsakov matrices not to be strict (see Definition 5). With this type of generalized Sarymsakov matrices, we establish sufficient conditions for the convergence of the product of an infinite sequence of stochastic matrices to a rank-one matrix, which are novel in view of the results available in the existing literature (see Theorem 6 and Corollary 2), and then apply the conditions to doubly stochastic matrices (see Theorem 7). We also establish necessary and sufficient conditions for deciding whether a compact set of doubly stochastic matrices is a consensus set or not (see Theorems 10 and 11).

The rest of this paper is organized as follows. Some preliminaries are introduced in Section II. Section III introduces the SIA index and Type-I generalized Sarymsakov matrices, and studies the properties of the set of stochastic matrices with SIA indices no larger than $k$ (see Section III-A), where $k$ is a positive integer, constructs a set of stochastic matrices, larger than the set of Sarymsakov matrices, whose compact subsets are all consensus sets (see Section III-B), and discusses pattern-symmetric stochastic matrices (see Section III-C). In Section IV, the class of Type-II generalized Sarymsakov matrices are introduced, sufficient conditions are provided for the convergence of the left-product of an infinite sequence of matrices from the class to a rank-one matrix (see Section IV-A), and the results are applied to doubly stochastic matrices (see Section IV-B). Section V revisits the necessary and sufficient condition for deciding consensus, derived in [36], and

establishes a necessary and sufficient condition for deciding whether a set of doubly stochastic matrices is a consensus set. This paper ends with some concluding remarks in Section VI, and several appendices that contain complete proofs of several of the results in the main part.

## II. PRELIMINARIES

We begin with some notations and definitions. Let $n$ be a positive integer and $\mathcal{N}$ denote the set $\{1, 2, \ldots, n\}$. For any set $\mathcal{A} \subseteq \mathcal{N}$, we use $\bar{\mathcal{A}}$ to denote the complement of $\mathcal{A}$ with respect to $\mathcal{N}$. A square matrix $P = \left[ p_{ij} \right]_{n \times n}$ is said to be a *stochastic matrix* if $p_{ij} \geq 0$ for all $i, j \in \mathcal{N}$ and $\sum_{j=1}^{n} p_{ij} = 1$ for all $i \in \mathcal{N}$.

Consider an $n \times n$ nonnegative matrix $P$. For a set $\mathcal{A} \subseteq \mathcal{N}$, the set of *one-stage consequent indices* [39] of $\mathcal{A}$ is defined by

$$F_P(\mathcal{A}) = \{j : \ p_{ij} > 0 \text{ for some } i \in \mathcal{A}\}$$

which we call the *consequent function* of $P$. In the case when $\mathcal{A}$ is a singleton $\{i\}$, we write $F_P(i)$ instead of $F_P(\{i\})$ for simplicity. An important property of the consequent function $F_P$ is as follows.

*Lemma 1 (see Lemma 4.1 of [41]):* Let $P$ and $Q$ be two $n \times n$ nonnegative matrices. Then, $F_{PQ}(\mathcal{A}) = F_Q(F_P(\mathcal{A}))$ for all subsets $\mathcal{A} \subseteq \mathcal{N}$.

A stochastic matrix $P$ is indecomposable and aperiodic if $\lim_{k \to \infty} P^k = \mathbf{1}c^T$, where $\mathbf{1}$ is the $n$-dimensional column vector whose entries all equal 1, and $c = \left[ c_1 \ c_2 \ \cdots \ c_n \right]^T$ is some column vector satisfying $c_i \geq 0$ for all $i \in \mathcal{N}$ and $\sum_{i=1}^{n} c_i = 1$. Such matrices are called *SIA matrices* in the literature [35].

A stochastic matrix $P$ is said to belong to the *Sarymsakov class*, or equivalently, $P$ is a *Sarymsakov matrix*, if for any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset \tag{2}$$

or

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}| \tag{3}$$

where $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. We say that $P$ is a *scrambling matrix* if for any pair of distinct indices $i, j \in \mathcal{N}$, there holds $F_P(i) \cap F_P(j) \neq \emptyset$, which is equivalent to the property that there always exists an index $k \in \mathcal{N}$ such that both $p_{ik}$ and $p_{jk}$ are positive.

From the preceding definitions, it is clear that a scrambling matrix belongs to the Sarymsakov class. It has been shown in [39] that any product of $n - 1$ matrices of size $n \times n$ from the Sarymsakov class is a scrambling matrix. Since a scrambling matrix is SIA (see [47, Th. 4.11]), any Sarymsakov matrix must be an SIA matrix.

To better understand the notions of the consequent function $F_P$, the Sarymsakov matrix, and the scrambling matrix, we provide here a graphical description in terms of one node influencing another. For a given $n \times n$ stochastic matrix $P$, define a directed graph $\mathbb{G}(P)$ associated with $P$ as: $\mathbb{G}(P) = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E}$ is the edge set and $(j, i) \in \mathcal{E}$ if and only if $p_{ij} > 0$. In view of the consensus dynamics (1) with $P(k) \equiv P, k \geq 1$,

$(j, i) \in \mathcal{E}$ means that $j$ has influence on $i$ and $i$ takes $j$'s state into account when updating. Therefore, $F_P(\mathcal{A})$ is indeed the set of nodes having influence on the nodes in the set $\mathcal{A}$. Regarding the Sarymsakov matrix, (2) says that sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have influencing nodes in common; (3) says that sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have *no* influencing nodes in common but the number of influencers is greater than that of influences. A scrambling matrix is one for which each pair of distinct nodes share at least one common influencing node.

*Definition 1:* Let $\mathcal{P}$ be a set of $n \times n$ stochastic matrices. We say that $\mathcal{P}$ is a *consensus set* if for each infinite sequence of matrices $P(1), P(2), P(3), \ldots$ from $\mathcal{P}$, the product $P(k) \cdots P(2)P(1)$ converges to a rank-one matrix $\mathbf{1}c^T$ as $k \to \infty$.

Deciding whether a set of stochastic matrices is a consensus set or not is critical in establishing the convergence of the state of system (1) to a common value. Necessary and sufficient conditions for $\mathcal{P}$ to be a consensus set have been established [35], [36], [47]–[49]. Specifically, we will make use of the following result.

*Theorem 1 (see Theorem 3 of [49]):* Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices. The following conditions are equivalent.

1) $\mathcal{P}$ is a consensus set.
2) For each integer $k \geq 1$ and any $P(i) \in \mathcal{P}, 1 \leq i \leq k$, the matrix $P(1) \cdots P(k-1)P(k)$ is SIA.
3) There is an integer $\nu \geq 1$ such that for each $k \geq \nu$ and any $P(i) \in \mathcal{P}, 1 \leq i \leq k$, the matrix $P(1) \cdots P(k-1)P(k)$ is scrambling.
4) There is an integer $\mu \geq 1$ such that for each $k \geq \mu$ and any $P(i) \in \mathcal{P}, 1 \leq i \leq k$, the matrix $P(1) \cdots P(k-1)P(k)$ has a column with only positive elements.
5) There is an integer $\alpha \geq 1$ such that for each $k \geq \alpha$ and any $P(i) \in \mathcal{P}, 1 \leq i \leq k$, the matrix $P(1) \cdots P(k-1)P(k)$ belongs to the Sarymsakov class.

In view of condition (2) in Theorem 1, for a compact set $\mathcal{P}$ to be a consensus set, it is necessary that every matrix in $\mathcal{P}$ be SIA. If a set of SIA matrices is closed under matrix multiplication, then from condition (2), its compact subsets are all consensus sets. However, it is well known that the product of two SIA matrices may not be SIA [35]. The Sarymsakov class is the largest known set of stochastic matrices, which is closed under matrix multiplication. Whether there exists a larger class of SIA matrices, which is closed under matrix multiplication and contains the Sarymsakov class as a proper subset, has remained unknown. We will explore this issue by taking a closer look at the definition of the Sarymsakov class, and study the properties of classes of generalized Sarymsakov matrices that contain the Sarymsakov class as a subset.

## III. TYPE-I GENERALIZED SARYMSAKOV MATRICES

The key notion in the definition of the Sarymsakov class is the set of one-stage consequent indices. In this section, we generalize the notion to the set of $k$-*stage consequent indices*, and introduce a larger matrix set, which subsumes the Sarymsakov class, using the new notion.

For a stochastic matrix $P$ and a set $\mathcal{A} \subseteq \mathcal{N}$, the set of $k$-*stage consequent indices* of $\mathcal{A}$, written $F_P^k(\mathcal{A})$, is defined by

$$F_P^1(\mathcal{A}) = F_P(\mathcal{A})$$
$$F_P^k(\mathcal{A}) = F_P(F_P^{k-1}(\mathcal{A})), \quad k \geq 2.$$

It directly follows from Lemma 1 that $F_{P^k}(\mathcal{A}) = F_P^k(\mathcal{A})$ for any stochastic matrix $P$, any integer $k \geq 1$, and any subset $\mathcal{A} \subseteq \mathcal{N}$. With the above-mentioned notion, we introduce the following class of generalized Sarymsakov matrices, called Type-I generalized Sarymsakov matrices, which turns out to be equal to the class of SIA matrices (see Theorem 2).

*Definition 2 (see [49]):* A stochastic matrix $P$ is said to belong to the *class* $\mathcal{W}$ if for any two disjoint nonempty subsets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an integer $k \geq 1$ such that either

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset \tag{4}$$

or

$$F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset \ \text{ and } \ |F_P^k(\mathcal{A}) \cup F_P^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{5}$$

From a graphical point of view, $k$-stage consequent indices are nodes which influence (possibly indirectly) the set $\mathcal{A}$ in $k$ time steps. Regarding Type-I generalized Sarymsakov matrices (see Definition 2): (4) says that sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have at least one $k$-stage influencer in common; (5) says that sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have no $k$-stage influencing nodes in common, but the total number of $k$-stage influencers is greater than the total number of influences in $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

The intuition behind Definition 2 will be given shortly (see Remark 1).

It is easy to see that the Sarymsakov class is a subset of the class $\mathcal{W}$. The following theorem establishes the relationship between the matrices in the class $\mathcal{W}$ and SIA matrices.

*Theorem 2 (see Theorem 1 of [49]):* Class $\mathcal{W}$ is equal to the class of SIA matrices.

More can be said. The following corollary implies that the integer $k$ in (4) and (5) can be bounded.

*Corollary 1:* A stochastic matrix $P$ is SIA if and only if for any pair of disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exists an index $k$, $k \leq n(n-1)/2$, such that $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset$.

This corollary is an immediate consequence of the following result.

*Theorem 3 (see Theorem 4.4 of [50]):* A stochastic matrix $P$ is SIA if and only if for every pair of indices $i$ and $j$, there exists an integer $k$, $k \leq n(n-1)/2$, such that $F_P^k(i) \cap F_P^k(j) \neq \emptyset$.

