Distributed Online Convex Optimization With Time-Varying Coupled Inequality Constraints

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***Abstract*—This paper considers distributed online optimization with time-varying coupled inequality constraints. The global objective function is composed of local convex cost and regularization functions and the coupled constraint function is the sum of local convex functions. A distributed online primal-dual dynamic mirror descent algorithm is proposed to solve this problem, where the local cost, regularization, and constraint functions are held privately and revealed only after each time slot. Without assuming Slater's condition, we first derive regret and constraint violation bounds for the algorithm and show how they depend on the stepsize sequences, the accumulated dynamic variation of the comparator sequence, the number of agents, and the network connectivity. As a result, under some natural decreasing stepsize sequences, we prove that the algorithm achieves sublinear dynamic regret and constraint violation if the accumulated dynamic variation of the optimal sequence also grows sublinearly. We also prove that the algorithm achieves sublinear static regret and constraint violation under mild conditions. Assuming Slater's condition, we show that the algorithm achieves smaller bounds on the constraint violation. In addition, smaller bounds on the static regret are achieved when the objective function is strongly convex. Finally, numerical simulations are provided to illustrate the effectiveness of the theoretical results.**

***Index Terms*—Distributed optimization, dynamic mirror descent, online optimization, time-varying constraints.**

I. Introduction

We consider distributed online optimization with time-varying coupled inequality constraints, which is a sequential decision problem. Specifically, consider a network of n agents indexed by $i = 1, \dots, n$. For each i, let the local decision set $X_i \subseteq \mathbb{R}^{p_i}$ be a closed convex set with p_i being a positive integer. Let $\{f_{i,t} : X_i \to \mathbb{R}\}$, $\{r_{i,t} : X_i \to \mathbb{R}\}$, and $\{g_{i,t} : X_i \to \mathbb{R}^m\}$ be arbitrary sequences of local convex cost, regularization, and constraint functions over time $t = 1, 2, \dots,$ respectively, where m is a positive integer. At time t, each agent i selects a decision $x_{i,t} \in X_i$. After the selection, the agent receives its cost function $f_{i,t}$ and regularization $r_{i,t}$ together with its constraint function $g_{i,t}$, and obtains the loss $l_{i,t}(x_{i,t}) = f_{i,t}(x_{i,t}) + r_{i,t}(x_{i,t})$. Here the regularization function is used to influence the structure of the decisions. Examples of regularization include ℓ_1-regularization $r_{i,t}(x_i) = \lambda_i \|x_i\|_1$ and ℓ_2-regularization $r_{i,t}(x_i) = \frac{\lambda_i}{2}\|x_i\|$ with $\lambda_i > 0$. At the same moment, the agents exchange data with their neighbors over a time-varying directed graph. The network's objective is to choose a global decision sequence $\boldsymbol{x}_T = (x_1, \dots, x_T)$ with $x_t = \mathrm{col}(x_{1,t}, \dots, x_{n,t})$ so that the accumulated global loss $\sum_{t=1}^{T} l_t(x_t)$ is competitive with the loss of any comparator sequence $\boldsymbol{y}_T = (y_1, \dots, y_T)$ with $y_t = \mathrm{col}(y_{1,t}, \dots, y_{n,t})$ (i.e., the regret grows sublinearly in T) and at the same time the constraint violation grows sublinearly in T, where T is the total number of iterations and $l_t(x_t) = \sum_{i=1}^{n} l_{i,t}(x_{i,t})$ is the global loss function.

Specifically, the regret of a global decision sequence $\boldsymbol{x}_T$ with respect to a comparator sequence $\boldsymbol{y}_T$ is defined as

$$\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) = \sum_{t=1}^{T} l_t(x_t) - \sum_{t=1}^{T} l_t(y_t).$$

In the literature, there are two commonly used comparator sequences. One is the optimal dynamic decision sequence $\boldsymbol{y}_T = \boldsymbol{x}^*_T = (x^*_1, \dots, x^*_T)$ solving the following constrained convex optimization problem when the sequences of cost, regularization, and constraint functions are known a priori:

$$\min_{x_t \in \mathbb{X}} \sum_{t=1}^{T} l_t(x_t) \quad \text{s.t.} \quad g_t(x_t) \le \mathbf{0}_m, \quad \forall t = 1, \dots, T, \tag{1}$$

where $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \subseteq \mathbb{R}^p$ is the global decision set, $p = \sum_{i=1}^{n} p_i$, and $g_t(x_t) = \sum_{i=1}^{n} g_{i,t}(x_{i,t})$ is the coupled constraint function. In order to guarantee that problem (1) is feasible, we assume that for any $T \in \mathbb{N}_+$, the set of all feasible decision sequences $\mathcal{X}_T = \{(x_1, \dots, x_T) : x_t \in \mathbb{X}, g_t(x_t) \le \mathbf{0}_m, t = 1, \dots, T\}$ is non-empty. With this standing assumption, an optimal dynamic decision sequence to (1) always exists. In this case $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}^*_T)$ is called the *dynamic regret* for $\boldsymbol{x}_T$. Another comparator sequence is $\boldsymbol{y}_T = \check{\boldsymbol{x}}^*_T =$

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$(\check{x}_T^*, \ldots, \check{x}_T^*)$, where $\check{x}_T^*$ is the optimal static decision solving

$$\min_{x \in \mathbb{X}} \quad \sum_{t=1}^{T} l_t(x)$$
$$\text{s.t.} \quad g_t(x) \le \mathbf{0}_m, \quad \forall t = 1, \ldots, T. \tag{2}$$

Similar to above, in order to guarantee that problem (2) is feasible, we assume that for any $T \in \mathbb{N}_+$, the set of all feasible static decision sequences $\check{\mathcal{X}}_T = \{(x, \ldots, x) : x \in \mathbb{X}, g_t(x) \le \mathbf{0}_m, t = 1, \ldots, T\} \subseteq \mathcal{X}_T$ is non-empty. In this case $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*)$ is called the *static regret*. It is straightforward to see that $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) \le \mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*), \forall \boldsymbol{y}_T \in \mathcal{X}_T$, and that $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \le \mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)$. For a decision sequence $\boldsymbol{x}_T$, the commonly used constraint violation measure is

$$\left\| \left[\sum_{t=1}^{T} g_t(x_t) \right]_+ \right\|,$$

i.e., the accumulation of constraint violations. This definition implicitly allows constraint violations at some times to be compensated by strictly feasible decisions at other times. This is appropriate for constraints that have a cumulative nature such as energy budgets enforced through average power constraints.

This paper develops a distributed online algorithm to solve the problem of distributed online optimization with time-varying coupled inequality constraints with guaranteed performance measured by the regret and constraint violation. We are satisfied with low regret and constraint violation, by which we mean that both $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T)$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|$ grow sublinearly with T, i.e., there exist $\kappa_1, \kappa_2 \in (0, 1)$ such that $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) = \mathcal{O}(T^{\kappa_1})$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{\kappa_2})$. This implies that the upper bound of the time averaged difference between the accumulated cost of the decision sequence and the accumulated cost of any comparator sequences tends to zero as T goes to infinity. The same thing holds for the upper bound of the time averaged constraint violation. The novel algorithm we design explores the stepsize sequences in a way that allows the trade-off between how fast these two bounds tend to zero.

A. Motivating Example

As a motivating example, consider a multi-target tracking problem in which n agents follow n targets. Let $z_i(s), \tilde{z}_i(s)$ denote the positions of agent i and target i at time s, respectively. To model agent and target paths, we introduce a parameterization:

$$z_i(s) = \sum_{k=1}^{p_i} x_{i,t}[k] c_{k,t}(s),$$
$$\tilde{z}_i(s) = \sum_{k=1}^{p_i} \xi_{i,t}[k] c_{k,t}(s), s \in [t, t+1),$$

where $c_{k,t}(s)$ are vector functions that parameterize the space of possible trajectories over time $[t, t+1)$ and satisfy

$$\int_t^{t+1} \langle c_{k,t}(s), c_{l,t}(s) \rangle ds = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{else.} \end{cases}$$

The action spaces of agent i and target i are given by $x_{i,t} = [x_{i,t}[1], \ldots, x_{i,t}[p_i]]^\top \in X_i \subseteq \mathbb{R}^{p_i}$ and $\xi_{i,t} = [\xi_{i,t}[1], \ldots, \xi_{i,t}[p_i]]^\top \in \mathbb{R}^{p_i}$, respectively. At time t, agent i repositions itself by selecting an action $x_{i,t}$ such that it could stay as close as possible to target i during time $[t, t+1)$ and at the same time it wants the selection cost $\langle \pi_{i,t}, x_{i,t} \rangle$ to be as small as possible, where $\pi_{i,t} \in \mathbb{R}_+^{p_i}$ is the price vector. This goal can be captured by defining a local cost function

$$f_{i,t}(x_{i,t}) = \zeta_{i,1} \langle \pi_{i,t}, x_{i,t} \rangle + \zeta_{i,2} \int_t^{t+1} \|z_i(s) - \tilde{z}_i(s)\|^2 ds$$
$$= \zeta_{i,1} \langle \pi_{i,t}, x_{i,t} \rangle + \zeta_{i,2} \|x_{i,t} - \xi_{i,t}\|^2,$$

where $\zeta_{i,1}$ and $\zeta_{i,2}$ are nonnegative constants to trade-off the two subgoals. Here, target i's action $\xi_{i,t}$ and the price vector $\pi_{i,t}$ are observed only after the selection. Agents need to cooperatively take into account energy and communication constraints. For simplicity, we introduce linear local constraint functions $g_{i,t}(x_{i,t}) = D_{i,t} x_{i,t} - d_{i,t}$, where $D_{i,t} \in \mathbb{R}^{m \times p_i}$ and $d_{i,t} \in \mathbb{R}^m$ are time-varying and unknown at time t. These coupling constraints determine the limits on the available resources to be shared among the agents. Section V shows how this multi-target tracking problem can be solved by the algorithm proposed in this paper.

B. Literature Review

The problem of distributed online optimization with time-varying coupled inequality constraints is related to two bodies of literature: centralized online convex optimization with time-varying inequality constraints ($n = 1$) and distributed online convex optimization with time-varying coupled inequality constraints ($n \ge 2$). Depending on the characteristics of the constraint, there are two important special cases: optimization with static constraints ($g_{i,t} \equiv 0$ for all t and i) and time-invariant constraints ($g_{i,t} = g_i$ for all t and i). Below, we provide an overview of the related works.

Centralized online convex optimization with static set constraints was first studied by Zinkevich [1]. Specifically, he developed a projection-based online gradient descent algorithm and achieved $\mathcal{O}(\sqrt{T})$ static regret bound for an arbitrary sequence of convex objective functions with bounded subgradients. It was later shown that this is a tight bound up to constant factors [2]. The regret bound can be reduced under more stringent strong convexity conditions on the objective functions [2]–[5] or by allowing to query the gradient of the objective function multiple times [6]. When the static constrained sets are characterized by inequalities, the conventional projection-based online algorithms are difficult to implement and may be inefficient in practice due to high computational complexity of the projection operation. To overcome these difficulties, some researchers proposed primal-dual algorithms for centralized online convex optimization with time-invariant inequality constraints, e.g., [7]–[10]. The authors of [11] showed that the algorithms proposed in [7], [8] are general enough to handle time-varying inequality constraints. The authors of [12] used the modified saddle-point method to handle time-varying constraints. The papers [13], [14] used a virtual queue, which essentially is a modified Lagrange multiplier, to handle stochastic and time-varying constraints and the authors of [15] extended the algorithm proposed in [14] with bandit feedback. The authors of [16] studied online convex optimization with time-varying constraints in the continuous-time setting and showed that the static regret in continuous-time can be bounded by a constant independent of the time horizon, as opposed to the sublinear static regret observed in the discrete-time setting.

Distributed online convex optimization has been extensively studied, so here we only list some of the most relevant work. Firstly, the authors of [17]–[22] proposed distributed online algorithms to solve convex optimization problems with static set constraints and achieved sublinear regret. For instance, the authors of [21] proposed a decentralized variant of the dynamic mirror descent algorithm proposed in [23]. Mirror descent generalizes classical gradient descent to Bregman divergences and is suitable for solving high-dimensional convex optimization problems. The weighted majority algorithm in machine learning [24] can be viewed as a special case of mirror descent. Secondly, the paper [25] extended the adaptive algorithm proposed in [8] to a distributed setting to solve an online convex optimization problem with a static inequality constraint. Finally, the authors of [26], [27] proposed distributed primal-dual algorithms to solve an online convex optimization with static coupled inequality constraints. To the best of our knowledge, no existing papers considered distributed online convex optimization with time-varying constraints in the discrete-time setting. In the continuous-time setting, the authors of [28] extended the online saddle point algorithm proposed in [16] to a distributed version.

C. Main Contributions

Compared to the literature the contributions of this paper are summarized as follows.

1) We propose a novel distributed online primal-dual dynamic mirror descent algorithm. In this algorithm, each agent i maintains two local sequences: the local decision sequence $\{x_{i,t}\} \subseteq X_i$ and the local dual variable sequence $\{q_{i,t}\} \subseteq \mathbb{R}^m_+$. An agent averages its local dual variable with its in-neighbors in a consensus step, and takes into account the estimated dynamics of the optimal sequences. The proposed algorithm uses different non-increasing stepsize sequences $\{\alpha_t > 0\}$ and $\{\gamma_t > 0\}$ for the primal and dual updates, respectively, and a non-increasing sequence $\{\beta_t > 0\}$ to design penalty terms such that the dual variables are not growing too large. These sequences give some freedom in the regret and constraint violation bounds, as they allow the trade-off between how fast these two bounds tend to zero. The algorithm uses the subgradients of the local cost and constraint functions at the previous decision, but the total number of iterations or any other parameters related to the objective or constraint functions are not used.

2) Without assuming Slater's condition, i.e., that the feasible region has an interior point, we derive regret and constraint violation bounds for the algorithm and show how they depend on the stepsize sequences, the accumulated dynamic variation of the comparator sequence, the number of agents, and the network connectivity. The same regret bound was achieved by the centralized dynamic mirror descent proposed in [23] for static set constraints. With the stepsize sequences $\alpha_t = 1/t^c$, $\beta_t = 1/t^\kappa$, $\gamma_t = 1/t^{(1-\kappa)}$, where $c, \kappa \in (0,1)$ are user-defined trade-off parameters, we prove that our algorithm simultaneously achieves sublinear dynamic regret and constraint violation if the accumulated dynamic variation of the optimal sequence grows sublinearly. Moreover, if $c = \kappa$ we show that the algorithm achieves the same sublinear static regret and constraint violation bounds as in [8], i.e., $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}^*_T) = \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{1-\kappa/2})$. Compared with [7], [8], [10], [11], [27], which assumed the same assumption on the cost and constraint functions as this paper, the proposed algorithm has the following advantages. The parameter κ enables the user to trade-off static regret bound for constraint violation bound, while recovering the $\mathcal{O}(\sqrt{T})$ static regret bound and $\mathcal{O}(T^{3/4})$ constraint violation bound from [7], [11] as special cases. The algorithms proposed in [7], [8], [11] are centralized and the constraint functions in [7], [8] are time-invariant. Moreover, in [7], [11] the total number of iterations and in [7], [8], [11] the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to design the stepsizes. The proposed algorithm achieves smaller static regret and constraint violation bounds than [27], although time-invariant coupled inequality constraints were considered. The algorithm proposed in [10] achieved a better constraint violation bound than ours, but their algorithm is centralized and the constraint function is time-invariant.

3) Assuming Slater's condition and the stepsize sequences above with $c = 1 - \kappa$, we show that the dynamic regret bound is similar to the bound without assuming Slater's condition, but the constraint violation bound can be reduced to $\mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$. Our results are superior to [12] in the sense that the accumulated variation of constraints, $V(\{g_t\}_{t=1}^T) = \sum_{t=1}^T \max_{x\in\mathcal{X}} \|[g_{t+1}(x) - g_t(x)]_+\|$, appears in their bounds and more assumptions are needed. We show that our algorithm simultaneously achieves sublinear dynamic regret and constraint violation, if the accumulated variation of the optimal sequence grows sublinearly. Moreover, the static regret and constraint violation bounds grow as $\mathcal{O}(\sqrt{T})$, which is better than the results for the centralized algorithm in [14]. The authors of [26] achieved the same bounds, but they assumed that the coupled inequality constraints are time-invariant and they explicitly assumed boundedness of the dual variable sequence. The conditions to guarantee this assumption are not so obvious since the dual variable sequence is generated by the algorithm. In this paper, we show that the dual variable sequence is indeed bounded.

4) When the local objective functions are assumed to be strongly convex, we show that, also without Slater's condition, the proposed algorithm achieves $\mathcal{O}(T^\kappa)$ static regret bound and $\mathcal{O}(T^{1-\kappa/2})$ constraint violation bound. Moreover, we find that the constraint violation bound can be reduced to $\mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$ when Slater's condition holds.

The comparison between this paper and the literature is summarized in Table I.

TABLE I
COMPARISON OF THIS PAPER TO SOME RELATED WORKS ON ONLINE CONVEX OPTIMIZATION

References	Problem type	Constraint type	Regret and constraint violation bounds
[7]	Centralized	$g(x) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(\sqrt{T})$, $\|[\sum_{t=1}^T g(x_t)]_+\| = \mathcal{O}(T^{3/4})$
[8]	Centralized	$g(x) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$, $\|[\sum_{t=1}^T g(x_t)]_+\| = \mathcal{O}(T^{1-\kappa/2}), \kappa \in (0,1)$
[10]	Centralized	$g(x) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(\sqrt{T})$, $\sum_{t=1}^T \|[g(x_t)]_+\|^2 = \mathcal{O}(\sqrt{T})$
[11]	Centralized	$g_t(x) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(\sqrt{T})$, $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{3/4})$
[12]	Centralized	$g_t(x) \le \mathbf{0}_m$ and Slater's condition	$\text{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*) = \mathcal{O}(\max\{T^{1/3}\sum_{t=1}^T \|x_t^* - x_{t-1}^*\|, T^{1/3}V(\{g_t\}_{t=1}^T), T^{2/3}\})$, $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{2/3})$,
[14]	Centralized	$g_t(x) \le \mathbf{0}_m$ and Slater's condition	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*)/T \le c\epsilon$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T \le c\epsilon$ for $T \ge 1/\epsilon^2$
[26]	Distributed	$g(x) = \sum_{i=1}^n g_i(x_i) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(\sqrt{T})$, $\|[\sum_{t=1}^T g(x_t)]_+\| = \mathcal{O}(\sqrt{T})$ if dual variables generated by the proposed algorithm are bounded
[27]	Distributed	$g(x) = \sum_{i=1}^n g_i(x_i) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(T^{1/2+2\kappa})$, $\|[\sum_{t=1}^T g(x_t)]_+\| = \mathcal{O}(T^{1-\kappa/2}), \kappa \in (0,1/4)$
This paper	Distributed	$g_t(x) = \sum_{i=1}^n g_{i,t}(x_i) \le \mathbf{0}_m$	$\text{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*) = \mathcal{O}(\max\{T^{\kappa}\sum_{t=1}^{T-1} \|x_{t+1}^* - x_t^*\|, T^{\max\{1-\kappa,\kappa\}}\})$, $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{1-\kappa/2})$ (without Slater's condition), $\|[\sum_{t=1}^T g_t(x_t)]_+\| = \mathcal{O}(T^{\max\{1-\kappa,\kappa\}})$ (with Slater's condition), $\kappa \in (0,1)$

D. Outline

The rest of this paper is organized as follows. Section II introduces the preliminaries. Section III provides the distributed primal-dual dynamic mirror descent algorithm. Section IV analyses the bounds of the regret and constraint violation for the algorithm. Section V gives numerical simulations. Finally, Section VI concludes the paper. Proofs are given in the Appendix.

Notations: All inequalities and equalities are understood componentwise. $\mathbb{R}^n$ and $\mathbb{R}^n_+$ stand for the set of n-dimensional vectors and nonnegative vectors, respectively. $\mathbb{N}_+$ denotes the set of positive integers. $[n]$ represents the set $\{1,\dots,n\}$ for any $n \in \mathbb{N}_+$. $\|\cdot\|$ ($\|\cdot\|_1$) denotes the Euclidean norm (1-norm) for vectors and the induced 2-norm (1-norm) for matrices. $\langle x, y\rangle$ represents the standard inner product of two vectors x and y. $x^\top$ is the transpose of the vector or matrix x. I_n is the n-dimensional identity matrix. $\mathbf{1}_n$ ($\mathbf{0}_n$) denotes the column one (zero) vector of dimension n. $\text{col}(z_1,\dots,z_k)$ is the concatenated column vector of vectors $z_i \in \mathbb{R}^{n_i}, i \in [k]$. $[z]_+$ represents the component-wise projection of a vector $z \in \mathbb{R}^n$ onto $\mathbb{R}^n_+$. $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the ceiling and floor functions, respectively. $\log(\cdot)$ is the natural logarithm. Given two scalar sequences $\{\alpha_t, t \in \mathbb{N}_+\}$ and $\{\beta_t > 0, t \in \mathbb{N}_+\}$, $\alpha_t = \mathcal{O}(\beta_t)$ means that there exists a constant $a > 0$ such that $\alpha_t \le a\beta_t$ for all t, while $\alpha_t = \mathbf{o}(t)$ means that there exist two constants $a > 0$ and $\kappa \in (0,1)$ such that $\alpha_t \le at^{\kappa}$ for all t.

II. PRELIMINARIES

In this section, we present some definitions, properties, and assumptions related to graph theory, projections, subgradients, and Bregman divergence.

A. Graph Theory

Interactions between agents are modeled by a time-varying directed graph. Specifically, at time t, agents communicate with each other according to a directed graph $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$, where $\mathcal{V} = [n]$ is the agent set and $\mathcal{E}_t \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A directed edge $(j,i) \in \mathcal{E}_t$ means that agent i can receive data broadcasted by agent j at time t. Let $\mathcal{N}_i^{\text{in}}(\mathcal{G}_t) = \{j \in [n] \mid (j,i) \in \mathcal{E}_t\}$ and $\mathcal{N}_i^{\text{out}}(\mathcal{G}_t) = \{j \in [n] \mid (i,j) \in \mathcal{E}_t\}$ be the sets of in- and out-neighbors, respectively, of agent i at time t. A directed path is a sequence of consecutive directed edges, and a graph is called strongly connected if there is at least one directed path from any agent to any other agent in the graph. The adjacency matrix $W_t \in \mathbb{R}^{n\times n}$ at time t fulfills $[W_t]_{ij} > 0$ if $(j,i) \in \mathcal{E}_t$ or $i = j$, and $[W_t]_{ij} = 0$ otherwise.

The following mild assumption is made on the graph.

Assumption 1: For any $t \in \mathbb{N}_+$, the graph $\mathcal{G}_t$ satisfies the following conditions:

1) There exists a constant $w \in (0,1)$, such that $[W_t]_{ij} \ge w$ if $[W_t]_{ij} > 0$.
2) The adjacency matrix W_t is doubly stochastic, i.e., $\sum_{i=1}^n [W_t]_{ij} = \sum_{j=1}^n [W_t]_{ij} = 1, \forall i,j \in [n]$.
3) There exists an integer $\iota > 0$ such that the graph $(\mathcal{V}, \cup_{l=0,\dots,\iota-1}\mathcal{E}_{t+l})$ is strongly connected.

B. Projections

For a set $\mathcal{S} \subseteq \mathbb{R}^p$, $\mathcal{P}_{\mathcal{S}}(\cdot)$ is the projection operator

$$\mathcal{P}_{\mathcal{S}}(y) = \arg\min_{x\in\mathcal{S}} \|x - y\|^2, \quad \forall y \in \mathbb{R}^p.$$

This projection always exists and is unique when $\mathcal{S}$ is closed and convex [29]. For simplicity, we use $[\cdot]_+$ to denote $\mathcal{P}_{\mathcal{S}}(\cdot)$ when

$\mathcal{S} = \mathbb{R}^p_+$, which satisfies

$$\|[x]_+ - [y]_+\| \le \|x - y\|, \forall x, y \in \mathbb{R}^p. \tag{3}$$

Moreover, if a function $f : \mathrm{Dom} \to \mathbb{R}$ is convex, then $[f]_+$ is also convex.

C. Subgradients

Definition 1: Let $f : \mathrm{Dom} \to \mathbb{R}$ be a function with $\mathrm{Dom} \subset \mathbb{R}^p$. A vector $g \in \mathbb{R}^p$ is called a subgradient of f at $x \in \mathrm{Dom}$ if

$$f(y) \ge f(x) + \langle g, y - x\rangle, \forall y \in \mathrm{Dom}. \tag{4}$$

The set of all subgradients of f at x, denoted $\partial f(x)$, is called the subdifferential of f at x.

When the function f is convex and differentiable, then its subdifferential at any point x only has a single element, which is exactly its gradient, denoted $\nabla f(x)$. With a slight abuse of the notation, we use $\nabla f(x)$ to denote the subgradient of f at x also when f is not differentiable. Then, $\partial f(x) = \{\nabla f(x)\}$. If f is a closed convex function, then $\partial f(x)$ is non-empty for any $x \in \mathrm{Dom}$ [30]. Similarly, for a vector function $f = [f_1, \dots, f_m]^\top : \mathrm{Dom} \to \mathbb{R}^m$, its subgradient at $x \in \mathrm{Dom}$ is denoted as

$$\nabla f(x) = \begin{bmatrix} (\nabla f_1(x))^\top \\ (\nabla f_2(x))^\top \\ \vdots \\ (\nabla f_m(x))^\top \end{bmatrix} \in \mathbb{R}^{m \times p}.$$

We make the following standing assumption on the cost, regularization, and constraint functions.