*Remark 1:* Theorem 3 reveals the key feature of the SIA matrices, namely that a stochastic matrix is an SIA matrix as long as for each pair of distinct indices, their sets of some finite stage of consequent indices contain a common index. Definition 2 naturally extends the class of Sarymsakov matrices to a larger class that turns out to be the set of SIA matrices. Indeed, Definition 2 and Theorem 2 imply that given an SIA matrix and for each pair of distinct indices, which is a special case of a pair of nonempty disjoint subsets of $\mathcal{N}$, if (4) does not hold, then the cardinalities of their sets of $k$-stage consequent indices must increase because of (5). Since the matrix is of finite

dimensions, the sets of some finite stage of consequent indices must contain a common index, which verifies the property of the SIA matrices. ∎

*Example 1:* Consider the following stochastic matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and two disjoint nonempty sets $\mathcal{A} = \{2\}, \tilde{\mathcal{A}} = \{3\}$. It is straightforward to verify that $F_P(\mathcal{A}) = \{1\}$ and $F_P(\tilde{\mathcal{A}}) = \{2\}$, which implies that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset$ and $|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Therefore, $P$ is not a Sarymsakov matrix. However, the facts that $F_P^2(\mathcal{A}) = \{1, 2, 3\}$ and $F_P^2(\tilde{\mathcal{A}}) = \{2\}$ imply that $F_P^2(\mathcal{A}) \cap F_P^2(\tilde{\mathcal{A}}) \neq \emptyset$. This means that (4) holds for $k = 2$. For every other pair of disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, it can be verified that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$. Thus, although $P$ is not a Sarymsakov matrix, $P$ is an SIA matrix from Corollary 1. ∎

From the above-mentioned example and Corollary 1, the class of SIA matrices may contain a large number of matrices that do not belong to the Sarymsakov class. Starting from the Sarymsakov class, with $k = 1$ in (4) and (5), we relax the constraint on the value of the integer $k$ in (4) and (5) (i.e., allowing for $k \leq 2, k \leq 3, \ldots$), and obtain a larger set containing the Sarymsakov class. We formalize the idea below and study whether the derived set is closed under matrix multiplication or not.

Fix a positive integer $n$ and denote all possible unordered pairs of disjoint nonempty sets of $\mathcal{N}$ by $(\mathcal{A}_1, \tilde{\mathcal{A}}_1), (\mathcal{A}_2, \tilde{\mathcal{A}}_2), \ldots, (\mathcal{A}_m, \tilde{\mathcal{A}}_m)$, where $m$ is a finite number.

*Definition 3:* Let $P \in \mathbb{R}^{n \times n}$ be an SIA matrix. For each pair of disjoint nonempty sets $\mathcal{A}_i, \tilde{\mathcal{A}}_i \subseteq \mathcal{N}, i \in \{1, 2, \ldots, m\}$, let $s_i$ be the smallest integer such that either (4) or (5) holds. The *SIA index* $s$ of $P$ is $s = \max\{s_1, s_2, \ldots, s_m\}$.

We provide an example to further elaborate on Definition 3.

*Example 2:* Consider again the matrix $P$ given in Example 1. The number of all possible unordered pairs of disjoint nonempty sets of $\mathcal{N}$ is 6. For the pair of nonempty sets $\mathcal{A} = \{2\}, \tilde{\mathcal{A}} = \{3\}$, from the discussions in Example 1, one knows that the smallest integer such that (4) or (5) holds is 2. For all other pairs of nonempty sets $\mathcal{A}, \tilde{\mathcal{A}}$, the smallest integer is 1. We, therefore, conclude that the SIA index of $P$ is $s = 2$. ∎

From Corollary 1, for any SIA matrix $P$ of size $n \times n$, its SIA index $s$ is bounded above by $n(n-1)/2$. Assume that the largest value of the SIA indices of all $n \times n$ SIA matrices is $l$, which depends on the order $n$. For our purposes, we define the following subsets of the class of SIA matrices. For each $k \in \{1, 2, \ldots, l\}$, let

$$\mathcal{V}_k = \{P \in \mathbb{R}^{n \times n} | P \text{ is SIA with SIA index } k\} \tag{6}$$

and

$$\mathcal{W}_k = \cup_{r=1}^k \mathcal{V}_r. \tag{7}$$

It is clear that $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_l$, and $\mathcal{W}_1 = \mathcal{V}_1$ is the set of $n \times n$ Sarymsakov matrices. Moreover, Theorem 2 implies that $\mathcal{W}_l$ is the set of $n \times n$ SIA matrices. The relationships among the set of Sarymsakov matrices, the sets $\mathcal{W}_i$, and the set of SIA matrices are illustrated in Fig. 1.
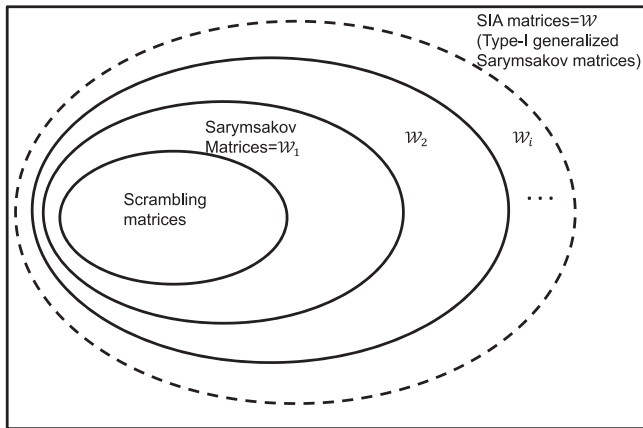
Fig. 1. Relationships among the set of SIA matrices, the sets $\mathcal{W}_i$, and the set of SIA matrices.

It is straightforward to check that when $n = 2$, all SIA matrices are scrambling matrices and hence belong to the Sarymsakov class. When $n \geq 3$, the set $\mathcal{V}_{n-1}$ is nonempty. To see this, consider the following example.

*Example 3:* Let

$$P = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

be an $n \times n$ stochastic matrix. For an index $i \in \mathcal{N}$, $i \neq n$, it is easy to check that $F_P^{n-1}(i) = \mathcal{N}$. Hence, for any two nonempty disjoint sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, it must be true that $F_P^{n-1}(\mathcal{A}) \cap F_P^{n-1}(\tilde{\mathcal{A}}) \neq \emptyset$, which implies that $P$ is an SIA matrix. Consider the specific pair of sets $\mathcal{A} = \{n\}$, $\tilde{\mathcal{A}} = \{n-1\}$. Then, $F_P^{n-2}(n) = \{2\}$, $F_P^{n-2}(n-1) = \{1\}$, and $F_P^{n-1}(n) \cap F_P^{n-1}(n-1) \neq \emptyset$, which imply that $P \in \mathcal{V}_{n-1}$. From this example, we know that a lower bound for $l$ is $n-1$. ∎

*Lemma 2:* For $n \geq 2$, the maximum SIA index $l$ of all $n \times n$ SIA matrices satisfies $n - 1 \leq l \leq n(n-1)/2$.

In the next three sections, we first discuss the properties of $\mathcal{W}_i, i \in \{1, 2, \ldots, l\}$, then construct a set of stochastic matrices, which consists of a specific pattern of SIA matrices and all Sarymsakov matrices, and is closed under matrix multiplication, and finally discuss the class of "pattern-symmetric matrices."

### A. Properties of $\mathcal{W}_i$

The following theorem, which is one of the main results of this paper, reveals an important property of the sets $\mathcal{W}_i$, $i \in \{1, 2, \ldots, l\}$.

*Theorem 4:* Suppose that $n \geq 3$. Among the sets $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_l$, the set $\mathcal{W}_1$ is the only set that is closed under matrix multiplication.

The proof of Theorem 4 is given in Appendix A.

Note that a compact subset $\mathcal{P}$ of $\mathcal{W}_1$ is a consensus set. However, if $\mathcal{P}$ is a compact set consisting of matrices in $\mathcal{V}_i$,

$i \geq 2$, as defined in (6), $\mathcal{P}$ may not be a consensus set any more as can be seen from Lemma 6 in the proof of Theorem 4. Although a set of stochastic matrices can be a consensus set even if it is not closed under matrix multiplication, the closure property under matrix multiplication is important in that if a set of SIA matrices has this property, then from condition (2) in Theorem 1, all of its compact subsets are consensus sets. So this property leads to a sufficient condition to identify consensus sets that will be useful in practice. Naturally the larger such a set becomes, the more choices for its subsets one will have, and thus more freedom to construct consensus sets. The Sarymsakov class is the largest known set that is closed under matrix multiplication. Theorem 4 reveals why it is challenging to explore a set larger than the set of Sarymsakov matrices.

In the literature, there has been work on defining another class of stochastic matrices that is a subset of the SIA matrices and larger than the set of scrambling matrices (see [47, Ch. 4]), as follows.

*Definition 4:* (see [47, Ch. 4]) A stochastic matrix $P$ is said to belong to the class $\mathcal{G}$ if $P$ is SIA and for any SIA matrix $Q$, $QP$ is SIA.

The following proposition establishes the relationship between the class $\mathcal{G}$ and the Sarymsakov class, whose proof is given in Appendix B.

*Proposition 1:* For $n \geq 3$, the class $\mathcal{G}$ is a proper subset of the class of Sarymsakov matrices $\mathcal{W}_1$.

### B. Set Closed Under Matrix Multiplication

In this section, we construct a subset of $\mathcal{W}$, which is closed under matrix multiplication. This subset consists of the set $\mathcal{W}_1$ and a specific pattern of matrices in $\mathcal{V}_2$, introduced in more precise terms as follows.

Let $R$ be a matrix in $\mathcal{V}_2$, which satisfies the property that for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) \neq \emptyset \tag{8}$$

or

$$F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| \geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{9}$$

Such a matrix exists as can be seen from the example given as follows.

*Example 4:* Let

$$R^* = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}. \tag{10}$$

To verify that $R^*$ satisfies the above-mentioned condition, it is enough to consider the pair of sets $\mathcal{A} = \{2\}$ and $\tilde{\mathcal{A}} = \{3\}$, since for any other pair of $\mathcal{A}$ and $\tilde{\mathcal{A}}$, there holds $F_{R^*}(\mathcal{A}) \cap F_{R^*}(\tilde{\mathcal{A}}) \neq \emptyset$. Note that $|F_{R^*}(2) \cup F_{R^*}(3)| = |\{1, 2\}| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$ and $F_{R^*}^2(2) \cap F_{R^*}^2(3) = \{1\}$. Thus, $R^*$ satisfies the condi-

tion. It is worth emphasizing that any stochastic matrix that has the same zero-nonzero pattern as $R^*$ satisfies the condition. ∎

Given a stochastic matrix $R$, let

$$\mathcal{C}(R) = \{P | P \text{ is a stochastic matrix and}$$

$$\text{has the same zero-nonzero pattern as } R\}.$$

*Theorem 5:* Suppose that $R$ is a matrix in $\mathcal{V}_2$ such that for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either (8) or (9) holds. Then, the set $\mathcal{T} = \mathcal{W}_1 \cup \mathcal{C}(R)$ is closed under matrix multiplication, and any compact subset of $\mathcal{T}$ is a consensus set.

The proof of Theorem 5 is given in Appendix C.

For a set consisting of the set $\mathcal{W}_1$, and two or more different patterns of matrices in $\mathcal{V}_2$, which satisfy the property that for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either (8) or (9) holds, whether the set is closed under matrix multiplication or not depends on those matrices in $\mathcal{V}_2$.

*Example 5:* Let

$$R_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}. \tag{11}$$

Note that for each $R_i$, $i = 1, 2, 3$, either (8) or (9) holds for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$. Let $\mathcal{T}_1 = \mathcal{W}_1 \cup \{\mathcal{C}(R_1), \mathcal{C}(R_2)\}$ and $\mathcal{T}_2 = \mathcal{W}_1 \cup \{\mathcal{C}(R_1), \mathcal{C}(R_3)\}$. It is straightforward to verify that $\mathcal{T}_1$ is not closed under multiplication, and in addition

$$R_1 R_2 = \begin{bmatrix} + & + & + \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

is not an SIA matrix. However, $\mathcal{T}_2$ is closed under multiplication. To see this, note that $R_1^2, R_3^2, R_1 R_3, R_3 R_1$ are all scrambling matrices and hence belong to the Sarymsakov class. Note that the product of a Sarymsakov matrix and $R_1$ or $R_3$ is still a Sarymsakov matrix. It then follows that for any $P_1, P_2 \in \mathcal{T}_2$, the product $P_2 P_1$ is a Sarymsakov matrix. By induction, $\mathcal{T}_2$ is closed under matrix multiplication. ∎

### C. Pattern-Symmetric Matrices

In this section, we focus on a class of $n \times n$ "pattern-symmetric" stochastic matrices, where by a *pattern-symmetric* matrix we mean a square nonnegative matrix $P = \begin{bmatrix} p_{ij} \end{bmatrix}_{n \times n}$ such that

$$p_{ij} > 0 \text{ if and only if } p_{ji} > 0 \text{ for all } i \neq j. \tag{13}$$

A linear consensus process (1) with bidirectional interactions between neighboring agents induces update matrices satisfying (13), which arises often in the literature [4], [12], [17]. The stochastic matrices satisfying (13) have the following property.

*Proposition 2:* Suppose that $P$ is an SIA matrix and satisfies (13). Then, $P \in \mathcal{W}_2$, and if, in addition, $P$ is symmetric, then $P \in \mathcal{W}_1$.

*Proof:*

1) Suppose that, to the contrary, $P$ is not in $\mathcal{W}_2$. Then, there must exist two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that

$$F_P^2(\mathcal{A}) \cap F_P^2(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P^2(\mathcal{A}) \cup F_P^2(\tilde{\mathcal{A}})| \leq |\mathcal{A} \cup \tilde{\mathcal{A}}|.$$

From (13), for any nonempty set $\mathcal{C} \subseteq \mathcal{N}$, there holds $\mathcal{C} \subseteq F_P^2(\mathcal{C})$, which implies that $|F_P^2(\mathcal{A}) \cup F_P^2(\tilde{\mathcal{A}})| \geq |\mathcal{A} \cup \tilde{\mathcal{A}}|$. It follows that $|F_P^2(\mathcal{A}) \cup F_P^2(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Then, $F_P^2(\mathcal{A}) = \mathcal{A}$ and $F_P^2(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$, which implies that $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset$ for any positive integer $k$. This contradicts the fact that $P$ is an SIA matrix in view of Corollary 1. Therefore, $P \in \mathcal{W}_2$.

2) Suppose that, to the contrary, $P \notin \mathcal{W}_1$. Then, there exist two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \leq |\mathcal{A} \cup \tilde{\mathcal{A}}|.$$

Since for any set $\mathcal{C} \subseteq \mathcal{N}$

$$\sum_{i \in \mathcal{C}, j \in F_P(\mathcal{C})} p_{ij} = |\mathcal{C}| = \sum_{i \in \mathcal{C}, j \in F_P(\mathcal{C})} p_{ji} \leq |F_P(\mathcal{C})|$$

it follows that $|F_P(\mathcal{A})| = |\mathcal{A}|$ and $|F_P(\tilde{\mathcal{A}})| = |\tilde{\mathcal{A}}|$. This implies that

$$\sum_{i \in \mathcal{A}, j \in F_P(\mathcal{A})} p_{ji} = |F_P(\mathcal{A})|.$$

Combined with the fact that $\mathcal{A} \subseteq F_P^2(\mathcal{A})$, there holds $F_P^2(\mathcal{A}) = \mathcal{A}$. Similarly, $F_P^2(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$. Thus, $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset$ for any positive integer $k$. This contradicts the fact that $P$ is SIA. Therefore, $P \in \mathcal{W}_1$. ∎

For symmetric stochastic matrices, conditions for deciding whether a set of such matrices is a consensus set or not have existed in the literature. Specifically, it has been established in [36, Example 7] that a compact set $\mathcal{P}$ of symmetric stochastic matrices is a consensus set if and only if $P$ is an SIA matrix for every $P \in \mathcal{P}$. Note that the necessary condition holds for any consensus set. From Proposition 2, a symmetric stochastic matrix $P$ is SIA if and only if $P$ is a Sarymsakov matrix. Then, the sufficient condition follows immediately from the fact that the Sarymsakov class is closed under matrix multiplication.

The above-mentioned condition for symmetric stochastic matrices cannot be extended to nonsymmetric stochastic matrices that satisfy (13). To see this, note that a stochastic matrix satisfying (13) is not necessarily a Sarymsakov matrix. Hence, in view of Theorem 4, the product of two such matrices may not be SIA.

*Example 6:* Consider the set consisting of the following two matrices:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

It is straightforward to verify that both $P_1$ and $P_2$ satisfy (13), but $P_1 \in \mathcal{W}_2$, $P_1 \notin \mathcal{W}_1$. In addition, $(P_1 P_2)^k$ does not converge to a rank-one matrix as $k \to \infty$. ∎

## IV. TYPE-II GENERALIZED SARYMSAKOV MATRICES

We have shown in Theorem 5 that the class of Sarymsakov matrices plus some specific SIA matrices constitute a set of stochastic matrices that is closed under matrix multiplication and contains $\mathcal{W}_1$. The property (9) of the matrix $R$ turns out to be critical in the analysis. We next consider a class of matrices containing all such matrices, called Type-II generalized Sarymsakov matrices, whose definition is as follows.

*Definition 5:* A stochastic matrix $P$ is said to belong to the *class* $\mathcal{M}$ if for any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, either

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset \qquad (14)$$

or

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \qquad (15)$$

The definition of the class $\mathcal{M}$ relaxes that of the Sarymsakov class $\mathcal{W}_1$ by allowing the inequality in (3) not to be strict. Thus, it is clear that $\mathcal{W}_1$ is a subset of $\mathcal{M}$. More can be said.

*Lemma 3:* The set $\mathcal{M}$ is closed under matrix multiplication.

*Proof:* Let $P, Q \in \mathcal{M}$. For any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, suppose that $F_{PQ}(\mathcal{A}) \cap F_{PQ}(\tilde{\mathcal{A}}) = \emptyset$. It follows from (15) that

$$\begin{aligned} |F_{PQ}(\mathcal{A}) \cup F_{PQ}(\tilde{\mathcal{A}})| &= |F_Q(F_P(\mathcal{A})) \cup F_Q(F_P(\tilde{\mathcal{A}}))| \\ &\geq |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \\ &\geq |\mathcal{A} \cup \tilde{\mathcal{A}}| \end{aligned}$$

which implies that $PQ \in \mathcal{M}$. ∎

Although the sets $\mathcal{M}$ and $\mathcal{W}_1$ are both closed under matrix multiplication and have similar definitions, their elements can have significantly different properties. Specifically, a matrix in $\mathcal{M}$ is not necessarily SIA. For example, permutation matrices[1] belong to the class $\mathcal{M}$ since for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there hold

$$F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset \text{ and } |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|. \qquad (16)$$

But it can be verified that permutation matrices are not SIA. The relationships among Type-I generalized Sarymsakov matrices $\mathcal{W}$, Type-II generalized Sarymsakov matrices $\mathcal{M}$, and the Sarymsakov matrices are illustrated in Fig. 2.

*Remark 2:* One may conjecture that the set $\mathcal{M} \cap \mathcal{W}$ is closed under matrix multiplication, which is, however, false, as shown by the following counterexample. Consider the two matrices $R_1$ and $R_2$ given in (11), which are both SIA and in $\mathcal{M}$. But their product, as shown in (12), is not SIA. ∎

In the sequel, we will explore sufficient conditions for the convergence of infinite sequences of products of stochastic matrices from $\mathcal{M}$, and their applications to doubly stochastic matrices.
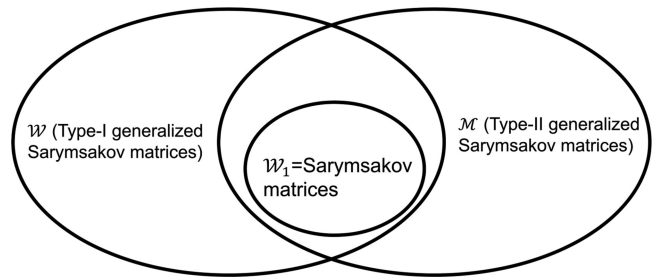


Fig. 2. Relationships among Type-I generalized Sarymsakov matrices $\mathcal{W}$, Type-II generalized Sarymsakov matrices $\mathcal{M}$, and the Sarymsakov matrices.

### A. Sufficient Condition for Consensus

The following theorem provides a sufficient condition for the convergence of infinite sequences of products of stochastic matrices from a compact subset of $\mathcal{M}$.

*Theorem 6:* Let $\mathcal{P}$ be a compact subset of $\mathcal{M}$ and let $P(1)$, $P(2), \ldots$ be an infinite sequence of matrices from $\mathcal{P}$. Suppose that $j_1, j_2, \ldots$ is a strictly increasing, infinite sequence of the indices such that $P(j_r) \in \mathcal{P}' \subseteq \mathcal{P} \cap \mathcal{W}_1$, $r = 1, 2, \ldots$, where $\mathcal{P}'$ is a compact set. Then, $P(k) \cdots P(2)P(1)$ converges to a rank-one matrix as $k \to \infty$ if there exists a positive integer $T$ such that $j_{r+1} - j_r \leq T$ for all $r \geq 1$.

The proof of Theorem 6 is given in Appendix D.

*Remark 3:* Set $T_r = j_{r+1} - j_r$ for each $r \geq 1$. Suppose that $T_r$ is not uniformly upper bounded. Then, $\cup_{r=1}^{\infty} \mathcal{Q}_{T_r}$ ($\mathcal{Q}_{T_r}$ is defined similarly to $\mathcal{Q}_T$ in (32) in the proof of Theorem 6) is not necessarily compact so that the conditions in Theorem 1 do not apply. Thus, in this case, the result of Theorem 6 may not hold. ∎

*Remark 4:* For a set of stochastic matrices $\mathcal{P}$, consider two assumptions: (A1) $\mathcal{P}$ is a compact set, and (A2) the positive entries of all the matrices in $\mathcal{P}$ are uniformly lower bounded by a positive scalar. In this paper, we mainly consider the assumption (A1). In Theorems 1 and 6, if the assumption that $\mathcal{P}$ is a compact set is replaced by (A2), then the same conclusions still hold [47]. However, it is worth noting that (A1) does not imply (A2), and (A2) does not imply (A1) either. For example, consider the following set:

$$\mathcal{P}_1 = \bigcup_{n \geq 2}^{\infty} \left\{ \begin{bmatrix} 1 - \frac{1}{n} & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\} \bigcup \left\{ \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}.$$

The set $\mathcal{P}_1$ is compact; however, the positive entries do not have a uniform positive lower bound. On the other hand, consider

$$\mathcal{P}_2 = \bigcup_{n \geq 2}^{\infty} \left\{ \begin{bmatrix} \frac{1}{2} - \frac{1}{n} & \frac{1}{2} + \frac{1}{n} \\ 0 & 1 \end{bmatrix} \right\}.$$

The positive entries of all the matrices in $\mathcal{P}_2$ have a uniform positive lower bound $\frac{1}{6}$, but $\mathcal{P}_2$ is not compact. ∎

*Remark 5:* In the existing studies of the discrete-time consensus process (1), it is usually assumed that the diagonal

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[1] A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column, and zeros elsewhere. Permutation matrices are stochastic and include the identity matrix as a special case.