Assumption 2:

1) The set X_i is convex and compact for all $i \in [n]$.
2) $\{f_{i,t}\}$, $\{r_{i,t}\}$, and $\{g_{i,t}\}$ are convex and uniformly bounded on X_i, i.e., there exists a constant $F > 0$ such that

$$|f_{i,t}(x)| \le F, |r_{i,t}(x)| \le F,$$
$$\|g_{i,t}(x)\| \le F, \forall t \in \mathbb{N}_+, \forall i \in [n], \forall x \in X_i. \tag{5}$$

3) $\{\nabla f_{i,t}\}$, $\{\nabla r_{i,t}\}$, and $\{\nabla g_{i,t}\}$ exist and they are uniformly bounded on X_i, i.e., there exists a constant $G > 0$ such that

$$\|\nabla f_{i,t}(x)\| \le G, \|\nabla r_{i,t}(x)\| \le G,$$
$$\|\nabla g_{i,t}(x)\| \le G, \forall t \in \mathbb{N}_+, \forall i \in [n], \forall x \in X_i. \tag{6}$$

D. Bregman Divergence

Each agent $i \in [n]$ uses the Bregman divergence $\mathcal{D}_{\psi_i}(x, y)$ to measure the distance between $x \in X_i$ and $y \in X_i$, where

$$\mathcal{D}_{\psi_i}(x, y) = \psi_i(x) - \psi_i(y) - \langle \nabla\psi_i(y), x - y\rangle, \tag{7}$$

and $\psi_i : X_i \to \mathbb{R}$ is a differentiable and strongly convex function with convexity parameter $\sigma_i > 0$. Then, we have $\psi_i(x) \ge \psi_i(y) + \langle \nabla\psi_i(y), x - y\rangle + \frac{\sigma_i}{2}\|x - y\|^2$. Thus,

$$\mathcal{D}_{\psi_i}(x, y) \ge \frac{\underline{\sigma}}{2}\|x - y\|^2, \tag{8}$$

where $\underline{\sigma} = \min\{\sigma_1, \dots, \sigma_n\}$. Hence, $\mathcal{D}_{\psi_i}(\cdot, y)$ is a strongly convex function with convexity parameter $\underline{\sigma}$ for all $y \in X_i$. Additionally, (7) implies that for all $i \in [n]$ and $x, y, z \in X_i$,

$$\langle y - x, \nabla\psi_i(z) - \nabla\psi_i(y)\rangle$$
$$= \mathcal{D}_{\psi_i}(x, z) - \mathcal{D}_{\psi_i}(x, y) - \mathcal{D}_{\psi_i}(y, z). \tag{9}$$

Two well-known examples of Bregman divergence are Euclidean distance $\mathcal{D}_{\psi_i}(x, y) = \|x - y\|^2$ (with X_i an arbitrary convex and compact set in $\mathbb{R}^{p_i}$) generated from $\psi_i(x) = \|x\|^2$, and the Kullback-Leibler (KL) divergence $\mathcal{D}_{\psi_i}(x, y) = -\sum_{j=1}^p x_j \log\frac{y_j}{x_j}$ between two p_i-dimensional standard unit vectors (with X_i the p_i-dimensional probability simplex in $\mathbb{R}^{p_i}$) generated from $\psi_i(x) = \sum_{j=1}^p (x_j \log x_j - x_j)$. One mild assumption on the Bregman divergence is stated as follows.

Assumption 3: For all $i \in [n]$ and $y \in X_i$, $\mathcal{D}_{\psi_i}(\cdot, y) : X_i \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $K > 0$ such that

$$|\mathcal{D}_{\psi_i}(x_1, y) - \mathcal{D}_{\psi_i}(x_2, y)| \le K\|x_1 - x_2\|, \forall x_1, x_2 \in X_i. \tag{10}$$

This assumption is satisfied when ψ_i is Lipschitz on X_i. From Assumptions 2 and 3 it follows that

$$\mathcal{D}_{\psi_i}(x, y) \le d(X)K, \forall x, y \in X_i, \forall i \in [n], \tag{11}$$

where $d(X)$ is a positive constant such that

$$\|x - y\| \le d(X), \forall x, y \in X. \tag{12}$$

To end this section, we introduce a generalized definition of strong convexity.

Definition 2: (Definition 2 in [31]) A convex function $f : \mathrm{Dom} \to \mathbb{R}$ is μ-strongly convex over the convex set Dom with respect to a strongly convex and differentiable function ψ with $\mu > 0$ if for all $x, y \in \mathrm{Dom}$,

$$f(x) \ge f(y) + \langle x - y, \nabla f(y)\rangle + \mu\mathcal{D}_\psi(x, y).$$

This definition generalizes the usual definition of strong convexity by replacing the Euclidean distance with the Bregman divergence.

III. Distributed Online Primal-Dual Dynamic Mirror Descent Algorithms

In this section, we propose a distributed online primal-dual dynamic mirror descent algorithm for solving the problem of distributed online optimization with time-varying coupled inequality constraints. In the next section, we derive regret and constraint violation bounds for this algorithm.

The augmented Lagrangian function associated with the considered problem at each time t is

$$\mathcal{A}_t(x_t, u_t) = f_t(x_t) + r_t(x_t) + u_t^\top g_t(x_t) - \frac{\beta_{t+1}}{2}\|u_t\|^2, \tag{13}$$

where $\{u_t \in \mathbb{R}^m_+\}$ is the dual variable or Lagrange multiplier vector sequence and $\{\beta_t > 0\}$ is the regularization sequence. Inspired by the dynamic mirror descent [23], which is a generalization of the composite objective mirror descent algorithm [32], a centralized online primal-dual dynamic mirror descent algorithm to solve the considered problem is

$$\tilde{x}_{t+1} = \arg\min_{x\in X}\{\alpha_{t+1}(\langle x, \nabla f_t(x_t) + (\nabla g_t(x_t))^\top u_t\rangle + r_t(x_t)) + \mathcal{D}_\psi(x, x_t)\}, \tag{14a}$$

$$u_{t+1} = [u_t + \gamma_{t+1}(g_t(x_t) - \beta_{t+1}u_t)]_+, \tag{14b}$$

$$x_{t+1} = \Phi_{t+1}(\tilde{x}_{t+1}), \tag{14c}$$

where $\{\alpha_t > 0\}$ and $\{\gamma_t > 0\}$ are the stepsize sequences used in the primal and dual updates, respectively; ψ is a strongly convex function to define the Bregman divergence $\mathcal{D}_\psi(\cdot,\cdot)$; and $\Phi_t : X \to X$ is a dynamic model and characterizes a prior knowledge of the considered problem, akin to developing a state space model for stochastic filters [23], and if the prior knowledge is lacking then Φ_t is simply set to the identity mapping. When r_t is a constant mapping and Φ_t is the identity mapping, then the centralized online algorithm (14) is Algorithm 1 in [11]. The potential drawback of that algorithm is that the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to choose the stepsize sequences. In order to avoid using these upper bounds, inspired by the algorithm proposed in [14], we slightly modify the dual update equation (14b) as

$$u_{t+1} = [u_t + \gamma_{t+1}(g_t(x_t) + \nabla g_t(x_t)(x_{t+1} - x_t) - \beta_{t+1}u_t)]_+. \tag{15}$$

Then we modify the centralized online primal-dual dynamic mirror descent algorithm (14a), (15), and (14c) to a distributed manner, which is given in pseudo-code as Algorithm 1. The key difficulty caused by the distributed setting is that each agent does not know the global dual variable. In order to overcome this, the consensus step (16) is introduced such that each agent has an estimation of the global dual variable. In Algorithm 1, the sequences $\{\alpha_t, \beta_t, \gamma_t\}$ play a key role in deriving the regret and constraint violation bounds. They allow the trade-off between how fast these two bounds tend to zero, as will be seen in the next section. With some modifications, all the results in this paper still hold if the coordinated sequences $\alpha_t, \beta_t, \gamma_t$ are replaced by uncoordinated ones $\alpha_{i,t}, \beta_{i,t}, \gamma_{i,t}$. The minimization problem (18) is the composite objective mirror descent [32] and is strongly convex, so it is solvable at a linear convergence rate and closed-form solutions are available in special cases. For example, if $r_{i,t}$ is a constant mapping and Euclidean distance is used as the Bregman distance, i.e., $\mathcal{D}_{\psi_i}(x, y) = \|x - y\|^2$, then (18) can be solved by the projection $\tilde{x}_{i,t} = \mathcal{P}_{X_i}(x_{i,t-1} - \frac{\alpha_t}{2}a_{i,t})$.

In order to execute Algorithm 1, at each iteration t, each agent i needs to know the regularization function at the previous time $t-1$, i.e., $r_{i,t-1}(\cdot)$. This is in many situations a mild assumption since regularization functions are normally predefined to influence the structure of the decision. Furthermore, $g_{i,t-1}(x_{i,t-1})$, $\nabla f_{i,t-1}(x_{i,t-1})$, and $\nabla g_{i,t-1}(x_{i,t-1})$ rather than the full knowledge of $f_{i,t-1}(\cdot)$ and $g_{i,t-1}(\cdot)$ are needed, similar to the assumption on most online algorithms in the literature, cf., [7], [8], [10], [11], [27]. Note that the total number of iterations or any parameters related to the objective or constraint functions, such as upper bounds of the objective and constraint functions or their subgradients, are not used in the algorithm. Also note that no local information related to the primal is exchanged between the agents, but only local dual variables.

Algorithm 1: Distributed Online Primal-Dual Dynamic Mirror Descent.

Input: non-increasing sequences $\{\alpha_t\}$, $\{\beta_t\}$, $\{\gamma_t\} \subseteq (0, 1]$; differentiable and strongly convex functions $\{\psi_i, i \in [n]\}$.
Initialize: $x_{i,1} \in X_i$ and $q_{i,1} = \mathbf{0}_m, \forall i \in [n]$.
for $t = 2, \dots, T$ **do**
  for $i = 1, \dots, n$ in parallel **do**
    Observe $\nabla f_{i,t-1}(x_{i,t-1})$, $\nabla g_{i,t-1}(x_{i,t-1})$, $g_{i,t-1}(x_{i,t-1})$, and $r_{i,t-1}(\cdot)$;
    Determine $\Phi_{i,t}(\cdot)$;
    Update

$$\tilde{q}_{i,t} = \sum_{j=1}^{n}[W_{t-1}]_{ij}q_{j,t-1}, \tag{16}$$

$$a_{i,t} = \nabla f_{i,t-1}(x_{i,t-1}) + (\nabla g_{i,t-1}(x_{i,t-1}))^\top \tilde{q}_{i,t}, \tag{17}$$

$$\tilde{x}_{i,t} = \arg\min_{x\in X_i}\{\alpha_t\langle x, a_{i,t}\rangle + \alpha_t r_{i,t-1}(x) + \mathcal{D}_{\psi_i}(x, x_{i,t-1})\}, \tag{18}$$

$$b_{i,t} = \nabla g_{i,t-1}(x_{i,t-1})(\tilde{x}_{i,t} - x_{i,t-1}) + g_{i,t-1}(x_{i,t-1}), \tag{19}$$

$$q_{i,t} = [\tilde{q}_{i,t} + \gamma_t(b_{i,t} - \beta_t\tilde{q}_{i,t})]_+, \tag{20}$$

$$x_{i,t} = \Phi_{i,t}(\tilde{x}_{i,t}); \tag{21}$$

    Broadcast $q_{i,t}$ to $\mathcal{N}_i^{\rm out}(\mathcal{G}_t)$ and receive $[W_t]_{ij}q_{j,t}$ from $j \in \mathcal{N}_i^{\rm in}(\mathcal{G}_t)$.
  end for
end for
Output: $\boldsymbol{x}_T$.

The dynamic mapping $\Phi_{i,t}$ used in (21) plays the role of a prediction, which is a decentralized variant of the dynamical model Φ_t introduced in [23] and a generalization of the time-invariant linear mapping A used in [21]. If the optimal sequence of agent i has the dynamics $x^*_{i,t} = \Phi^*_{i,t}(x^*_{i,t-1})$ for some true dynamic mapping $\Phi^*_{i,t} : X_i \to X_i$, then $\Phi_{i,t}$ can be viewed as an estimate of $\Phi^*_{i,t}$. If $\Phi_{i,t}$ is equal or close enough to $\Phi^*_{i,t}$, then $x^*_{i,t} - \Phi_{i,t}(x^*_{i,t-1}) = \Phi^*_{i,t}(x^*_{i,t-1}) - \Phi_{i,t}(x^*_{i,t-1})$ is small. $\Phi_{i,t}$ is chosen as the identity mapping if at time t agent i has no knowledge about the dynamics of the optimal sequence.

To end this section, an assumption on the dynamic mapping $\Phi_{i,t}$ is introduced.

Assumption 4: For any $t \in \mathbb{N}_+$ and $i \in [n]$, the dynamic mapping $\Phi_{i,t}$ is nonexpansive, i.e.,

$$\mathcal{D}_{\psi_i}(\Phi_{i,t}(x), \Phi_{i,t}(y)) \le \mathcal{D}_{\psi_i}(x, y), \forall x, y \in X_i. \tag{22}$$

The assumption is used to exclude the situation that any poor prediction made at one step could be exacerbated as the algorithm moves forward. The same assumption can also be found in [21], [23]. An example of the mapping $\Phi_{i,t}$ that satisfies his assumption is the identity mapping.

IV. Regret and Constraint Violation Bounds

This section presents the main results on regret and constraint violation bounds for Algorithm 1, but first some preliminary results are given.

A. Preliminary Results

Firstly, we present two results on the regularized Bregman projection.