entries of each $P(k)$ are positive, and the nonzero entries of each $P(k)$ are uniformly bounded below by some positive constant [3], [4], [6]–[8], [10], [12], [18], [51]. The sufficient conditions for reaching a consensus are then given in terms of a joint graphical connectivity, namely there exists an infinite sequence of time instants $t_1, t_2, \ldots$ such that the union of the graphs of the stochastic matrices $P(t)$ across each interval $[t_i, t_{i+1})$ has a directed spanning tree and there exists a positive integer $T$ for which $t_{r+1} - t_r \leq T$ for all $r \geq 1$, although the form of the connectivity may vary slightly from publication to publication. These assumptions guarantee that each product $P(kT) \cdots P((k-1)T + 2)P((k-1)T + 1)$, $k \geq 1$, is a stochastic matrix with positive diagonal entries and its graph has a directed spanning tree. Moreover, it can be easily shown that such a product is indeed a Sarymsakov matrix. Then, reaching a consensus is implied by condition (2) in Theorem 1. The difference between Theorem 6 and those existing results [3], [4], [6]–[8], [10], [12], [18], [51] is that the stochastic matrices $P(t)$ considered in this paper are not required to have positive diagonal entries (but instead to belong to the class $\mathcal{M}$). This relaxation is important in the sense that when each agent in a multiagent network updates its own variable, it can completely ignore the current value of its own variable, which provides more freedom in the design of each agent's local update rule. It is worth noting that the uniform bound on the time instants of the appearance of a Sarymsakov matrix in Theorem 6 plays a similar role to the above-mentioned joint graphical connectivity in the existing literature, and thus also guarantees that each $P(kT) \cdots P((k-1)T + 2)P((k-1)T + 1)$, $k \geq 1$, is a Sarymsakov matrix. ∎

*Remark 6:* There exist other results on the discrete-time consensus process (1) that do not require the assumptions (A1) or (A2) in Remark 4. The absolute infinite flow condition is necessary and sufficient for the ergodicity of a chain of doubly stochastic matrices [52] and, in addition, is necessary and sufficient for the ergodicity of a chain of stochastic matrices under the balanced asymmetry condition [53]. The notion has also been used to study the ergodicity of random chains of stochastic matrices [54]. ∎

In the case when the set $\mathcal{P}$ is a finite set, we have the following corollary that is a direct consequence of Theorem 6.

*Corollary 2:* Let $\mathcal{P}$ be a finite subset of $\mathcal{M}$ and let $P(1)$, $P(2), \ldots$ be an infinite sequence of matrices from $\mathcal{P}$. Suppose that $j_1, j_2, \ldots$ is a strictly increasing, infinite sequence of the indices such that $P(j_1), P(j_2), \ldots$ are Sarymsakov matrices. Then, $P(k) \cdots P(2)P(1)$ converges to a rank-one matrix as $k \to \infty$ if there exists a positive integer $T$ such that $j_{r+1} - j_r \leq T$ for all $r \geq 1$.

### B. Applications to Doubly Stochastic Matrices

A square nonnegative matrix is called *doubly stochastic* if its row sums and column sums all equal one. Thus, the set of doubly stochastic matrices is a proper subset of stochastic matrices. In fact, the following result shows that the set of doubly stochastic matrices is also a proper subset of $\mathcal{M}$.

*Proposition 3:* If $P$ is a doubly stochastic matrix, then $P \in \mathcal{M}$.

This proposition is an immediate consequence of the following property of doubly stochastic matrices.

*Lemma 4:* Let $P$ be a doubly stochastic matrix. Then, for any nonempty set $\mathcal{A} \subseteq \mathcal{N}$, there holds $|F_P(\mathcal{A})| \geq |\mathcal{A}|$.

*Proof:* From the Birkhoff–von Neumann Theorem (see [55, Th. 8.7.1]), $P$ is doubly stochastic if and only if $P$ is a convex combination of permutation matrices, i.e., $P = \sum_{i=1}^{n!} \alpha_i P_i$, where $\sum_{i=1}^{n!} \alpha_i = 1$, $a_i \geq 0$ for all $i \in \{1, 2, \ldots, n!\}$, and each $P_i$ is a permutation matrix. For each permutation matrix $P_i$, there holds $|F_{P_i}(\mathcal{A})| = |\mathcal{A}|$ for any set $\mathcal{A} \subseteq \mathcal{N}$. In view of the Birkhoff–von Neumann Theorem, it must be true that

$$F_P(\mathcal{A}) = \cup_{\alpha_i \neq 0} F_{P_i}(\mathcal{A}).$$

It then immediately follows that $|F_P(\mathcal{A})| \geq |\mathcal{A}|$. ∎

From the above-mentioned lemma, it is easy to see that for any doubly stochastic matrix $P$, either (14) or (15) holds. Hence, doubly stochastic matrices belong to the set $\mathcal{M}$.

The following result establishes a relationship between doubly stochastic matrices and Sarymsakov matrices, which is helpful for establishing a similar result to Theorem 6.

*Proposition 4:* Let $P$ be a doubly stochastic matrix. Then, $P$ is a Sarymsakov matrix if and only if for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, there holds $|F_P(\mathcal{A})| > |\mathcal{A}|$.

*Proof:* The sufficiency part is clearly true. It remains, therefore, to prove the necessity. Suppose that, to the contrary, there exists a nonempty set $\mathcal{A} \subsetneq \mathcal{N}$ such that $|F_P(\mathcal{A})| \leq |\mathcal{A}|$. It follows from Lemma 4 that

$$|F_P(\mathcal{A})| = |\mathcal{A}| = \sum_{i \in \mathcal{A}, j \in F_P(\mathcal{A})} p_{ij}. \qquad (17)$$

Since $P$ is doubly stochastic, $\sum_{i \in \mathcal{N}, j \in F_P(\mathcal{A})} p_{ij} = |F_P(\mathcal{A})|$. Hence

$$\sum_{i \in \bar{\mathcal{A}}, j \in F_P(\mathcal{A})} p_{ij} = \sum_{i \in \mathcal{N}, j \in F_P(\mathcal{A})} p_{ij} - \sum_{i \in \mathcal{A}, j \in F_P(\mathcal{A})} p_{ij}$$
$$= |F_P(\mathcal{A})| - |\mathcal{A}|$$
$$= 0. \qquad (18)$$

It follows that $F_P(\bar{\mathcal{A}}) \subseteq \overline{F_P(\mathcal{A})}$. Note that Lemma 4 implies that

$$|F_P(\bar{\mathcal{A}})| \geq |\bar{\mathcal{A}}| = n - |\mathcal{A}| = |\overline{F_P(\mathcal{A})}|.$$

It follows that $|F_P(\bar{\mathcal{A}})| = n - |\mathcal{A}|$ and $F_P(\bar{\mathcal{A}}) = \overline{F_P(\mathcal{A})}$. Then

$$F_P(\mathcal{A}) \cap F_P(\bar{\mathcal{A}}) = \emptyset, \ \text{ and } \ |F_P(\mathcal{A}) \cup F_P(\bar{\mathcal{A}})| = n = |\mathcal{A} \cup \bar{\mathcal{A}}|$$

which contradicts the fact that $P$ is a Sarymsakov matrix. Therefore, it must be true that for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, there holds $|F_P(\mathcal{A})| > |\mathcal{A}|$. ∎

*Theorem 7:* Let $\mathcal{P}$ be a set of doubly stochastic matrices, and let $P(1), P(2), \ldots$ be an infinite sequence of matrices from $\mathcal{P}$. Suppose that $k_1, k_2, \ldots$ is a strictly increasing, infinite sequence of the indices such that $P(k_r) \in \mathcal{P}' \subseteq \mathcal{P} \cap \mathcal{W}_1$, $r = 1, 2, \ldots$, where $\mathcal{P}'$ is a compact set. Then, $P(k) \cdots P(2)P(1)$ converges to $\mathbf{1}\mathbf{1}^T/n$ as $k \to \infty$.

The proof of Theorem 7 is given in Appendix E.

*Remark 7:* The assumption on uniform boundedness of $k_{r+1} - k_r$, $r \geq 1$, is removed for the case of doubly stochastic matrices compared with Theorem 6. The above-mentioned result claims that as long as the sequence of doubly stochastic matrices contains infinitely many Sarymsakov matrices chosen from a compact subset of the Sarymsakov class, then the infinite product of this sequence converges to the rank-one matrix $\mathbf{1}\mathbf{1}^T / n$.

Proposition 4 provides a condition to decide whether a doubly stochastic matrix belongs to $\mathcal{W}_1$ or not. For a doubly stochastic matrix, a necessary and sufficient condition for the matrix in $\mathcal{W}$ is stated as follows.

*Proposition 5:* Let $P$ be a doubly stochastic matrix. $P$ is an SIA matrix if and only if for every nonempty set $\mathcal{A} \subsetneq \mathcal{N}$, there exists a positive integer $k$ such that $|F_P^k(\mathcal{A})| > |\mathcal{A}|$.

The proof of the proposition makes use of the following result.

*Lemma 5:* Let $P$ be a doubly stochastic matrix. For two disjoint nonempty subsets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, if $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$, then $|F_P(\mathcal{A})| > |\mathcal{A}|$ and $|F_P(\tilde{\mathcal{A}})| > |\tilde{\mathcal{A}}|$.

*Proof:* Suppose to the contrary that $|F_P(\mathcal{A})| = |\mathcal{A}|$ or $|F_P(\tilde{\mathcal{A}})| = |\tilde{\mathcal{A}}|$. We first consider the case when $|F_P(\mathcal{A})| = |\mathcal{A}|$. Then obviously (17) holds. Since $P$ is doubly stochastic, (18) holds and implies that $p_{ij} = 0$ for $i \in \bar{\mathcal{A}}, j \in F_P(\mathcal{A})$. Since $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are disjoint sets, $\tilde{\mathcal{A}}$ is a subset of $\bar{\mathcal{A}}$. Therefore, for any $j \in F_P(\mathcal{A})$, there holds $j \notin F_P(\tilde{\mathcal{A}})$, which contradicts the fact that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$. We conclude that $|F_P(\mathcal{A})| > |\mathcal{A}|$. The conclusion that $|F_P(\tilde{\mathcal{A}})| > |\tilde{\mathcal{A}}|$ can be proved in a similar manner.                                                                 ∎

*Proof of Proposition 5:* (Necessity) For a nonempty subset $\mathcal{A} \subsetneq \mathcal{N}$, let $\tilde{\mathcal{A}} = \bar{\mathcal{A}}$. Since $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are disjoint sets, according to Corollary 1, there exists a positive integer $k$ such that $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) \neq \emptyset$. Noting that $F_P^{i+1}(\mathcal{A}) = F_P(F_P^i(\mathcal{A}))$, applying Lemmas 4 and 5 yields

$$|F_P^k(\mathcal{A})| > |F_P^{k-1}(\mathcal{A})| \geq |F_P^{k-2}(\mathcal{A})| \geq \cdots \geq |\mathcal{A}|.$$

(Sufficiency) For every two disjoint nonempty subsets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there exist positive integers $k_1$ and $k_2$ such that $|F_P^{k_1}(\mathcal{A})| > |\mathcal{A}|$ and $|F_P^{k_2}(\tilde{\mathcal{A}})| > |\tilde{\mathcal{A}}|$. Let $k = \max\{k_1, k_2\}$. If $F_P^k(\mathcal{A}) \cap F_P^k(\tilde{\mathcal{A}}) = \emptyset$, then using Lemma 4, there holds

$$|F_P^k(\mathcal{A})| \geq |F_P^{k_1}(\mathcal{A})| > |\mathcal{A}|$$

and

$$|F_P^k(\tilde{\mathcal{A}})| \geq |F_P^{k_2}(\tilde{\mathcal{A}})| > |\tilde{\mathcal{A}}|.$$

It then follows that $|F_P(\tilde{\mathcal{A}}) \cup F_P^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Therefore, $P$ is SIA.                                                                 ∎

For doubly stochastic matrices satisfying (13), more can be said.

*Proposition 6:* Let $P$ be a doubly stochastic matrix satisfying (13). If $P$ is SIA, then $P \in \mathcal{W}_1$.

*Proof:* Suppose that, to the contrary, $P$ is not a Sarymsakov matrix. In view of Proposition 4, there exists a set $\mathcal{A} \subseteq \mathcal{N}$ such that $|\mathcal{A}| = |F_P(\mathcal{A})|$. From the proof of Proposition 4, it follows that (17) and (18) hold, and $|\bar{\mathcal{A}}| = |F_P(\bar{\mathcal{A}})| = |\overline{F_P(\mathcal{A})}|$. Note that (13) and (17) imply that $p_{ij} = 0$ for any $i \in F_P(\mathcal{A})$ and $j \in \bar{\mathcal{A}}$. Thus, $F_P^2(\mathcal{A}) \subseteq \mathcal{A}$. Combining this and the fact

that $\mathcal{A} \subseteq F_P^2(\mathcal{A})$, it follows that $\mathcal{A} = F_P^2(\mathcal{A})$. Similarly, there holds $\bar{\mathcal{A}} = F_P^2(\bar{\mathcal{A}})$. Thus, $F_P^k(\mathcal{A}) \cap F_P^k(\bar{\mathcal{A}}) = \emptyset$, which contradicts the assumption that $P$ is an SIA matrix. Therefore, $P$ must be a Sarymsakov matrix.                                                                 ∎

## V. NECESSARY AND SUFFICIENT CONDITIONS FOR DECIDING CONSENSUS

To elucidate the importance of the class of Sarymsakov matrices, in this section, we first provide an alternative proof for a necessary and sufficient combinatorial condition, the "avoiding set condition," established in [36] for deciding whether or not a compact set of stochastic matrices is a consensus set and then carry out the discussion to doubly stochastic matrices.

*Theorem 8 (see Theorem 2.2 of [36]):* A compact set $\mathcal{P}$ of $n \times n$ stochastic matrices is not a consensus set if and only if there exist two sequences of nonempty subsets of $\mathcal{N}, S_1, S_2, \ldots, S_l$ and $S_1', S_2', \ldots, S_l'$ of length $l \leq 3^n - 2^{n+1} + 1$, and a sequence of matrices $P(1), P(2), \ldots, P(l)$ from $\mathcal{P}$ such that

$$S_i \cap S_i' = \emptyset \quad \text{for all} \quad i \in \{1, 2, \ldots, l\}$$

and for all $i \in \{1, 2, \ldots, l-1\}$

$$F_{P(i)}(S_i) \subseteq S_{i+1}, \quad F_{P(l)}(S_l) \subseteq S_1$$
$$F_{P(i)}(S_i') \subseteq S_{i+1}', \quad F_{P(l)}(S_l') \subseteq S_1'.$$

*Remark 8:* From [50, Th. 4.7], the values of $\nu$ and $\alpha$, respectively, in conditions (3) and (5) of Theorem 1 can be chosen as $\frac{1}{2}(3^n - 2^{n+1} + 1)$. For our purposes, we choose $\nu = \alpha = 3^n - 2^{n+1} + 1$. The reason for relaxing this upper bound will be clear shortly. Hence, a specific conclusion based on condition (5) of Theorem 1 yields that a compact set $\mathcal{P}$ of $n \times n$ stochastic matrices is not a consensus set if and only if there exists a sequence of matrices $Q(1), Q(2), \ldots, Q(m)$ from $\mathcal{P}$ such that $Q(1) \cdots Q(m-1)Q(m)$ is not a Sarymsakov matrix with $m \geq 3^n - 2^{n+1} + 1$.                                                                 ∎

In view of Remark 8, Theorem 8 is a direct consequence of the following result, whose proof makes use of the properties of Sarymsakov matrices and is given in Appendix F.

*Theorem 9:* Let $\mathcal{P}$ be a compact set of $n \times n$ stochastic matrices. Then, there exists a sequence of matrices $Q(1), Q(2), \ldots, Q(m)$ from $\mathcal{P}$ such that $Q(1) \cdots Q(m-1)Q(m)$ is not a Sarymsakov matrix with $m \geq 3^n - 2^{n+1} + 1$ if and only if there exist two sequences of nonempty subsets of $\mathcal{N}, S_1, S_2, \ldots, S_l$ and $S_1', S_2', \ldots, S_l'$ of length $l \leq 3^n - 2^{n+1} + 1$, and a sequence of matrices $P(1), P(2), \ldots, P(l)$ from $\mathcal{P}$ such that

$$S_i \cap S_i' = \emptyset \quad \text{for all} \quad i \in \{1, 2, \ldots, l\} \tag{19}$$

and for all $i \in \{1, 2, \ldots, l-1\}$

$$F_{P(i)}(S_i) \subseteq S_{i+1}, \quad F_{P(l)}(S_l) \subseteq S_1$$
$$F_{P(i)}(S_i') \subseteq S_{i+1}', \quad F_{P(l)}(S_l') \subseteq S_1'. \tag{20}$$

For doubly stochastic matrices, the necessary and sufficient condition for deciding consensus can be obtained using Proposition 5. We first prove the following result.

*Theorem 10:* Let $\mathcal{P}$ be a compact set of $n \times n$ doubly stochastic matrices, and let

$$b(n) \triangleq \begin{pmatrix} n \\ \lfloor \frac{n-1}{2} \rfloor \end{pmatrix}$$

where $\lfloor n - 1/2 \rfloor$ is the greatest integer that is no larger than $n - 1/2$ and

$$\begin{pmatrix} n \\ \lfloor \frac{n-1}{2} \rfloor \end{pmatrix} = \frac{n!}{\lfloor \frac{n-1}{2} \rfloor! \left( n - \lfloor \frac{n-1}{2} \rfloor \right)!}.$$

Then, $\mathcal{P}$ is a consensus set if and only if for each $k \geq b(n)$ and any $P(i) \in \mathcal{P}$, $1 \leq i \leq k$, the matrix $P(1) \cdots P(k-1)P(k)$ belongs to the Sarymsakov class.

*Proof:* In view of Theorem 1, it is sufficient to prove the necessity. Suppose, therefore, that $\mathcal{P}$ is a consensus set. Assume that there exists a matrix $P(1) \cdots P(k-1)P(k)$ with $k \geq b(n)$ and $P(1) \cdots P(k-1)P(k)$ is not a Sarymsakov matrix. Note that the product of doubly stochastic matrices remains a doubly stochastic matrix. By Lemma 4 and Proposition 4, there exists a nonempty subset $\mathcal{A} \subsetneq \mathcal{N}$ such that

$$|F_{P(1)\cdots P(k-1)P(k)}(\mathcal{A})| = |\mathcal{A}|. \tag{21}$$

Let $\mathcal{A}_0 = \mathcal{A}$ and $A_i = F_{P(i)}(\mathcal{A}_{i-1})$ for $1 \leq i \leq k$. Hence, Lemma 4 and (21) imply that $|\mathcal{A}_0| = |\mathcal{A}_1| = \cdots = |\mathcal{A}_k|$.

We check the total number of nonempty proper subsets with the same cardinality of $\mathcal{N}$. If $n$ is an even number, this number is at most $\binom{n}{n/2}$; if $n$ is an odd number, this number is at most $\binom{n}{n-1/2}$. In both cases, this number is at most $b(n)$. Since $k \geq b(n)$, there exist two indices $j, l$, $0 \leq j < l \leq k$, such that $\mathcal{A}_j = \mathcal{A}_l$. It follows that $(P(j+1) \cdots P(l))^s$ is not a Sarymsakov matrix for each $s = 1, 2, \ldots$ and $\mathcal{P}$ is not a consensus set by Theorem 1. ∎

*Remark 9:* Theorem 10 shows that "$\alpha$" in condition (5) in Theorem 1 can be taken as $b(n)$ when all the matrices in $\mathcal{P}$ are doubly stochastic matrices, instead of $1/2(3^n - 2^{n+1} + 1)$ for general stochastic matrices. ∎

*Theorem 11:* Let $\mathcal{P}$ be a compact set of $n \times n$ doubly stochastic matrices. Then, $\mathcal{P}$ is not a consensus set if and only if there exist a sequence of nonempty subsets of $\mathcal{N}$, $S_1, S_2, \ldots, S_l$ of length $l \leq b(n)$, and a sequence of matrices $P(1), P(2), \ldots, P(l)$ from $\mathcal{P}$ such that for all $i \in \{1, 2, \ldots, l-1\}$

$$F_{P(i)}(S_i) \subseteq S_{i+1}, \quad F_{P(l)}(S_l) \subseteq S_1. \tag{22}$$

*Proof:* (Necessity) Suppose that $\mathcal{P}$ is not a consensus set. By Theorem 10, there exists a sequence of matrices $Q(1), Q(2), \ldots, Q(m)$ from $\mathcal{P}$ such that $Q(1) \cdots Q(m-1)Q(m)$ is not a Sarymsakov matrix with $m \geq b(n)$. Then, from Proposition 4, there exists a nonempty set $\mathcal{A} \subsetneq \mathcal{N}$ such that

$$|F_{Q(1)\cdots Q(m)}(\mathcal{A})| = |\mathcal{A}|.$$

Let $S_1 = \mathcal{A}$, $S_{i+1} = F_{Q(1)\cdots Q(i)}(\mathcal{A})$, for all $i \in \{1, 2, \ldots, m\}$. It follows that $|S_1| = |S_2| = \cdots = |S_{m+1}|$. Note that the number of proper subsets with the same cardinality of $\mathcal{N}$ is at most $b(n)$. Since $m \geq b(n)$, there must exist two sets that are the same, i.e., $S_k = S_r$ for $1 \leq k < r \leq b(n) + 1$. Without loss of generality, assume that $k = 1$. Then, $S_1, S_2, \ldots, S_{r-1}$ with

$r - 1 \leq b(n)$ and the sequence of matrices $Q(1), Q(2), \ldots, Q(r-1)$ satisfy the condition (22).