Lemma 1: Suppose that $\psi : \mathbb{R}^p \to \mathbb{R}^p$ is a strongly convex function with convexity parameter $\sigma > 0$ and $h : \mathrm{Dom} \to \mathrm{Dom}$ is a convex function with Dom being a convex and closed set in $\mathbb{R}^p$. Moreover, assume that $\nabla h(x), \forall x \in \mathrm{Dom}$, exists and there exists $G_h > 0$ such that $\|\nabla h(x)\| \le G_h, \forall x \in \mathrm{Dom}$. Given $z \in \mathrm{Dom}$, the regularized Bregman projection

$$y = \arg\min_{x\in\mathrm{Dom}}\{h(x) + \mathcal{D}_\psi(x,z)\}, \tag{23}$$

satisfies the following inequalities

$$\langle y - x, \nabla h(y)\rangle \le \mathcal{D}_\psi(x,z) - \mathcal{D}_\psi(x,y) - \mathcal{D}_\psi(y,z), \forall x \in \mathrm{Dom}, \tag{24}$$

$$\|y - z\| \le \frac{G_h}{\sigma}. \tag{25}$$

Proof: See Appendix A. ∎

Note that (24) extends Lemma 6 in [21] and (25) presents an upper bound on the deviation of the optimal point from a fixed point for the regularized Bregman projection. Next we state some results on the local dual variables.

Lemma 2: Suppose Assumptions 1–2 hold. For all $i \in [n]$ and $t \in \mathbb{N}_+$, $\tilde{q}_{i,t}$ and $q_{i,t}$ generated by Algorithm 1 satisfy

$$\|q_{i,t}\| \le \frac{F}{\beta_t}, \|\tilde{q}_{i,t+1}\| \le \frac{F}{\beta_t}, \tag{26}$$

$$\|\tilde{q}_{i,t+1} - \bar{q}_t\| \le n\tau B_1 \sum_{s=1}^{t-1}\gamma_{s+1}\lambda^{t-1-s}, \tag{27}$$

$$\frac{\Delta_{t+1}}{2\gamma_{t+1}} \le \frac{n(B_1)^2}{2}\gamma_{t+1} + [\bar{q}_t - q]^\top g_t(x_t) + E_1(t) + E_2(t) + n\left(\frac{G^2\alpha_{t+1}}{\underline{\sigma}} + \frac{\beta_{t+1}}{2}\right)\|q\|^2, \tag{28}$$

where $\bar{q}_t = \frac{1}{n}\sum_{i=1}^n q_{i,t},\ \tau = (1 - w/2n^2)^{-2} > 1,\ \lambda = (1 - w/2n^2)^{1/\iota}$,

$$\Delta_t = \sum_{i=1}^n \|q_{i,t} - q\|^2 - (1 - \beta_t\gamma_t)\sum_{i=1}^n \|q_{i,t-1} - q\|^2, \tag{29}$$

$B_1 = 2F + Gd(X)$, q is an arbitrary vector in $\mathbb{R}^m_+$, $E_1(t) = n^2\tau B_1 F\sum_{s=1}^t \gamma_{s+1}\lambda^{t-s}$, and

$$E_2(t) = \frac{\underline{\sigma}}{4\alpha_{t+1}}\sum_{i=1}^n \|\tilde{x}_{i,t+1} - x_{i,t}\|^2 + \sum_{i=1}^n [\tilde{q}_{i,t+1}]^\top \nabla g_{i,t}(x_{i,t})(\tilde{x}_{i,t+1} - x_{i,t}).$$

Proof: See Appendix B. ∎

An upper bound of the local dual variables is given in (26) even without Slater's condition. (27) is a standard estimate from the consensus protocol with perturbations and time-varying communication graphs [26] and presents an upper bound on the deviation of the local estimate from the average value of the local dual variables at each iteration. (28) gives an upper bound on the regularized drift of the local dual variables Δ_t, which extends Lemma 3 in [23] from a centralized setting to a distributed one. Next, we provide an upper bound on the regret for one update step.

Lemma 3: Suppose Assumptions 1–4 hold. For all $i \in [n]$, let $\{x_t\}$ be the sequence generated by Algorithm 1 and $\{y_t\}$ be an arbitrary sequence in X, then

$$[\bar{q}_t]^\top g_t(x_t) + l_t(x_t) - l_t(y_t) \le [\bar{q}_t]^\top \frac{4nG^2\alpha_{t+1}}{\underline{\sigma}} + \frac{K}{\alpha_{t+1}}\sum_{i=1}^n \|y_{i,t+1} - \Phi_{i,t+1}(y_{i,t})\| + g_t(y_t) + 2E_1(t) - E_2(t) + E_3(t), \forall t \in \mathbb{N}_+, \tag{30}$$

where

$$E_3(t) = \frac{1}{\alpha_{t+1}}\sum_{i=1}^n \left[\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) - \mathcal{D}_{\psi_i}(y_{i,t+1}, x_{i,t+1})\right].$$

Proof: See Appendix C. ∎

Finally, we derive regret and constraint violation bounds for Algorithm 1.

Lemma 4: Suppose Assumptions 1–4 hold. For any $T \in \mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1. Then, for any comparator sequence $\boldsymbol{y}_T \in \mathcal{X}_T$,

$$\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T) \le \frac{KV_\Phi(\boldsymbol{y}_T)}{\alpha_T} - \frac{1}{2}\sum_{t=1}^T\sum_{i=1}^n \left[\frac{1}{\gamma_t} - \frac{1}{\gamma_{t+1}} + \beta_{t+1}\right]\|q_{i,t}\|^2 + C_{1,1}\sum_{t=1}^T \gamma_{t+1} + C_{1,2}\sum_{t=1}^T \alpha_{t+1} + \sum_{t=1}^T E_3(t), \tag{31}$$

and

$$\left\|\left[\sum_{t=1}^T g_t(x_t)\right]_+\right\|^2 \le E_4(T)\left\{2nFT + \frac{KV_\Phi^*}{\alpha_T}\right.$$

$$-\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\frac{1}{\gamma_t}-\frac{1}{\gamma_{t+1}}+\beta_{t+1}\right)\|q_{i,t}-q_c\|^2$$
$$+C_{1,1}\sum_{t=1}^{T}\gamma_{t+1}+C_{1,2}\sum_{t=1}^{T}\alpha_{t+1}+\sum_{t=1}^{T}E_3(t)\Bigg\}, \quad (32)$$

where $V_\Phi(\boldsymbol{y}_T)=\sum_{t=1}^{T-1}\sum_{i=1}^{n}\|y_{i,t+1}-\Phi_{i,t+1}(y_{i,t})\|$ is the accumulated dynamic variation of the sequence $\boldsymbol{y}_T$ with respect to $\{\Phi_{i,t}\}$, $C_{1,1}=\frac{3n^2\tau B_1 F}{1-\lambda}+\frac{n(B_1)^2}{2}$, $C_{1,2}=\frac{4nG^2}{\underline{\sigma}}$ are constants independent of T, $V_\Phi^*=\min_{\boldsymbol{y}_T\in\mathcal{X}_T}V_\Phi(\boldsymbol{y}_T)$ is the minimum accumulated dynamic variation of all feasible sequences, $E_4(T)=4n[\frac{1}{\gamma_1}+\sum_{t=1}^{T}(\frac{G^2\alpha_{t+1}}{\underline{\sigma}}+\frac{\beta_{t+1}}{2})]$, and $q_c=\frac{2[\sum_{t=1}^{T}g_t(x_t)]_+}{E_4(T)}$.

Proof: See Appendix D. ■

Note that the dependence on the stepsize sequences, the accumulated dynamic variation of the comparator sequence, the number of agents, and the network connectivity is characterized in (31) and (32). The accumulated variation of constraints or the point-wise maximum variation of consecutive constraints defined in [12] do, however, not appear in (31) and (32). This regret bound is the same as the regret bound achieved by the centralized dynamic mirror descent in [23], while [23] only considered static set constraints. The term V_Φ^* in (32) can be replaced by $V_\Phi(\boldsymbol{y}_T)$ since $V_\Phi^*\le V_\Phi(\boldsymbol{y}_T)$. Moreover, if all $\{\Phi_{t,i}\}$ are the identity mapping, then $V_\Phi^*=\min_{\boldsymbol{y}_T\in\check{\mathcal{X}}_T}V_\Phi(\boldsymbol{y}_T)=V_\Phi(\check{\boldsymbol{x}}_T^*)=0$.

In order to obtain sublinear regret and constraint violation bounds, the sequences $\{\alpha_t\},\{\beta_t\},\{\gamma_t\}$ should be properly chosen. Firstly, note that α_t appears in both the denominator and numerator of (31) and (32), so we should let $\alpha_t=\mathcal{O}(\frac{1}{t^c})$ with $c\in(0,1)$ because otherwise one of the terms that contained α_t will grow linearly or superlinearly. Then, note that the dual sequence is not upper-bounded, so we should let $\frac{1}{\gamma_{t+1}}-\frac{1}{\gamma_t}-\beta_{t+1}\alpha_{t+1}\le 0$. In the next section, we characterize the regret and constraint violation bounds based on such sequences.

B. Dynamic Regret and Constraint Violation Bounds

This section states the main results on dynamic regret and constraint violation bounds for Algorithm 1. The succeeding theorem characterizes the bounds based on some natural decreasing stepsize sequences.

Theorem 1: Suppose Assumptions 1–4 hold. For any $T\in\mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1 with

$$\alpha_t=\frac{1}{t^c},\beta_t=\frac{1}{t^\kappa},\ \gamma_t=\frac{1}{t^{1-\kappa}},\ \forall t\in\mathbb{N}_+, \quad (33)$$

where $\kappa\in(0,1)$ and $c\in(0,1)$ are constants. Then,

$$\mathrm{Reg}(\boldsymbol{x}_T,\boldsymbol{x}_T^*)\le C_1\,T^{\max\{1-c,c,\kappa\}}+2KT^cV_\Phi(\boldsymbol{x}_T^*), \quad (34)$$

$$\left\|\left[\sum_{t=1}^{T}g_t(x_t)\right]_+\right\|^2\le C_2T^{\max\{2-c,2-\kappa\}}$$
$$+KC_{2,1}T^{\max\{1,1+c-\kappa\}}V_\Phi^*, \quad (35)$$

where $C_1=\frac{C_{1,1}}{\kappa}+\frac{C_{1,2}}{1-c}+2nd(X)K$, $C_2=C_{2,1}(2nF+C_1)$, and $C_{2,1}=2n(\frac{2G^2}{(1-c)\underline{\sigma}}+\frac{1}{1-\kappa}+2)$ are constants independent of T.

Proof: See Appendix E. ■

Sublinear dynamic regret and constraint violation is thus achieved if $V_\Phi(\boldsymbol{x}_T^*)$ grows sublinearly. If, in this case, there exists a constant $\nu\in[0,1)$, such that $V_\Phi(\boldsymbol{x}_T^*)=\mathcal{O}(T^\nu)$, then setting $c\in(0,1-\nu)$ in Theorem 1 gives $\mathrm{Reg}(\boldsymbol{x}_T,\boldsymbol{x}_T^*)=\mathbf{o}(T)$ and $\|[\sum_{t=1}^{T}g_t(x_t)]_+\|=\mathbf{o}(T)$. $V_\Phi(\boldsymbol{x}_T^*)$ depends on the dynamic mapping $\Phi_{i,t}$. In practice, agents may not know what is a good estimate of $\Phi_{i,t}$ and $\Phi_{i,t}$ may change stochastically. It is for future research how to estimate $\Phi_{i,t}$ from a finite or parametric class of candidates.

From (35), we can see that the constraint violation bound is strictly greater than $\mathcal{O}(\sqrt{T})$ since $\max\{2-c,2-\kappa\}>1$. In the following we show that an $\mathcal{O}(\sqrt{T})$ bound on constraint violation can be achieved if all $\{\Phi_{i,t}\}$ are the identity mapping and the constraint functions $\{g_{i,t}\}$ satisfy Slater's condition, which was also assumed in [12], [14].

Assumption 5: (Slater's condition) There exists a constant $\varepsilon>0$ and a vector $x_c\in X$, such that

$$g_t(x_c)\le-\varepsilon\mathbf{1}_m, t\in\mathbb{N}_+. \quad (36)$$

Theorem 2: Suppose Assumptions 1–5 hold. For any $T\in\mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1 with all $\{\Phi_{i,t}\}$ being the identity mapping, and

$$\alpha_t=\frac{1}{t^{1-\kappa}},\beta_t=\frac{1}{t^\kappa},\ \gamma_t=\frac{1}{t^{1-\kappa}},\forall t\in\mathbb{N}_+, \quad (37)$$

where $\kappa\in(0,1)$. Then,

$$\mathrm{Reg}(\boldsymbol{x}_T,\boldsymbol{x}_T^*)\le C_1\,T^{\max\{1-\kappa,\kappa\}}+2KT^{1-\kappa}V_I(x_T^*), \quad (38)$$

$$\left\|\left[\sum_{t=1}^{T}g_t(x_t)\right]_+\right\|\le C_3T^{\max\{1-\kappa,\kappa\}}, \quad (39)$$

where $V_I(x_T^*)=\sum_{t=1}^{T-1}\|x_{t+1}^*-x_t^*\|$ is the accumulated variation of the optimal sequence $\boldsymbol{x}_T^*$, $C_3=n[2B_2+\frac{B_2}{1-\kappa}+\frac{G^2(B_2+2)\sqrt{m}}{\underline{\sigma}\kappa}]$, $B_2=\max\{2\varepsilon+2\sqrt{\varepsilon^2+nd(X)K},\frac{2B_3}{\varepsilon}\}$, and $B_3=2F+C_{1,1}$ are constants independent of T.

Proof: See Appendix F. ■

From (39), we note that under Slater's condition the constraint violation bound is not affected by the optimal sequences or the point-wise maximum variation of consecutive constraints, which is different from the bounds obtained in [12]. From (38), it follows that sublinear dynamic regret could be achieved if $V_I(\boldsymbol{x}_T^*)$ grows sublinearly with a known upper bound. Then, there exists a constant $\nu\in[0,1)$, such that $V_I(\boldsymbol{x}_T^*)=\mathcal{O}(T^\nu)$, so setting $\kappa\in(\nu,1)$ in Theorem 2 gives $\mathrm{Reg}(\boldsymbol{x}_T,\boldsymbol{x}_T^*)=\mathbf{o}(T)$ and $\|[\sum_{t=1}^{T}g_t(x_t)]_+\|=\mathbf{o}(T)$. Under the additional assumption that the accumulated variation of constraints grows sublinearly with a known upper bound, similar results have been achieved by the modified centralized online saddle-point method proposed in [12]. However, [12] assumed not only that the time-varying constraint functions satisfy Slater's condition but also that the slack constant is larger than the point-wise maximum variation of consecutive constraints. The latter assumption is not always satisfied. Moreover, in [12] the total number of iterations T needs to be known in advance.

C. Static Regret and Constraint Violation Bounds

This section states the main results on static regret and constraint violation bounds for Algorithm 1. When considering static regret, $\{\Phi_{i,t}\}$ should be set to the identity mapping since the static optimal sequence is used as the comparator sequence. In this case, replacing $\boldsymbol{x}_T^*$ by the static sequence $\check{\boldsymbol{x}}_T^*$ in Theorem 1 gives the following results on the bounds of static regret and constraint violation.

Corollary 1: Under the same conditions as stated in Theorem 1 with all $\{\Phi_{i,t}\}$ being the identity mapping and $c = \kappa$, it holds that

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_1\, T^{\max\{1-\kappa,\kappa\}}, \tag{40}$$

$$\left\| \left[\sum_{t=1}^{T} g_t(x_t) \right]_+ \right\| \leq \sqrt{C_2} T^{1-\kappa/2}. \tag{41}$$

Proof: Substituting $c = \kappa$ in Theorem 1 gives the results. ■

From Corollary 1, we know that Algorithm 1 achieves the same static regret and constraint violation bounds as in [8]. As discussed in [8], $\kappa \in (0, 1)$ is a user-defined parameter which enables the trade-off between the static regret bound and the constraint violation bound. Corollary 1 recovers the $\mathcal{O}(\sqrt{T})$ static regret bound and $\mathcal{O}(T^{3/4})$ constraint violation bound from [7], [11] when $\kappa = 0.5$. Moreover, the result extends the $\mathcal{O}(T^{2/3})$ bound for both static regret and constraint violation achieved in [7] for linear constraint functions. However, the algorithms proposed in [7], [8], [11] are centralized and the constraint functions considered in [7], [8] are time-invariant. Moreover, in [7], [11] the total number of iterations and in [7], [8], [11] the upper bounds of the objective and constraint functions and their subgradients need to be known in advance to choose the stepsize sequences. Furthermore, Corollary 1 achieves smaller static regret and constraint violation bounds than [27], although [27] considered time-invariant coupled inequality constraints. However, [27] did not require the time-varying directed graph to be balanced. Although the algorithm proposed in [10] achieved more strict constraint violation bound than our Algorithm 1, that algorithm assumed time-invariant constraint functions and the centralized computations.

Similarly, replacing $\boldsymbol{x}_T^*$ by the static sequence $\check{\boldsymbol{x}}_T^*$ in Theorem 2 gives the following results on the bounds of static regret and constraint violation.

Corollary 2: Under the same conditions as stated in Theorem 2, it holds that

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_1\, T^{\max\{1-\kappa,\kappa\}}, \tag{42}$$

$$\left\| \left[\sum_{t=1}^{T} g_t(x_t) \right]_+ \right\| \leq C_3 T^{\max\{1-\kappa,\kappa\}}. \tag{43}$$

Setting $\kappa = 0.5$ in Corollary 2 gives $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) = \mathcal{O}(\sqrt{T})$ and $\|[\sum_{t=1}^{T} g_t(x_t)]_+\| = \mathcal{O}(\sqrt{T})$. Hence, Algorithm 1 achieves stronger results than [14] and the same results as [13], [26]. However, the algorithms proposed in [13], [14] are centralized and in [13] it is assumed that the constraint functions are independent and identically distributed. Moreover, in [26] the coupled inequality constraints are time-invariant and the boundedness of the dual variable sequence generated by the proposed algorithm is explicitly assumed.

The static regret bounds in Corollaries 1 and 2 can be reduced, if a generalized strong convexity of the local objective functions $f_{i,t} + r_{i,t}$ is assumed. We put the strong convexity assumption on the local cost functions $f_{i,t}$ so $r_{i,t}$ can be simply convex, such as an ℓ_1-regularization.

Assumption 6: For any $i \in [n]$ and $t \in \mathbb{N}_+$, $\{f_{i,t}\}$ are μ_i-strongly convex over X_i with respect to ψ_i with $\mu_i > 0$.

Theorem 3: Suppose Assumptions 1–6 hold. For any $T \in \mathbb{N}_+$, let $\boldsymbol{x}_T$ be the sequence generated by Algorithm 1 with

$$\alpha_t = \frac{1}{t^{\max\{1-\kappa,\kappa\}}}, \beta_t = \frac{1}{t^{\kappa}},\ \gamma_t = \frac{1}{t^{1-\kappa}},\ \forall t \in \mathbb{N}_+, \tag{44}$$

where $\kappa \in (0, 1)$. Then,

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq \max\{C_1, C_4\} T^{\kappa}, \tag{45}$$

$$\left\| \left[\sum_{t=1}^{T} g_t(x_t) \right]_+ \right\| \leq \sqrt{C_2} T^{1-\kappa/2}, \tag{46}$$

where $C_4 = \frac{n(B_1)^2}{2\kappa} + \frac{B_1 C_{1,1}}{\kappa} + \frac{C_{1,2}}{\kappa} + 2nd(X)K(B_4)^{1-\kappa}$, $B_4 = \lceil \frac{1}{(\underline{\mu})^{\frac{1}{\kappa}}} \rceil$, and $\underline{\mu} = \min\{\mu_1, \dots, \mu_n\}$ are constants independent of T.

Proof: See Appendix G. ■

Corollary 3: Under the same conditions as stated in Theorem 2, if Assumption 6 also holds. Then,

$$\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_4 T^{\kappa}, \tag{47}$$

$$\left\| \left[\sum_{t=1}^{T} g_t(x_t) \right]_+ \right\| \leq C_3 T^{\max\{1-\kappa,\kappa\}}. \tag{48}$$

Proof: (47) follows from the first step in the proof of (45) and (48) follows from (39). ■

With some minor modifications, the results stated in Theorem 3 and Corollary 3 still hold if Assumption 6 is replaced by the assumption that for any $i \in [n]$ and $t \in \mathbb{N}_+$, $f_{i,t}$ or $r_{i,t}$ is μ_i-strongly convex over X_i with respect to ψ_i with $\mu_i > 0$.

V. Numerical Simulations

This section evaluates the performance of Algorithm 1 in solving the multi-target tracking problem introduced in Section I-A. In the simulations, for each agent $i \in [n]$, $\Phi_{i,t}$ is set as the identity mapping and the strongly convex function $\psi_i(x) = \sigma\|x\|^2$ is used to define the Bregman divergence $\mathcal{D}_{\psi_i}$. Thus, $\mathcal{D}_{\psi_i}(x, y) = \sigma\|x - y\|^2, \forall i \in [n]$. The stepsize sequences given (44) are used. Moreover, agent i could use a regularization function $r_{i,t}(x_{i,t}) = \lambda_{i,1}\|x_{i,t}\|_1 + \lambda_{i,2}\|x_{i,t}\|^2$ to influence the structure of its action, where $\lambda_{i,1}$ and $\lambda_{i,2}$ are nonnegative constants. At each time t, an undirected graph is used as the communication graph. Specifically, connections between vertices are random and the probability of two vertices being

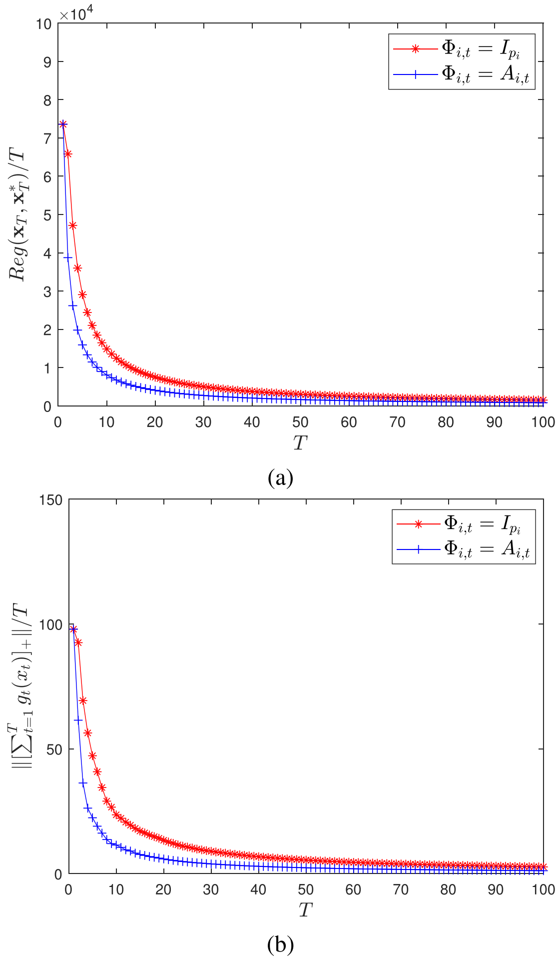

Fig. 1. Comparison of different $\Phi_{i,t}$: (a) Evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$; (b) Evolutions of $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$.

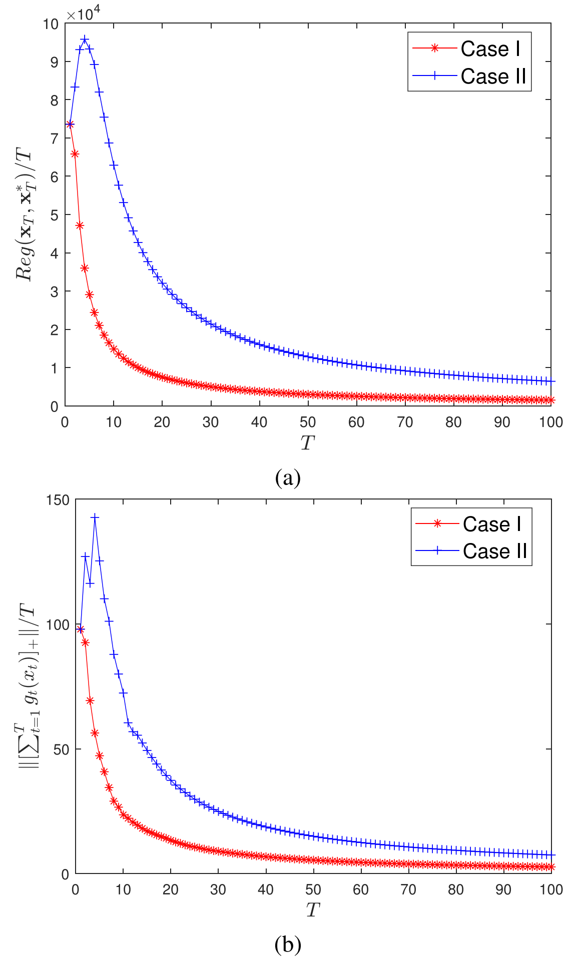

Fig. 2. (a) Evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$. (b) Evolutions of $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$.

connected is ρ. To guarantee that Assumption 1 holds, edges $(i, i+1), i \in [n-1]$ are added and $[W_t]_{ij} = \frac{1}{n}$ if $(j,i) \in \mathcal{E}_t$ and $[W_t]_{ii} = 1 - \sum_{j \in \mathcal{N}_i^{\mathrm{in}}(\mathcal{G}_t)} [W_t]_{ij}$.

We assume $n = 50$, $m = 5$, $\sigma = 10$, $p_i = 6$, $X_i = [0,5]^{p_i}$, $\zeta_{i,1} = \lambda_{i,1} = 1$, $\zeta_{i,2} = \lambda_{i,2} = 30$, $i \in [n]$, and $\rho = 0.2$. Each component of $\pi_{i,t}$ is drawn from the discrete uniform distribution in [0,10] and each component of $D_{i,t}$ is drawn from the discrete uniform distribution in $[-5, 5]$. We let $\xi_{i,t} = [2(\zeta_{i,2} + \lambda_{i,2})x_{i,t}^0 + \zeta_{i,1}\pi_{i,t} + \lambda_{i,1}\mathbf{1}_{p_i}]/(2\zeta_{i,2})$, where $x_{i,t+1}^0 = A_{i,t}x_{i,t}^0$ with $A_{i,t}$ being a doubly stochastic matrix and $x_{i,1}^0$ being a vector that is uniformly drawn from X_i. In order to guarantee the constraints are feasible, we let $d_{i,t} = D_{i,t}x_{i,t}^0$.