(Sufficiency) Suppose that a sequence $S_1, S_2, \ldots, S_l$ and a sequence of matrices $P(1), P(2), \ldots, P(l)$ exist. Let $S_1 = \mathcal{A}$. Then similar to the proof of Theorem 9 in Appendix A, we have

$$F_{P(1)\cdots P(l)}(\mathcal{A}) \subseteq F_{P(l)}(S_l) \subseteq S_1 = \mathcal{A}.$$

In view of the fact that $|F_{P(1)\cdots P(l)}(\mathcal{A})| \geq |\mathcal{A}|$, it is clear that $F_{P(1)\cdots P(l)}(\mathcal{A}) = \mathcal{A}$. Hence, $F^k_{P(1)\cdots P(l)}(\mathcal{A}) = \mathcal{A}$ for all integers $k \geq 1$ and, therefore, $(P(1) \cdots P(l))^k$ is not a Sarymsakov matrix by Proposition 4. $\mathcal{P}$ is not a consensus set by Theorem 10. ∎

*Remark 10:* It has been shown in [36] that deciding whether a finite set of stochastic matrices is a consensus set or not is NP-hard. Theorem 11 can be used to decide whether a finite set of doubly stochastic matrices is a consensus set and may be helpful for checking the complexity. ∎

## VI. CONCLUSION

In this paper, we have introduced two classes of generalized Sarymsakov matrices and studied their products. Type-I generalized Sarymsakov matrices are defined using the notion of the SIA index. We have shown that the set of all SIA matrices with SIA indices no larger than $k$ is closed under matrix multiplication only when $k = 1$, which is the set of Sarymsakov matrices. We have constructed a larger subset of SIA matrices than the class of Sarymsakov matrices that is closed under matrix multiplication, and of which any compact subset is a consensus set. For Type-II generalized Sarymsakov matrices, we have provided sufficient conditions for the convergence of the product of an infinite sequence of matrices from this class to a rank-one matrix, and discussed their application to doubly stochastic matrices. We have established a combinatorial necessary and sufficient condition for deciding whether or not a compact set of doubly stochastic matrices is a consensus set.

The results obtained in this paper underscore the critical role of the Sarymsakov class in the set of SIA matrices, and the importance of the generalized Sarymsakov classes in constructing consensus sets and convergent infinite sequences of stochastic matrices. Establishing an even larger set than the one constructed in this paper, which is closed under matrix multiplication and whose compact subsets are all consensus sets, can be very challenging and is a subject for future research. Another important future direction is to study the complexity problem of deciding consensus sets for specific classes of stochastic matrices.

## APPENDIX A

Theorem 4 is an immediate consequence of the forthcoming Lemma 6. To state the lemma, we need to define a matrix $Q$ in terms of a matrix $P \in \mathcal{V}_i$, $i \geq 2$, as follows.

For a given matrix $P \in \mathcal{V}_i$, $i \geq 2$, from the definition of the Sarymsakov class, there exist two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that $F_P(\mathcal{A}) \cap F_P(\tilde{\mathcal{A}}) = \emptyset$ and

$$|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \leq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{23}$$

Define a matrix $Q = [q_{ij}]_{n \times n}$ by setting

$$q_{ij} = \begin{cases} \frac{1}{|\mathcal{A}|}, & i \in F_P(\mathcal{A}), \ j \in \mathcal{A} \\ 0, & i \in F_P(\mathcal{A}), \ j \in \bar{\mathcal{A}} \\ \frac{1}{|\tilde{\mathcal{A}}|}, & i \in F_P(\tilde{\mathcal{A}}), \ j \in \tilde{\mathcal{A}} \\ 0, & i \in F_P(\tilde{\mathcal{A}}), \ j \in \bar{\tilde{\mathcal{A}}} \\ \frac{1}{n}, & \text{otherwise.} \end{cases} \quad (24)$$

Note that whether a stochastic matrix is SIA or not only depends on the positions of its nonzero entries, not on their values. Thus, we can construct other matrices, based on $Q$ in (24), by adjusting the values of the positive entries of $Q$, as long as each row sum equals 1 and the zero-nonzero pattern does not change.

*Lemma 6:* Suppose that $n \geq 3$. Then, for any $i \in \{2, 3, \ldots, l\}$ and any stochastic matrix $P \in \mathcal{V}_i$, the matrix $Q$ given in (24) belongs to the set $\mathcal{W}_2$, and $PQ, QP$ are not SIA. In addition, $Q \in \mathcal{V}_2$ if (23) holds with the equality sign, and $Q \in \mathcal{V}_1$ if the inequality in (23) is strict.

*Proof of Lemma 6:* $Q$ is obviously a stochastic matrix. Note that for any index $i \in \mathcal{N}$, the set of its one-stage consequent indices $F_Q(i)$ can only be $\mathcal{A}$, $\tilde{\mathcal{A}}$, or $\mathcal{N}$. We first show that $Q$ belongs to the set $\mathcal{W}_2$.

Consider two arbitrary disjoint nonempty sets $\mathcal{C}, \tilde{\mathcal{C}} \subseteq \mathcal{N}$. Then, one of the following cases must occur.

 a) $\mathcal{C} \cup \tilde{\mathcal{C}}$ contains some element in $\overline{F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})}$.
 b) $\mathcal{C} \cup \tilde{\mathcal{C}} \subseteq F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})$, $\mathcal{C} \cap F_P(\mathcal{A}) \neq \emptyset$, and $\tilde{\mathcal{C}} \cap F_P(\mathcal{A}) \neq \emptyset$.
 c) $\mathcal{C} \cup \tilde{\mathcal{C}} \subseteq F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})$, $\mathcal{C} \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$, and $\tilde{\mathcal{C}} \cap F_P(\tilde{\mathcal{A}}) \neq \emptyset$.
 d) $\mathcal{C} \subseteq F_P(\mathcal{A})$ and $\tilde{\mathcal{C}} \subseteq F_P(\tilde{\mathcal{A}})$.
 e) $\mathcal{C} \subseteq F_P(\tilde{\mathcal{A}})$ and $\tilde{\mathcal{C}} \subseteq F_P(\mathcal{A})$.

We treat the five cases separately.

*Case (a):* From the definition of the matrix $Q$ in (24), one of $F_Q(\mathcal{C})$ and $F_Q(\tilde{\mathcal{C}})$ must be $\mathcal{N}$, which implies that $F_Q(\mathcal{C}) \cap F_Q(\tilde{\mathcal{C}}) \neq \emptyset$.

*Case (b):* It is easy to see that $\mathcal{A}$ is a subset of both $F_Q(\mathcal{C})$ and $F_Q(\tilde{\mathcal{C}})$. Hence, $F_Q(\mathcal{C}) \cap F_Q(\tilde{\mathcal{C}}) \neq \emptyset$.

*Case (c):* Similar to case (b), $\tilde{\mathcal{A}}$ is a subset of both $F_Q(\mathcal{C})$ and $F_Q(\tilde{\mathcal{C}})$. Hence, $F_Q(\mathcal{C}) \cap F_Q(\tilde{\mathcal{C}}) \neq \emptyset$.

*Case (d):* From the definition of $Q$

$$F_Q(\mathcal{C}) = \mathcal{A}, \quad F_Q(\tilde{\mathcal{C}}) = \tilde{\mathcal{A}}. \quad (25)$$

Following (23)

$$|F_Q(\mathcal{C}) \cup F_Q(\tilde{\mathcal{C}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}| \geq |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \geq |\mathcal{C} \cup \tilde{\mathcal{C}}|.$$

If $|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| > |\mathcal{C} \cup \tilde{\mathcal{C}}|$, then $|F_Q(\mathcal{C}) \cup F_Q(\tilde{\mathcal{C}})| > |\mathcal{C} \cup \tilde{\mathcal{C}}|$. If $|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{C} \cup \tilde{\mathcal{C}}|$, then we consider the following two cases, separately.

 d1) $|\mathcal{A} \cup \tilde{\mathcal{A}}| > |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})|$.
 d2) $|\mathcal{A} \cup \tilde{\mathcal{A}}| = |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})|$.

*Case (d1):* It immediately follows that $|F_Q(\mathcal{C}) \cup F_Q(\tilde{\mathcal{C}})| > |\mathcal{C} \cup \tilde{\mathcal{C}}|$.

*Case (d2):* Since

$$|\mathcal{A} \cup \tilde{\mathcal{A}}| = |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{C} \cup \tilde{\mathcal{C}}|$$

there hold $\mathcal{C} = F_P(\mathcal{A})$ and $\tilde{\mathcal{C}} = F_P(\tilde{\mathcal{A}})$. We further look at the sets of two-stage consequent indices of $\mathcal{C}$ and $\tilde{\mathcal{C}}$, and obtain from (25) that

$$F_Q^2(\mathcal{C}) = F_Q(\mathcal{A}), \quad F_Q^2(\tilde{\mathcal{C}}) = F_Q(\tilde{\mathcal{A}}).$$

We claim that $F_Q(\mathcal{A}) \cap F_Q(\tilde{\mathcal{A}}) \neq \emptyset$, which implies that $k = 2$ is the smallest integer such that (4) holds for this pair of sets, $\mathcal{C}$ and $\tilde{\mathcal{C}}$, and the matrix $Q$. To establish the claim, suppose that, to the contrary, $F_Q(\mathcal{A}) \cap F_Q(\tilde{\mathcal{A}}) = \emptyset$. Since for any $i \in \mathcal{N}$, $F_Q(i)$ can only be $\mathcal{A}$, $\tilde{\mathcal{A}}$, or $\mathcal{N}$, the fact that $F_Q(\mathcal{A}) \cap F_Q(\tilde{\mathcal{A}}) = \emptyset$ implies that either

$$F_Q(\mathcal{A}) = \mathcal{A}, \quad F_Q(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}} \quad (26)$$

or

$$F_Q(\mathcal{A}) = \tilde{\mathcal{A}}, \quad F_Q(\tilde{\mathcal{A}}) = \mathcal{A}. \quad (27)$$

If (26) holds, then it is inferred from the structure of the matrix $Q$ that $\mathcal{A} \subseteq F_P(\mathcal{A})$ and $\tilde{\mathcal{A}} \subseteq F_P(\tilde{\mathcal{A}})$. Combining with the fact that $|F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| = |\mathcal{A} \cup \tilde{\mathcal{A}}|$, it must be true that $F_P(\mathcal{A}) = \mathcal{A}$ and $F_P(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$. It then follows that $F_P^k(\mathcal{A}) = \mathcal{A}$ and $F_P^k(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$ for any positive integer $k$. In view of Corollary 1, $P$ is not an SIA matrix. We conclude that $F_Q^2(\mathcal{C}) \cap F_Q^2(\tilde{\mathcal{C}}) \neq \emptyset$. If (27) holds, then one can similarly show that $F_Q^2(\mathcal{C}) \cap F_Q^2(\tilde{\mathcal{C}}) \neq \emptyset$.

*Case (e):* The discussion is similar to that in case (d).

Therefore, summarizing the discussions in all five cases, we have shown that $Q \in \mathcal{V}_2$ if (23) holds with the equality sign, and $Q \in \mathcal{V}_1$ if the inequality in (23) is strict.

We next consider the matrix product $PQ$. For the pair of sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$, there hold

$$F_{PQ}(\mathcal{A}) = F_Q(F_P(\mathcal{A})) = \mathcal{A}, \quad F_{PQ}(\tilde{\mathcal{A}}) = F_Q(F_P(\tilde{\mathcal{A}})) = \tilde{\mathcal{A}}. \quad (28)$$

Thus, for any positive integer $k$, $F_{PQ}^k(\mathcal{A}) = \mathcal{A}$ and $F_{PQ}^k(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$, which implies that $PQ$ is not an SIA matrix. Similarly, there hold

$$F_{QP}(F_P(\mathcal{A})) = F_P(\mathcal{A}), \quad F_{QP}(F_P(\tilde{\mathcal{A}})) = F_P(\tilde{\mathcal{A}}) \quad (29)$$

which implies that $QP$ is not an SIA matrix. ∎

## APPENDIX B

*Proof of Proposition 1:* It is clear that $\mathcal{G}$ is a subset of $\mathcal{W}$. For any $P \in \mathcal{V}_i$, $i \geq 2$, $P$ is not an element of $\mathcal{G}$, since there exists an SIA matrix $Q$ such that $QP$ is not SIA from Lemma 6. Hence, $\mathcal{G}$ is a subset of $\mathcal{W}_1$.