A. Dynamics of Optimal Sequences

Under the above settings, we have that $x_{i,t}^* = x_{i,t}^0$. To investigate the dependence of the dynamic regret and constraint violation with $\Phi_{i,t}$, we run Algorithm 1 for two cases: $\Phi_{i,t}$ is the identity mapping and the linear mapping $A_{i,t}$. Figs. 1(a) and (b) show the evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$, respectively, and we can see that knowing the dynamics of the optimal sequence leads to smaller dynamic regret and constraint violation.

B. Regularization Function

To highlight the dependence of the dynamic regret and constraint violation with the regularization function, we run Algorithm 1 for two cases. Case I: $f_{i,t}(x_i) = \zeta_{i,1}\langle \pi_{i,t}, x_i\rangle + \zeta_{i,2}\|H_{i,t}x_i - y_{i,t}\|^2$, $r_{i,t}(x_i) = \lambda_{i,1}\|x_i\|_1 + \lambda_{i,2}\|x_i\|^2$ and Case II: $f_{i,t}(x_i) = \zeta_{i,1}\langle \pi_{i,t}, x_i\rangle + \zeta_{i,2}\|H_{i,t}x_i - y_{i,t}\|^2 + \lambda_{i,1}\|x_i\|_1 + \lambda_{i,2}\|x_i\|^2$, $r_{i,t}(x_i) = 0$. Figs. 2(a) and (b) show the evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$, respectively, for these two cases. From these two figures, we can see that having the regularization term explicitly leads to smaller dynamic regret and constraint violation.

C. Effects of Parameter κ

To investigate the dependence of the dynamic regret and constraint violation with the parameter κ, we run Algorithm 1 with $\kappa = 0.1, 0.3, 0.5, 0.7, 0.9$. Figs. 3(a) and (b) show effects of κ on $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$, respectively, when $T = 100, 500, 1000$. From these two figures, we

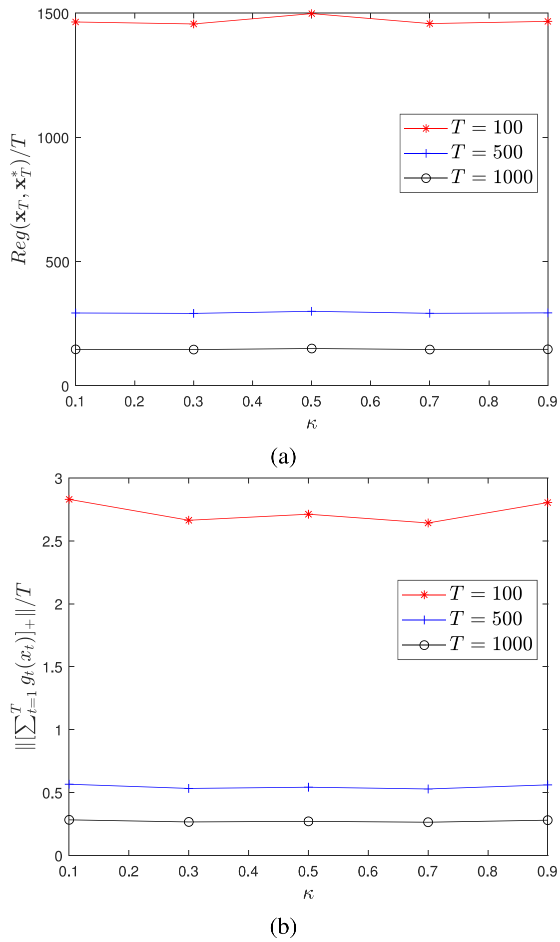

Fig. 3. Effects of parameter κ on (a) $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and (b) $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$ when $T = 100, 500, 1000$.

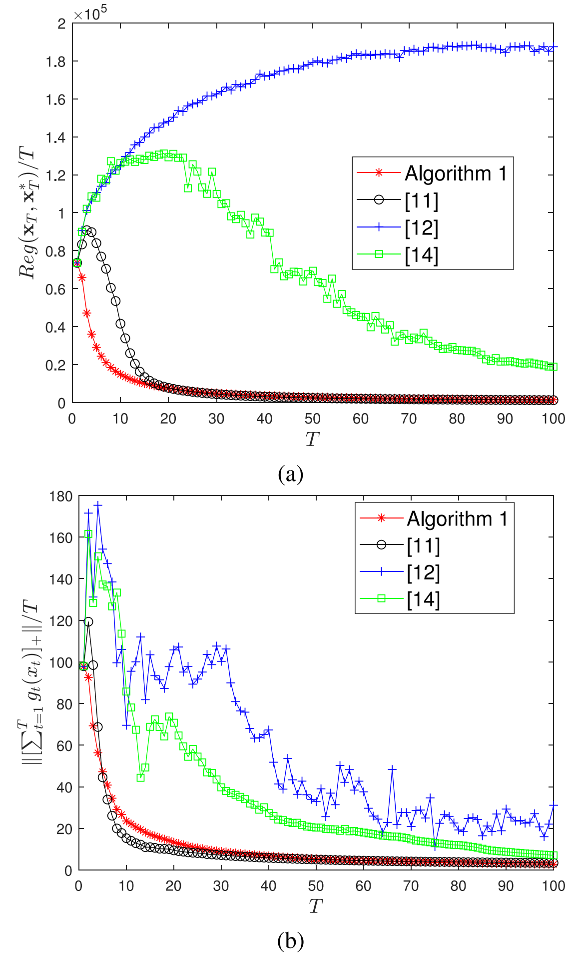

Fig. 4. Comparison of other algorithms: (a) Evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$; (b) Evolutions of $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$.

can see that κ almost does not affect $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$ when T is large (e.g., $T \geq 500$). This phenomenon is not contradictory to the theoretical results shown in Theorem 3 since the theoretical results provide upper bounds of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$.

D. Comparison to Other Algorithms

Since there are no distributed online algorithms to solve the problem of distributed online optimization with time-varying coupled inequality constraints, we compare Algorithm 1 with the centralized online algorithms in [11], [12], [14]. Here, Algorithm 1 in [11] with $\alpha = 10$, $\delta = 1$, and $\mu = 1/\sqrt{T}$, Algorithm 1 in [12] with $\alpha = \mu = T^{-1/3}$, and the virtual queue algorithm in [14] with $V = \sqrt{T}$ and $\alpha = V^2$ are used. Figs. 4(a) and (b) show the evolutions of $\mathrm{Reg}(\boldsymbol{x}_T, \boldsymbol{x}_T^*)/T$ and $\|[\sum_{t=1}^T g_t(x_t)]_+\|/T$, respectively, for these algorithms. From these two figures, we can see that in this example Algorithm 1 achieves smaller dynamic regret and constraint violation than the algorithms in [12], [14] and almost the same values as the algorithm in [11].

VI. Conclusion

In this paper, we considered an online convex optimization problem with time-varying coupled inequality constraints. We proposed a distributed online primal-dual dynamic mirror descent algorithm to solve this problem. We derived regret and constraint violation bounds for the algorithm and showed how they depend on the stepsize sequences, the accumulated dynamic variation of the comparator sequence, the number of agents, and the network connectivity. We proved that the algorithm achieves sublinear regret and constraint violation for both arbitrary and strongly convex objective functions. We showed that the results in this paper can be cast as extensions of existing literature. Future research directions include considering a strict form of the constraint violations, extending the algorithm with bandit feedback, and learning the dynamics of the optimal sequence.

Appendix

A. Proof of Lemma 1

i) Denote $\tilde{h}(x) = h(x) + \mathcal{D}_\psi(x, z)$. Then $\tilde{h}$ is a convex function on Dom. Thus the optimality condition (23), i.e.,

$y = \arg\min_{x \in \mathrm{Dom}} \tilde{h}(x)$, implies $\langle y - x, \nabla\tilde{h}(y)\rangle \le 0, \forall x \in \mathrm{Dom}$. Substituting $\nabla\tilde{h}(y) = \nabla h(y) + \nabla\psi(y) - \nabla\psi(z)$ into the above inequality yields

$$\begin{aligned}&\langle y - x, \nabla h(y)\rangle \le \langle y - x, \nabla\psi(z) - \nabla\psi(y)\rangle\\ &= \mathcal{D}_\psi(x, z) - \mathcal{D}_\psi(x, y) - \mathcal{D}_\psi(y, z), \forall x \in \mathrm{Dom},\end{aligned}$$

where the equality holds since (9). Hence, (24) holds.

ii) $\tilde{h}(x)$ is strongly convex with convexity parameter σ since $\mathcal{D}_\psi$ is strongly convex. It is known that if $\tilde{h} : \mathrm{Dom} \to \mathbb{R}$ is a strongly convex function and is minimized at the point $x^{\min} \in \mathrm{Dom}$, then

$$\tilde{h}(x^{\min}) \le \tilde{h}(x) - \frac{\sigma}{2}\|x - x^{\min}\|^2, \forall x \in \mathrm{Dom}.$$

Thus the optimality condition of (23) implies

$$h(y) + \mathcal{D}_\psi(y, z) \le h(z) + \mathcal{D}_\psi(z, z) - \frac{\sigma}{2}\|z - y\|^2.$$

Noting that $\mathcal{D}_\psi(y, z) \ge \frac{\sigma}{2}\|z - y\|^2$ and $\mathcal{D}_\psi(z, z) = 0$, and rearranging the above inequality gives

$$\sigma\|z - y\|^2 \le \frac{\sigma}{2}\|z - y\|^2 + \mathcal{D}_\psi(y, z) \le h(z) - h(y). \quad (49)$$

From (4) and $\|\nabla h(x)\| \le G_h, \forall x \in \mathrm{Dom}$, we have

$$h(z) - h(y) \le \langle \nabla h(z), z - y\rangle \le G_h\|z - y\|. \quad (50)$$

Thus, combining (49) and (50) yields (25).

B. *Proof of Lemma 2*

i) We prove (26) by induction.

It is straightforward to see that $q_{i,1} = \tilde{q}_{i,2} = \mathbf{0}_m, \forall i \in [n]$, thus $\|q_{i,1}\| \le \frac{F}{\beta_1}, \|\tilde{q}_{i,2}\| \le \frac{F}{\beta_1}, \forall i \in [n]$. Assume that (26) is true at time t for all $i \in [n]$. We show that it remains true at time $t+1$. (4) and (19) imply

$$\begin{aligned}&(1 - \gamma_{t+1}\beta_{t+1})\tilde{q}_{i,t+1} + \gamma_{t+1}b_{i,t+1}\\ &\le (1 - \gamma_{t+1}\beta_{t+1})\tilde{q}_{i,t+1} + \gamma_{t+1}g_{i,t}(\tilde{x}_{i,t+1}). \quad (51)\end{aligned}$$

Since $\|[x]_+\| \le \|y\|$ for all $x \le y$, (20), (51), and (5) imply

$$\begin{aligned}&\|q_{i,t+1}\| \le (1 - \gamma_{t+1}\beta_{t+1})\|\tilde{q}_{i,t+1}\| + \gamma_{t+1}\|g_{i,t}(\tilde{x}_{i,t+1})\|\\ &\le (1 - \gamma_{t+1}\beta_{t+1})\frac{F}{\beta_t} + \gamma_{t+1}F\\ &\le (1 - \gamma_{t+1}\beta_{t+1})\frac{F}{\beta_{t+1}} + \gamma_{t+1}F = \frac{F}{\beta_{t+1}}, \forall i \in [n],\end{aligned}$$

where the last inequality holds due to the sequence $\{\beta_t\}$ is non-increasing. The convexity of norms and $\sum_{j=1}^n [W_t]_{ij} = 1$ yield

$$\begin{aligned}\|\tilde{q}_{i,t+2}\| &\le \sum_{j=1}^n [W_t]_{ij}\|q_{j,t+1}\| \le \sum_{j=1}^n [W_t]_{ij}\frac{F}{\beta_{t+1}}\\ &= \frac{F}{\beta_{t+1}}, \forall i \in [n].\end{aligned}$$

Thus, (26) follows.

ii) We can rewrite (20) as

$$q_{i,t+1} = \sum_{j=1}^n [W_t]_{ij}q_{j,t} + \epsilon^q_{i,t},$$

where $\epsilon^q_{i,t} = [(1 - \gamma_{t+1}\beta_{t+1})\tilde{q}_{i,t+1} + \gamma_{t+1}b_{i,t+1}]_+ - \tilde{q}_{i,t+1}$. From (5), (6), and (12), we have

$$\begin{aligned}\|b_{i,t+1}\| &\le \|g_{i,t}(x_{i,t})\| + \|\nabla g_{i,t}(x_{i,t})\|\|(\tilde{x}_{i,t+1} - x_{i,t})\|\\ &\le F + Gd(X), \forall i \in [n]. \quad (52)\end{aligned}$$

Thus, (3), (26), and (52) give

$$\begin{aligned}\|\epsilon^q_{i,t}\| &\le \| - \gamma_{t+1}\beta_{t+1}\tilde{q}_{i,t+1} + \gamma_{t+1}b_{i,t+1}\|\\ &\le B_1\gamma_{t+1}, \forall i \in [n]. \quad (53)\end{aligned}$$

Then, Lemma 2 in [26], $q_{i,1} = \mathbf{0}_m, \forall i \in [n]$, and (53) yield

$$\|q_{i,t+1} - \bar{q}_{t+1}\| \le n\tau B_1\sum_{s=1}^t \gamma_{s+1}\lambda^{t-s}, \forall i \in [n].$$

So (27) follows since $\sum_{j=1}^n [W_t]_{ij} = 1$ and $\|\tilde{q}_{i,t+1} - \bar{q}_t\| = \|\sum_{j=1}^n [W_t]_{ij}q_{j,t} - \bar{q}_t\| \le \sum_{j=1}^n [W_t]_{ij}\|q_{j,t} - \bar{q}_t\|$.

iii) Applying (3) to (20) gives

$$\begin{aligned}\|q_{i,t} - q\|^2 &\le \left\|(1 - \beta_t\gamma_t)\tilde{q}_{i,t} + \gamma_t b_{i,t} - q\right\|^2\\ &= \|\tilde{q}_{i,t} - q\|^2 + (\gamma_t)^2\|b_{i,t} - \beta_t\tilde{q}_{i,t}\|^2\\ &\quad + 2\gamma_t[\tilde{q}_{i,t}]^\top\nabla g_{i,t-1}(x_{i,t-1})(\tilde{x}_{i,t} - x_{i,t-1})\\ &\quad - 2\gamma_t q^\top\nabla g_{i,t-1}(x_{i,t-1})(\tilde{x}_{i,t} - x_{i,t-1})\\ &\quad + 2\gamma_t[\tilde{q}_{i,t} - q]^\top g_{i,t-1}(x_{i,t-1})\\ &\quad - 2\beta_t\gamma_t[\tilde{q}_{i,t} - q]^\top\tilde{q}_{i,t}. \quad (54)\end{aligned}$$

For the first term of the right-hand side of (54), by convexity of norms and $\sum_{j=1}^n [W_{t-1}]_{ij} = 1$, it can be concluded that

$$\begin{aligned}\|\tilde{q}_{i,t} - q\|^2 &= \left\|\sum_{j=1}^n [W_{t-1}]_{ij}q_{j,t-1} - \sum_{j=1}^n [W_{t-1}]_{ij}q\right\|^2\\ &\le \sum_{j=1}^n [W_{t-1}]_{ij}\|q_{j,t-1} - q\|^2. \quad (55)\end{aligned}$$

For the second term of the right-hand side of (54), (26) and (52) yield

$$(\gamma_t)^2\|b_{i,t} - \beta_t\tilde{q}_{i,t}\|^2 \le (B_1\gamma_t)^2. \quad (56)$$

For the fourth term of the right-hand side of (54), (6) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}&- 2\gamma_t q^\top\nabla g_{i,t-1}(x_{i,t-1})(\tilde{x}_{i,t} - x_{i,t-1})\\ &\le 2\gamma_t\left(\frac{G^2\alpha_t}{\sigma}\|q\|^2 + \frac{\sigma}{4\alpha_t}\|\tilde{x}_{i,t} - x_{i,t-1}\|^2\right). \quad (57)\end{aligned}$$

For the fifth term of the right-hand side of (54), we have

$$\begin{aligned}2\gamma_t[\tilde{q}_{i,t} - q]^\top g_{i,t-1}(x_{i,t-1}) &= 2\gamma_t[\bar{q}_{t-1} - q]^\top g_{i,t-1}(x_{i,t-1})\\ &\quad + 2\gamma_t[\tilde{q}_{i,t} - \bar{q}_{t-1}]^\top g_{i,t-1}(x_{i,t-1}). \quad (58)\end{aligned}$$

Moreover, from (5) and (27), we have

$$2\gamma_t[\tilde{q}_{i,t}-\bar{q}_{t-1}]^\top g_{i,t-1}(x_{i,t-1}) \leq 2\gamma_t\|\tilde{q}_{i,t}-\bar{q}_{t-1}\|\|g_{i,t-1}(x_{i,t-1})\| \leq \frac{2\gamma_t E_1(t-1)}{n}. \quad (59)$$

For the last term of the right-hand side of (54), neglecting the nonnegative term $\beta_t\gamma_t\|\tilde{q}_{i,t}\|^2$ gives

$$-2\beta_t\gamma_t[\tilde{q}_{i,t}-q]^\top\tilde{q}_{i,t} \leq \beta_t\gamma_t(\|q\|^2-\|\tilde{q}_{i,t}-q\|^2). \quad (60)$$

Then, combining (54)–(60), summing over $i\in[n]$, and dividing by $2\gamma_t$, and using $\sum_{i=1}^n[W_{t-1}]_{ij}=1, \forall t\in\mathbb{N}_+$ yields (28).

C. Proof of Lemma 3

From (4), we have

$$\begin{aligned} &l_{i,t}(x_{i,t})-l_{i,t}(y_{i,t}) \\ &= f_{i,t}(x_{i,t})-f_{i,t}(y_{i,t})+r_{i,t}(x_{i,t})-r_{i,t}(\tilde{x}_{i,t+1}) \\ &\quad + r_{i,t}(\tilde{x}_{i,t+1})-r_{i,t}(y_{i,t}) \\ &\leq \langle\nabla f_{i,t}(x_{i,t}),x_{i,t}-y_{i,t}\rangle+\langle\nabla r_{i,t}(x_{i,t}),x_{i,t}-\tilde{x}_{i,t+1}\rangle \\ &\quad + \langle\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle \\ &= \langle\nabla f_{i,t}(x_{i,t})+\nabla r_{i,t}(x_{i,t}),x_{i,t}-\tilde{x}_{i,t+1}\rangle \\ &\quad + \langle\nabla f_{i,t}(x_{i,t})+\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle. \end{aligned} \quad (61)$$

We now bound each of the two terms above. For the first term, (6) and the Cauchy-Schwarz inequality give

$$\begin{aligned} &\langle\nabla f_{i,t}(x_{i,t})+\nabla r_{i,t}(x_{i,t}),x_{i,t}-\tilde{x}_{i,t+1}\rangle \\ &\leq 2G\|x_{i,t}-\tilde{x}_{i,t+1}\| \\ &\leq \frac{\sigma}{4\alpha_{t+1}}\|x_{i,t}-\tilde{x}_{i,t+1}\|^2+\frac{4\,G^2\alpha_{t+1}}{\sigma}. \end{aligned} \quad (62)$$

For the second term, we have

$$\begin{aligned} &\langle\nabla f_{i,t}(x_{i,t})+\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle \\ &= \langle(\nabla g_{i,t}(x_{i,t}))^\top\tilde{q}_{i,t+1},y_{i,t}-\tilde{x}_{i,t+1}\rangle \\ &\quad + \langle a_{i,t+1}+\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle \\ &= \langle(\nabla g_{i,t}(x_{i,t}))^\top\tilde{q}_{i,t+1},y_{i,t}-x_{i,t}\rangle \\ &\quad + \langle(\nabla g_{i,t}(x_{i,t}))^\top\tilde{q}_{i,t+1},x_{i,t}-\tilde{x}_{i,t+1}\rangle \\ &\quad + \langle a_{i,t+1}+\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle. \end{aligned} \quad (63)$$

From (4) and $\tilde{q}_{i,t}\geq\mathbf{0}_m, \forall t\in\mathbb{N}_+, \forall i\in[n]$, we have

$$\begin{aligned} &\langle(\nabla g_{i,t}(x_{i,t}))^\top\tilde{q}_{i,t+1},y_{i,t}-x_{i,t}\rangle \\ &\leq [\tilde{q}_{i,t+1}]^\top g_{i,t}(y_{i,t})-[\tilde{q}_{i,t+1}]^\top g_{i,t}(x_{i,t}) \\ &= [\bar{q}_t]^\top[g_{i,t}(y_{i,t})-g_{i,t}(x_{i,t})] \\ &\quad + [\tilde{q}_{i,t+1}-\bar{q}_t]^\top[g_{i,t}(y_{i,t})-g_{i,t}(x_{i,t})]. \end{aligned} \quad (64)$$

Similar to (59), we have

$$[\tilde{q}_{i,t+1}-\bar{q}_t]^\top[g_{i,t}(y_{i,t})-g_{i,t}(x_{i,t})] \leq \frac{2E_1(t)}{n}. \quad (65)$$

Applying (24) to the update rule (18), we get

$$\begin{aligned} &\langle a_{i,t+1}+\nabla r_{i,t}(\tilde{x}_{i,t+1}),\tilde{x}_{i,t+1}-y_{i,t}\rangle \\ &\leq \frac{1}{\alpha_{t+1}}[\mathcal{D}_{\psi_i}(y_{i,t},x_{i,t})-\mathcal{D}_{\psi_i}(y_{i,t},\tilde{x}_{i,t+1}) \\ &\quad - \mathcal{D}_{\psi_i}(\tilde{x}_{i,t+1},x_{i,t})] \\ &= \frac{1}{\alpha_{t+1}}[\mathcal{D}_{\psi_i}(y_{i,t},x_{i,t})-\mathcal{D}_{\psi_i}(y_{i,t+1},x_{i,t+1}) \\ &\quad + \mathcal{D}_{\psi_i}(y_{i,t+1},x_{i,t+1})-\mathcal{D}_{\psi_i}(\Phi_{i,t+1}(y_{i,t}),x_{i,t+1}) \\ &\quad + \mathcal{D}_{\psi_i}(\Phi_{i,t+1}(y_{i,t}),x_{i,t+1})-\mathcal{D}_{\psi_i}(y_{i,t},\tilde{x}_{i,t+1}) \\ &\quad - \mathcal{D}_{\psi_i}(\tilde{x}_{i,t+1},x_{i,t})] \\ &\leq \frac{1}{\alpha_{t+1}}[\mathcal{D}_{\psi_i}(y_{i,t},x_{i,t})-\mathcal{D}_{\psi_i}(y_{i,t+1},x_{i,t+1}) \\ &\quad + K\|y_{i,t+1}-\Phi_{i,t+1}(y_{i,t})\|-\frac{\sigma}{2}\|\tilde{x}_{i,t+1}-x_{i,t}\|^2], \end{aligned} \quad (66)$$

where the last inequality holds since (21), (22), (10), and (8).

Combining (61)–(66) and summing over $i\in[n]$ yields (30).

D. Proof of Lemma 4

i) The definition of Δ_t given by (29) yields

$$\begin{aligned} -\frac{\Delta_t}{2\gamma_t} &= \frac{1}{2\gamma_t}\sum_{i=1}^n[(1-\beta_t\gamma_t)\|q_{i,t-1}-q\|^2-\|q_{i,t}-q\|^2] \\ &= \frac{1}{2}\sum_{i=1}^n\left[\frac{1}{\gamma_{t-1}}\|q_{i,t-1}-q\|^2-\frac{1}{\gamma_t}\|q_{i,t}-q\|^2\right] \\ &\quad + \frac{1}{2}\sum_{i=1}^n\left(\frac{1}{\gamma_t}-\frac{1}{\gamma_{t-1}}-\beta_t\right)\|q_{i,t-1}-q\|^2. \end{aligned} \quad (67)$$

For any nonnegative sequence $\zeta_1,\zeta_2,\dots$, it holds that

$$\sum_{t=1}^T\sum_{s=1}^t\zeta_{s+1}\lambda^{t-s}=\sum_{t=1}^T\zeta_{t+1}\sum_{s=0}^{T-t}\lambda^s\leq\frac{1}{(1-\lambda)}\sum_{t=1}^T\zeta_{t+1}. \quad (68)$$

Let $g_c:\mathbb{R}_+^m\to\mathbb{R}$ be a function defined as

$$g_c(q)=\left[\sum_{t=1}^T g_t(x_t)\right]^\top q - n\left[\frac{1}{\gamma_1}+\sum_{t=1}^T\left(\frac{G^2\alpha_{t+1}}{\sigma}+\frac{\beta_{t+1}}{2}\right)\right]\|q\|^2. \quad (69)$$

Combining (28) and (30), summing over $t\in[T]$, neglecting the nonnegative term $\|q_{i,T+1}-q\|^2$, and using (67)–(69), $\|q_{i,1}-q\|^2\leq 2\|q_{i,1}\|^2+2\|q\|^2=2\|q\|^2$, and $g_t(y_t)\leq\mathbf{0}_m, \boldsymbol{y}_T\in\mathcal{X}_T$ yields

$$\begin{aligned} &g_c(q)+\mathrm{Reg}(\boldsymbol{x}_T,\boldsymbol{y}_T) \\ &\leq C_{1,1}\sum_{t=1}^T\gamma_{t+1}+\frac{4nG^2}{\sigma}\sum_{t=1}^T\alpha_{t+1}+\sum_{t=1}^T E_3(t) \end{aligned}$$

$$- \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\frac{1}{\gamma_t} - \frac{1}{\gamma_{t+1}} + \beta_{t+1}\right)\|q_{i,t} - q\|^2$$
$$+ K\sum_{t=1}^{T}\sum_{i=1}^{n}\frac{\|y_{i,t+1} - \Phi_{i,t+1}(y_{i,t})\|}{\alpha_{t+1}}, \forall q \in \mathbb{R}^m_+. \quad (70)$$

Then, substituting $q = \mathbf{0}_m$ into (70), setting $y_{i,T+1} = \Phi_{i,T+1}(y_{i,T})$, noting that $\{\alpha_t\}$ is non-increasing, and rearranging the terms yields (31).

ii) Substituting $q = q_c$ into $g_c(q)$ gives

$$g_c(q_c) = \frac{\|[\sum_{t=1}^{T} g_t(x_t)]_+\|^2}{E_4(T)}. \quad (71)$$

Moreover, (5) gives

$$|\operatorname{Reg}(\boldsymbol{x}_T, \boldsymbol{y}_T)| \le 2nFT, \forall \boldsymbol{y}_T \in \mathcal{X}_T. \quad (72)$$

Substituting $q = q_c$ into (70), combining (71)–(72), and rearranging the terms gives (32).