For $n \geq 3$, let

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

It can be verified that $P \in \mathcal{W}_1$. We claim that $P \notin \mathcal{G}$. To establish the claim, consider the following matrix:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

Since the first column of $Q^2$ is positive, $Q$ is an SIA matrix. Note that

$$QP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ + & + & + & + & \cdots & + \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ + & + & + & + & \cdots & + \end{bmatrix}$$

where "$+$" denotes an element that is positive. For two disjoint nonempty sets $\mathcal{A} = \{1, 2\}$ and $\tilde{\mathcal{A}} = \{3\}$, there hold $F_{QP}^k(\mathcal{A}) = \mathcal{A}$ and $F_{QP}^k(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$ for any positive integer $k$, which implies that $QP$ is not an SIA matrix. Thus, $P$ is not in $\mathcal{G}$. ∎

## APPENDIX C

*Proof of Theorem 5:* Since the matrices in $\mathcal{C}(R)$ have the same zero-nonzero pattern, for $P \in \mathcal{T}$ and $R_1, R_2 \in \mathcal{C}(R)$, $R_1 P$ and $R_2 P$ have the same zero-nonzero pattern. To check the product of two matrices in $\mathcal{T}$, we only have to consider the product of a matrix in $\mathcal{T}$ and $R$.

We first show that for any matrix $P \in \mathcal{W}_1$, both $RP$ and $PR$ are in $\mathcal{W}_1$. Suppose that we are given two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that $F_{RP}(\mathcal{A}) \cap F_{RP}(\tilde{\mathcal{A}}) = \emptyset$. Since $F_{RP}(\mathcal{A}) = F_P(F_R(\mathcal{A}))$ and $F_{RP}(\tilde{\mathcal{A}}) = F_P(F_R(\tilde{\mathcal{A}}))$, from Lemma 1, $F_R(\mathcal{A}) \cap F_R(\tilde{\mathcal{A}}) = \emptyset$. Since $P$ is a Sarymsakov matrix

$$\begin{aligned} |F_{RP}(\mathcal{A}) \cup F_{RP}(\tilde{\mathcal{A}})| &= |F_P(F_R(\mathcal{A})) \cup F_P(F_R(\tilde{\mathcal{A}}))| \\ &> |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| \\ &\geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \end{aligned}$$

It follows that $RP$ is a Sarymsakov matrix. Similarly, for any two disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ such that $F_{PR}(\mathcal{A}) \cap F_{PR}(\tilde{\mathcal{A}}) = \emptyset$, there holds

$$\begin{aligned} |F_{PR}(\mathcal{A}) \cup F_{PR}(\tilde{\mathcal{A}})| &= |F_R(F_P(\mathcal{A})) \cup F_R(F_P(\tilde{\mathcal{A}}))| \\ &\geq |F_P(\mathcal{A}) \cup F_P(\tilde{\mathcal{A}})| \\ &> |\mathcal{A} \cup \tilde{\mathcal{A}}|. \end{aligned}$$

Therefore, $PR$ is also a Sarymsakov matrix.

We next show that $R^2 \in \mathcal{W}_1$. Since $R \in \mathcal{V}_2$, for any disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$, there must exist a positive integer $k \leq 2$ such that either

$$F_R^k(\mathcal{A}) \cap F_R^k(\tilde{\mathcal{A}}) \neq \emptyset \qquad (30)$$

or

$$F_R^k(\mathcal{A}) \cap F_R^k(\tilde{\mathcal{A}}) = \emptyset \ \text{ and } \ |F_R^k(\mathcal{A}) \cup F_R^k(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|. \tag{31}$$

In the case when (30) holds, it follows from Lemma 1 that $F_{R^2}(\mathcal{A}) \cap F_{R^2}(\tilde{\mathcal{A}}) \neq \emptyset$. In the case when (31) holds, suppose that $F_{R^2}(\mathcal{A}) \cap F_{R^2}(\tilde{\mathcal{A}}) = \emptyset$. If (31) holds for $k = 1$, then from the assumption on $R$, there holds

$$|F_{R^2}(\mathcal{A}) \cup F_{R^2}(\tilde{\mathcal{A}})| \geq |F_R(\mathcal{A}) \cup F_R(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|.$$

If (31) holds for $k = 2$, then it immediately follows that $|F_{R^2}(\mathcal{A}) \cup F_{R^2}(\tilde{\mathcal{A}})| > |\mathcal{A} \cup \tilde{\mathcal{A}}|$. Hence, $R^2 \in \mathcal{W}_1$.

Since the set $\mathcal{W}_1$ is closed under matrix multiplication, for any $P_1, P_2 \in \mathcal{T}$, the product $P_2 P_1$ is a Sarymsakov matrix. By induction, it follows that $P_k \cdots P_2 P_1 \in \mathcal{W}_1$ for any integer $k \geq 2$ and $P_i \in \mathcal{T}, i \in \{1, 2, \ldots, k\}$, which implies that $\mathcal{T}$ is closed under matrix multiplication. Then, it follows from Theorem 1 (5) that any compact subset of $\mathcal{T}$ is a consensus set. ∎

## APPENDIX D

*Proof of Theorem 6:* Let $k_0$ be an integer such that $(k_0 - 1)T + 1 \geq j_1$. Since $j_{r+1} - j_r \leq T$ for all $r \geq 1$, for any integer $k \geq k_0$, the matrix sequence $P((k-1)T+1), P((k-1)T+2), \ldots, P(kT)$ contains at least one Sarymsakov matrix, i.e., there exists an integer $i_k$ depending on $k$, $1 \leq i_k \leq T$, such that $P((k-1)T + i_k) \in \mathcal{W}_1$.

We claim that for every integer $k \geq k_0$, the product $P(kT) \cdots P((k-1)T + 2)P((k-1)T + 1)$ is a Sarymsakov matrix. To establish the claim, we consider those pairs of disjoint nonempty sets $\mathcal{A}, \tilde{\mathcal{A}} \subseteq \mathcal{N}$ satisfying

$$F_{P(kT)\cdots P((k-1)T+1)}(\mathcal{A}) \cap F_{P(kT)\cdots P((k-1)T+1)}(\tilde{\mathcal{A}}) = \emptyset.$$

Since $P((k-1)T + i_k) \in \mathcal{W}_1$, combining with the properties of the class $\mathcal{M}$, it follows that

$$\begin{aligned} &|F_{P(kT)\cdots P((k-1)T+1)}(\mathcal{A}) \cup F_{P(kT)\cdots P((k-1)T+1)}(\tilde{\mathcal{A}})| \\ &\geq |F_{P(kT)\cdots P((k-1)T+2)}(\mathcal{A}) \cup F_{P(kT)\cdots P((k-1)T+2)}(\tilde{\mathcal{A}})| \\ &\geq \cdots \\ &\geq |F_{P(kT)\cdots P((k-1)T+i_k)}(\mathcal{A}) \\ &\quad \cup F_{P(kT)\cdots P((k-1)T+i_k)}(\tilde{\mathcal{A}})| \\ &> |F_{P(kT)\cdots P((k-1)T+i_k+1)}(\mathcal{A}) \\ &\quad \cup F_{P(kT)\cdots P((k-1)T+i_k+1)}(\tilde{\mathcal{A}})| \\ &\geq \cdots \\ &\geq |\mathcal{A} \cup \tilde{\mathcal{A}}|. \end{aligned}$$

Therefore, $P(kT) \cdots P((k-1)T+2)P((k-1)T+1)$ is a Sarymsakov matrix.

Define

$$\mathcal{Q}_T = \Big\{ P_T \cdots P_2 P_1 \mid P_i \in \mathcal{P} \text{ for all } i \in \{1, 2, \ldots, T\}$$

$$P_s \in \mathcal{P}' \text{ for some } 1 \leq s \leq T \Big\}. \tag{32}$$

Since both $\mathcal{P}$ and $\mathcal{P}'$ are compact sets, so is $\mathcal{Q}_T$. Note that from the above-mentioned discussion, the product $P(kT)\cdots P((k-1)T+2)P((k-1)T+1) \in \mathcal{Q}_T$ for all $k \geq k_0$ and all matrices in $\mathcal{Q}_T$ are Sarymsakov matrices. From Theorem 1

$$\lim_{k\to\infty} P(kT)\cdots P((k_0-1)T+2)P((k_0-1)T+1) = \mathbf{1}c^T$$

for some nonnegative normalized column vector $c$. For any integer $s \geq 1$, there exists an integer $k$ such that $kT+1 \leq s < (k+1)T$. Let $||\cdot||$ be the induced infinity norm on $\mathbb{R}^{n\times n}$. Then

$$||P(s)\cdots P(2)P(1) - \mathbf{1}c^T P((k_0-1)T)\cdots P(1)||$$
$$= ||P(s)\cdots P(1)$$
$$\quad - P(s)\cdots P(kT+1)\mathbf{1}c^T P((k_0-1)T)\cdots P(1)||$$
$$\leq ||P(s)\cdots P(kT+1)||$$
$$\quad \cdot ||P(kT)\cdots P((k_0-1)T+1) - \mathbf{1}c^T||$$
$$\quad \cdot ||P((k_0-1)T)\cdots P(1)||$$
$$\leq ||P(kT)\cdots P((k_0-1)T+1) - \mathbf{1}c^T||.$$

Thus, the matrix product $P(s)\cdots P(2)P(1)$ converges to a rank-one matrix as $s$ goes to infinity. ∎

## APPENDIX E

The proof of Theorem 7 makes use of some notions and [52, Th. 5], which we review first.

A chain of matrices $\{P(k)\}$ is a stochastic chain if $P(k)$ is a stochastic matrix for all $k \geq 1$, and it is a doubly stochastic chain if $P(k)$ is a doubly stochastic matrix for all $k \geq 1$. A chain $P(k)$ is ergodic if

$$\lim_{k\to\infty} P(k-1)P(k-2)\cdots P(s) = \mathbf{1}c^T(s)$$

for all $s \geq 1$, where $c(s)$ is a stochastic vector for all $s \geq 1$. Let $\{\mathcal{S}(k)\}$ be a sequence of proper index subsets of $\mathcal{N}, k \geq 1$. The sequence $\{\mathcal{S}(k)\}$ is regular if $\mathcal{S}(k)$ have the same nonzero cardinality, i.e., $|\mathcal{S}(k)| = |\mathcal{S}(1)|$ for all $k \geq 1$ and $0 < |\mathcal{S}(1)| < n$.

For a stochastic chain $\{P(k)\}$ and a regular sequence $\{\mathcal{S}(k)\}$, let the flow associated with the entries of the matrix $P(k)$ across $\mathcal{S}(k+1)$ and $\mathcal{S}(k)$ be defined as follows:

$$P_{\mathcal{S}(k+1),\mathcal{S}(k)}(k) = \sum_{\substack{i\in\mathcal{S}(k+1)\\ j\in\mathcal{S}(k)}} p_{ij}(k) + \sum_{\substack{i\in\mathcal{S}(k+1)\\ j\in\overline{\mathcal{S}(k)}}} p_{ij}(k) \quad (33)$$

for $k \geq 1$, where $\overline{\mathcal{S}(k)}$ is the complement of $\mathcal{S}(k)$. Let the total flow of the chain $\{P(k)\}$ over $\{\mathcal{S}(k)\}$ be defined as follows:

$$F(\{P(k)\}; \{\mathcal{S}(k)\}) = \sum_{k=1}^{\infty} P_{\mathcal{S}(k+1),\mathcal{S}(k)}(k). \quad (34)$$

A stochastic chain $\{P(k)\}$ has the absolute infinite flow property if $F(\{P(k)\}; \{\mathcal{S}(k)\}) = \infty$ for every regular sequence $\{\mathcal{S}(k)\}$.