E. Proof of Theorem 1

i) For any constant $\kappa < 1$ and $T \in \mathbb{N}_+$, it holds that

$$\sum_{t=1}^{T}\frac{1}{t^\kappa} \le \int_1^T \frac{1}{t^\kappa}dt + 1 = \frac{T^{1-\kappa} - \kappa}{1-\kappa} \le \frac{T^{1-\kappa}}{1-\kappa}. \quad (73)$$

Applying (73) to the third and forth terms of the right-hand side of (31) gives

$$C_{1,1}\sum_{t=1}^{T}\gamma_{t+1} \le \frac{C_{1,1}}{\kappa}T^\kappa, \quad (74)$$

$$C_{1,2}\sum_{t=1}^{T}\alpha_{t+1} \le \frac{C_{1,2}}{1-c}T^{1-c}. \quad (75)$$

Noting that $\{\alpha_t\}$ is non-increasing and (11), for any $s \in [T]$, we have

$$\sum_{t=s}^{T} E_3(t) =$$
$$\sum_{t=s}^{T}\sum_{i=1}^{n}\left[\frac{1}{\alpha_t}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) - \frac{1}{\alpha_{t+1}}\mathcal{D}_{\psi_i}(y_{i,t+1}, x_{i,t+1})\right]$$
$$+ \sum_{t=s}^{T}\sum_{i=1}^{n}\left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}\right)\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t})$$
$$\le \frac{1}{\alpha_s}\sum_{i=1}^{n}\mathcal{D}_{\psi_i}(y_{i,s}, x_{i,s}) - \frac{1}{\alpha_{T+1}}\sum_{i=1}^{n}\mathcal{D}_{\psi_i}(y_{i,T+1}, x_{i,T+1})$$
$$+ n\left(\frac{1}{\alpha_{T+1}} - \frac{1}{\alpha_s}\right)d(X)K \le \frac{nd(X)K}{\alpha_{T+1}}. \quad (76)$$

Combining (31) and (74)–(76), setting $y_{i,t} = x^*_{i,t}, \forall t \in [T]$, and noting that the second last term of the right-hand side of (31) is non-positive since $\frac{1}{\gamma_t} - \frac{1}{\gamma_{t+1}} + \beta_{t+1} > 0$ yields (34).

ii) Using (73) gives

$$E_4(T) \le C_{2,1}T^{\max\{1-c, 1-\kappa\}}. \quad (77)$$

Combining (32) and (74)–(77) and noting that the last term of the right-hand side of (32) is non-positive since $\frac{1}{\gamma_t} - \frac{1}{\gamma_{t+1}} + \beta_{t+1} > 0$ gives (35).

F. Proof of Theorem 2

i) Substituting $c = 1 - \kappa$ in (34) gives (38).

ii) We first show that $\|q_t\| \le B_2$ by induction, where $q_t = \operatorname{col}(q_{1,t}, \dots, q_{n,t})$.

It is straightforward to see that $\|q_1\| = 0 \le B_2$. Suppose that there exists $T_1 \in \mathbb{N}_+$ such that $\|q_t\| \le B_2, \forall t \in [T_1]$. We show that $\|q_{T_1+1}\| \le B_2$ by contradiction. Now suppose that $\|q_{T_1+1}\| > B_2$. Noting that $\|\bar{q}_{T_1+1}\|_1 = \|q_{T_1+1}\|_1 \ge \|q_{T_1+1}\| > B_2$ and $\|\bar{q}_1\|_1 = 0$, we know that there exists $t_0 \in [T_1]$ such that $\|\bar{q}_{t_0}\|_1 \le \frac{B_2}{2}$. Let $t_1 = \max\{t_0 : \|\bar{q}_{t_0}\|_1 \le \frac{B_2}{2}, t_0 \in [T_1]\}$. Combining (28) and (30), substituting $q = \mathbf{0}_m$ and $y_t = x_c$, setting $\{\Phi_{t,i}\}$ as the identity mapping, and using $|f_t(x_t) - f_t(x_0)| \le 2F$ and (36) yields

$$\|q_{t+1}\|^2 - (1 - \beta_{t+1}\gamma_{t+1})\|q_t\|^2$$
$$\le 2B_3\gamma_{t+1} + 2\gamma_{t+1}E_2(t+1) - 2\varepsilon\|\bar{q}_t\|_1\gamma_{t+1}. \quad (78)$$

Summing (78) over $t \in \{t_1, \dots, T_1\}$, using (11), $\alpha_t = \gamma_t = \frac{1}{t^{1-\kappa}}$ and $\beta_t \ge 0$, and noting that $\|q_{T_1+1}\| > B_2$, $\|q_{t_1}\| \le \|\bar{q}_{t_1}\|_1 \le \frac{B_2}{2}$, and $\|\bar{q}_t\|_1 > \frac{B_2}{2}, \forall t \in \{t_1+1, \dots, T_1\}$ gives

$$\frac{3(B_2)^2}{4} < \|q_{T_1+1}\|^2 - \|q_{t_1}\|^2 + \sum_{t=t_1}^{T_1}\beta_{t+1}\gamma_{t+1}\|q_t\|^2$$
$$\le 2B_3\sum_{t=t_1}^{T_1}\gamma_{t+1} + 2nd(X)K - 2\varepsilon\sum_{t=t_1}^{T_1}\|\bar{q}_t\|_1\gamma_{t+1}$$
$$\le \frac{2B_3}{\kappa}[(T_1+1)^\kappa - (t_1+1)^\kappa] + 2B_3 + 2nd(X)K$$
$$- \frac{\varepsilon B_2}{\kappa}[(T_1+1)^\kappa - (t_1+1)^\kappa] + \varepsilon B_2 - 2\varepsilon\|\bar{q}_{t_1}\|_1$$
$$\le 2nd(X)K + 2\varepsilon B_2 \le \frac{(B_2)^2}{2}, \quad (79)$$

which is a contradiction. Thus, $\|q_{T_1+1}\| \le B_2$.

We now show (39) holds. Applying (25) to the update (18) and noting $\|\tilde{q}_{i,t+1}\| \le \|q_t\| \le B_2$ gives

$$\|\tilde{x}_{i,t+1} - x_{i,t}\| \le \frac{\|\alpha_{t+1}a_{i,t+1}\| + \alpha_{t+1}G}{\underline{\sigma}}$$
$$\le \frac{G\alpha_{t+1}}{\underline{\sigma}}(B_2 + 2). \quad (80)$$

(16) and (20) give

$$q_{i,t+1} \ge (1 - \beta_{t+1}\gamma_{t+1})\sum_{j=1}^{n}[W_t]_{ij}q_{j,t} + \gamma_{t+1}b_{i,t+1}. \quad (81)$$

Summing (81) over $i \in [n]$, dividing by $n\gamma_{t+1}$, and using $\sum_{i=1}^{n}[W_t]_{ij} = 1, \forall t \in \mathbb{N}_+$, (6), (19), and (80) yields

$$\begin{aligned}\frac{\bar{q}_{t+1}}{\gamma_{t+1}} &\geq \left(\frac{1}{\gamma_{t+1}} - \beta_{t+1}\right)\bar{q}_t + \frac{1}{n}\sum_{i=1}^{n} b_{i,t+1} \\ &\geq \left(\frac{1}{\gamma_{t+1}} - \beta_{t+1}\right)\bar{q}_t + \frac{1}{n}g_t(x_t) \\ &\quad - \frac{G^2\alpha_{t+1}}{\underline{\sigma}}(B_2+2)\mathbf{1}_m. \end{aligned} \tag{82}$$

Summing (82) over $t \in [T]$ gives

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{T} g_t(x_t) &\leq \frac{\bar{q}_{T+1}}{\gamma_{T+1}} + \sum_{t=1}^{T}\beta_{t+1}\bar{q}_t \\ &\quad + \sum_{t=1}^{T}\frac{G^2\alpha_{t+1}}{\underline{\sigma}}(B_2+2)\mathbf{1}_m. \end{aligned} \tag{83}$$

Noting that $\|[x]_+\| \leq \|y\|$ for all $x \leq y$ and using $\|\bar{q}_t\| \leq \|q_t\| \leq B_2$ and (73) yields (39).

G. Proof of Theorem 3

i) We first show that $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_4\, T^{\kappa}$ when $\alpha_t = \frac{1}{t^{1-\kappa}}$. Under Assumption 6, (61) can be replaced by

$$\begin{aligned} & l_{i,t}(x_{i,t}) - l_{i,t}(y_{i,t}) \\ &\leq \langle \nabla f_{i,t}(x_{i,t}), x_{i,t} - y_{i,t}\rangle + \langle \nabla r_{i,t}(x_{i,t}), x_{i,t} - \tilde{x}_{i,t+1}\rangle \\ &\quad + \langle \nabla r_{i,t}(\tilde{x}_{i,t+1}), \tilde{x}_{i,t+1} - y_{i,t}\rangle - \underline{\mu}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) \\ &= \langle \nabla f_{i,t}(x_{i,t}) + \nabla r_{i,t}(x_{i,t}), x_{i,t} - \tilde{x}_{i,t+1}\rangle \\ &\quad + \langle \nabla f_{i,t}(x_{i,t}) + \nabla r_{i,t}(\tilde{x}_{i,t+1}), \tilde{x}_{i,t+1} - y_{i,t}\rangle \\ &\quad - \underline{\mu}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}). \end{aligned} \tag{84}$$

Thus, (30)–(32) still hold if replacing $E_3(t)$ by

$$\begin{aligned} E_5(t) = \sum_{i=1}^{n}\Big\{ &\frac{1}{\alpha_{t+1}}\Big[\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) \\ &- \mathcal{D}_{\psi_i}(y_{i,t+1}, x_{i,t+1})\Big] - \underline{\mu}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t})\Big\}. \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{t=1}^{T} E_5(t) \\ &= \sum_{t=1}^{T}\sum_{i=1}^{n}\left[\frac{1}{\alpha_t}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) - \frac{1}{\alpha_{t+1}}\mathcal{D}_{\psi_i}(y_{i,t+1}, x_{i,t+1})\right] \\ &\quad + \sum_{t=1}^{T}\sum_{i=1}^{n}\left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t} - \underline{\mu}\right)\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}). \end{aligned} \tag{85}$$

Noting that $\underline{\mu} > 0$, $\mathcal{D}_{\psi_i}(\cdot,\cdot) \geq 0$, and $\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t} - \underline{\mu} = \frac{t+1}{(t+1)^{\kappa}} - \frac{t}{t^{\kappa}} - \underline{\mu} < \frac{1}{t^{\kappa}} - \underline{\mu} \leq 0, \forall t \geq B_4$ and using (76) and (85) yields

$$\begin{aligned} &\sum_{t=1}^{T} E_5(t) = \sum_{t=1}^{B_4-1} E_3(t) + \sum_{t=B_4}^{T} E_5(t) \\ &\leq \frac{nd(X)K}{\alpha_{B_4}} + \sum_{t=B_4}^{T}\sum_{i=1}^{n}\Big(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t} - \underline{\mu}\Big)\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) \\ &\quad + \sum_{t=B_4}^{T}\sum_{i=1}^{n}\Big[\frac{1}{\alpha_t}\mathcal{D}_{\psi_i}(y_{i,t}, x_{i,t}) - \frac{1}{\alpha_{t+1}}\mathcal{D}_{\psi_i}(y_{i,t+1}, x_{i,t+1})\Big] \\ &\leq \frac{2nd(X)K}{\alpha_{B_4}}. \end{aligned} \tag{86}$$

Replacing (76) with (86) and along the same line as the proof of (34) in Theorem 1 gives that $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_4\, T^{\kappa}$ when $\alpha_t = \frac{1}{t^{1-\kappa}}$.

Next, we show that (45) holds. When $\kappa \in (0, 0.5)$, we have $\alpha_t = 1/t^{(1-\kappa)}$. Thus, from the above result, we have $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_4\, T^{\kappa}$. When $\kappa \in [0.5, 1)$, we have $\alpha_t = 1/t^{\kappa}$. Thus, (40) gives $\mathrm{Reg}(\boldsymbol{x}_T, \check{\boldsymbol{x}}_T^*) \leq C_1\, T^{\kappa}$. In conclusion, (45) holds.

ii) Substituting $c = 1 - \kappa$ when $\kappa \in (0, 0.5)$ and $c = \kappa$ when $\kappa \in [0.5, 1)$ in (35) gives (46).

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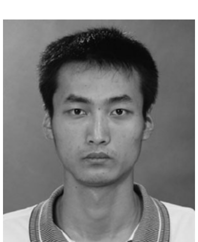

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