*Lemma 7 (see Theorem 5 of [52]):* A doubly stochastic chain $\{P(k)\}$ is ergodic if and only if it has the absolute infinite flow property.

*Proof of Theorem 7:* Since $P(k_r) \in \mathcal{P}', r = 1, 2, \ldots$, and $\mathcal{P}'$ is compact, there exists a subsequence of $P(k_r), r = 1, 2, \ldots$, still denoted as $P(k_r), r = 1, 2, \ldots$, and a matrix $P \in \mathcal{P}'$ such that $\lim_{r\to\infty} P(k_r) = P$. The infinite sequence of doubly stochastic matrices $P(1), P(2), \ldots$, forms a doubly stochastic chain $\{P(k)\}$. For this chain and any regular sequence $\{\mathcal{S}(k)\}$, note that

$$F(\{P(k)\}; \{\mathcal{S}(k)\})$$
$$\geq \sum_{r=1}^{\infty} \left( \sum_{\substack{i\in\mathcal{S}(k_r+1)\\ j\in\mathcal{S}(k_r)}} p_{ij}(k_r) + \sum_{\substack{i\in\mathcal{S}(k_r+1)\\ j\in\overline{\mathcal{S}(k_r)}}} p_{ij}(k_r) \right).$$

For the regular sequence $\{\mathcal{S}(k)\}$, there exist $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{N}$ and a subsequence of $k_1, k_2, \ldots$, still denoted as $k_1, k_2, \ldots$, such that $\mathcal{S}_{k_r} = \mathcal{S}_1$ and $\mathcal{S}_{k_r+1} = \mathcal{S}_2$. Then it follows that

$$F(\{P(k)\}; \{\mathcal{S}(k)\}) \geq \sum_{r=1}^{\infty} \left( \sum_{\substack{i\in\mathcal{S}_2\\ j\in\mathcal{S}_1}} p_{ij}(k_r) + \sum_{\substack{i\in\mathcal{S}_2\\ j\in\overline{\mathcal{S}_1}}} p_{ij}(k_r) \right).$$

For the Sarymsakov matrix $P$ and the two subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{N}$, we can show that

$$\sum_{\substack{i\in\mathcal{S}_2\\ j\in\overline{\mathcal{S}_1}}} p_{ij} + \sum_{\substack{i\in\mathcal{S}_2\\ j\in\mathcal{S}_1}} p_{ij} > 0. \quad (35)$$

Suppose, to the contrary, that the inequality does not hold. Then $p_{ij} = 0$ for $i \in \mathcal{S}_2, j \in \overline{\mathcal{S}_1}$. This implies that $F_P(\mathcal{S}_2) \subseteq \mathcal{S}_1$ and hence $|F_P(\mathcal{S}_2)| \leq |\mathcal{S}_1| = |\mathcal{S}_2|$, which contradicts the conclusion in Proposition 4. Therefore, (35) holds.

Since $\lim_{r\to\infty} P(k_r) = P$

$$\lim_{r\to\infty} \left[ \left( \sum_{\substack{i\in\mathcal{S}_2\\ j\in\overline{\mathcal{S}_1}}} p_{ij}(k_r) + \sum_{\substack{i\in\mathcal{S}_2\\ j\in\mathcal{S}_1}} p_{ij}(k_r) \right) - \left( \sum_{\substack{i\in\mathcal{S}_2\\ j\in\overline{\mathcal{S}_1}}} p_{ij} + \sum_{\substack{i\in\mathcal{S}_2\\ j\in\mathcal{S}_1}} p_{ij} \right) \right] = 0.$$

This, together with (35), implies that

$$\sum_{r=1}^{\infty} \left( \sum_{\substack{i\in\mathcal{S}_2\\ j\in\overline{\mathcal{S}_1}}} p_{ij}(k_r) + \sum_{\substack{i\in\mathcal{S}_2\\ j\in\mathcal{S}_1}} p_{ij}(k_r) \right) = \infty.$$

Hence, the stochastic chain $\{P(k)\}$ has the absolute infinite flow property. It follows from Lemma 7 that $\lim_{k\to\infty} P(k-1)P(k-2)\cdots P(1) = \mathbf{1}\mathbf{1}^T/n$. ∎

## APPENDIX F

*Proof of Theorem 9:* We first prove the necessity. Suppose, therefore, that $Q(1)\cdots Q(m-1)Q(m)$ is not a Sarymsakov matrix with $m \geq 3^n - 2^{n+1} + 1$. Then, from the definition of Sarymsakov matrices, there exist two disjoint nonempty sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ such that

$$F_{Q(1)\cdots Q(m)}(\mathcal{A}) \cap F_{Q(1)\cdots Q(m)}(\tilde{\mathcal{A}}) = \emptyset$$

and

$$|F_{Q(1)\cdots Q(m)}(\mathcal{A}) \cup F_{Q(1)\cdots Q(m)}(\tilde{\mathcal{A}})| \le |\mathcal{A} \cup \tilde{\mathcal{A}}|.$$

Since

$$F_{Q(1)\cdots Q(m)}(\mathcal{A}) = F_{Q(m)}(F_{Q(1)\cdots Q(m-1)}(\mathcal{A}))$$
$$F_{Q(1)\cdots Q(m)}(\tilde{\mathcal{A}}) = F_{Q(m)}(F_{Q(1)\cdots Q(m-1)}(\tilde{\mathcal{A}}))$$

there holds

$$F_{Q(1)\cdots Q(m-1)}(\mathcal{A}) \cap F_{Q(1)\cdots Q(m-1)}(\tilde{\mathcal{A}}) = \emptyset. \tag{36}$$

By induction, it must be true that for all $i \in \{1, 2, \ldots, m\}$

$$F_{Q(1)\cdots Q(i)}(\mathcal{A}) \cap F_{Q(1)\cdots Q(i)}(\tilde{\mathcal{A}}) = \emptyset. \tag{37}$$

To proceed, let

$$\begin{aligned}
S_1 &= \mathcal{A} \\
S_{i+1} &= F_{Q(1)\cdots Q(i)}(\mathcal{A}) \\
S_1' &= \tilde{\mathcal{A}} \\
S_{i+1}' &= F_{Q(1)\cdots Q(i)}(\tilde{\mathcal{A}})
\end{aligned}$$

for all $i \in \{1, 2, \ldots, m\}$. From (37) and Lemma 1, it follows that $S_i \cap S_i' = \emptyset$

$$\begin{aligned}
S_{i+1} &= F_{Q(1)\cdots Q(i)}(\mathcal{A}) \\
&= F_{Q(i)}(F_{Q(1)\cdots Q(i-1)}(\mathcal{A})) \\
&= F_{Q(i)}(S_i)
\end{aligned}$$

and similarly, $S_{i+1}' = F_{Q(i)}(S_i')$. Note that each pair of sets $S_i$ and $S_i'$ are disjoint and nonempty. By [50, Th. 4.7], the number of ordered partitions $(B_1, B_2, B_3)$ of $\mathcal{N}$ such that $B_1 \cup B_2 \cup B_3 = \mathcal{N}$, $B_1 \cap B_2 = \emptyset$, $B_1 \cap B_3 = \emptyset$, $B_2 \cap B_3 = \emptyset$, and $B_1$ and $B_2$ nonempty, is $3^n - 2^{n+1} + 1$. Consider the sequence of triples of sets $(S_1, S_1', \mathcal{N}\backslash(S_1 \cup S_1')), (S_2, S_2', \mathcal{N}\backslash(S_2 \cup S_2')), \ldots, (S_{m+1}, S_{m+1}', \mathcal{N}\backslash(S_{m+1} \cup S_{m+1}'))$. Since $m + 1 > 3^n - 2^{n+1} + 1$, there must exist two pairs of sets that are the same, i.e., $S_k = S_r$ and $S_k' = S_r'$ for $1 \le k < r \le 3^n - 2^{n+1} + 2$. Without loss of generality, assume that $k = 1$. Then, the two sequences of nonempty sets, $S_1, S_2, \ldots, S_{r-1}$ and $S_1', S_2', \ldots, S_{r-1}'$ with $r - 1 \le 3^n - 2^{n+1} + 1$, and the sequence of matrices $Q(1), Q(2), \ldots, Q(r-1)$ satisfy the conditions (19) and (20).

Now, we prove the sufficiency. Suppose, therefore, that such two sequences of sets, $S_1, \ldots, S_l$ and $S_1', \ldots, S_l'$, and a sequence of matrices $P(1), P(2), \ldots, P(l)$ exist. Let $S_1 = \mathcal{A}$ and $S_1' = \tilde{\mathcal{A}}$. Then, from (20), it follows that

$$F_{P(1)}(\mathcal{A}) = F_{P(1)}(S_1) \subseteq S_2$$
$$F_{P(1)}(\tilde{\mathcal{A}}) = F_{P(1)}(S_1) \subseteq S_2'.$$

Furthermore

$$F_{P(1)P(2)}(\mathcal{A}) = F_{P(2)}(F_{P(1)}(S_1)) \subseteq F_{P(2)}(S_2) \subseteq S_3$$
$$F_{P(1)P(2)}(\tilde{\mathcal{A}}) \subseteq S_3'.$$

Continuing this process, it follows that for all $i \in \{1, 2, \ldots, l-1\}$

$$F_{P(1)\cdots P(i)}(\mathcal{A}) = F_{P(1)\cdots P(i)}(S_1) \subseteq S_{i+1}$$
$$F_{P(1)\cdots P(i)}(\tilde{\mathcal{A}}) = F_{P(1)\cdots P(i)}(S_1') \subseteq S_{i+1}'$$

and for $i = l$

$$F_{P(1)\cdots P(l)}(\mathcal{A}) \subseteq F_{P(l)}(S_l) \subseteq S_1 = \mathcal{A}$$
$$F_{P(1)\cdots P(l)}(\tilde{\mathcal{A}}) \subseteq F_{P(l)}(S_l') \subseteq S_1' = \tilde{\mathcal{A}}.$$

It immediately follows that for any positive integer $k$

$$F_{(P(1)\cdots P(l))^k}(\mathcal{A}) = F_{P(1)\cdots P(l)}^k(\mathcal{A}) \subseteq \mathcal{A}$$
$$F_{(P(1)\cdots P(l))^k}(\tilde{\mathcal{A}}) \subseteq \tilde{\mathcal{A}}.$$

Thus

$$F_{(P(1)\cdots P(l))^k}(\mathcal{A}) \cap F_{(P(1)\cdots P(l))^k}(\tilde{\mathcal{A}}) = \emptyset$$

and

$$|F_{(P(1)\cdots P(l))^k}(\mathcal{A}) \cup F_{(P(1)\cdots P(l))^k}(\tilde{\mathcal{A}})| \le |\mathcal{A} \cup \tilde{\mathcal{A}}|.$$

This implies that $(P(1)P(2)\cdots P(l))^k$ is not a Sarymsakov matrix for any positive integer $k$. Let $k$ be a positive integer such that $kl \ge 3^n - 2^{n+1} + 1$. Then, the matrix product $(P(1)P(2)\cdots P(l))^k$ is not a Sarymsakov matrix. ∎

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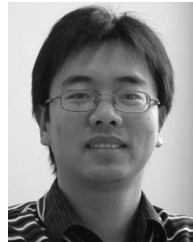
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