Optimization of the $\mathcal{H}_\infty$-norm of Dynamic Flow Networks

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Abstract— In this paper, we study the $\mathcal{H}_\infty$- norm of linear systems over graphs, which is used to model distribution networks. In particular, we aim to minimize the $\mathcal{H}_\infty$- norm subject to allocation of the weights on the edges. The optimization problem is formulated with LMI (Linear-Matrix-Inequality) constraints. For distribution networks with one port, i.e., SISO systems, we show that the $\mathcal{H}_\infty$- norm coincides with the effective resistance between the nodes in the port. Moreover, we derive an upper bound of the $\mathcal{H}_\infty$- norm, which is in terms of the algebraic connectivity of the graph on which the distribution network is defined.

I. INTRODUCTION

In this paper we study robustness of a basic model for the dynamics of a distribution network. Identifying the network with a undirected graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*. Furthermore, some of the vertices serve as terminals where an unknown flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. Many control protocols are designed for a distributed control structure (the control input corresponding to a given edge only depending on the difference of the state variables of the adjacent vertices) which will ensure that the state variables associated to all vertices will converge to the same value, i.e., reach consensus, [4],[13].

In this paper, we consider the distribution network controlled by proportional controllers on the edges and study the robustness property with respect to the controller gain, i.e., the edges weights. In particular, we are interested in minimizing the $\mathcal{H}_\infty$- norm by allocating the edge weights.

The distribution networks can be seen as linear time-invariant port-Hamiltonian systems [1], [23], but also resides in the category of state-space symmetric systems [24], [12], [18], [25], [17]. One important property of the state-space symmetric system is that its $\mathcal{H}_\infty$- norm is attained at the zero frequency [22], which is employed to solve the current problem.

The contributions of this paper are: The problem of minimizing the $\mathcal{H}_\infty$- norm of the distribution networks subject to the allocation of the edge weights is formulated and written

with LMIs as constraints. Moreover, we give an interpretation of the Riccati inequality which regards definitness of a Laplacian to a graph containing both positive and negative weights on the edges. As a consequence of the interpretation, it is shown for distribution networks with one port (to be defined), that the $\mathcal{H}_\infty$- norm (or induced $\mathcal{L}_2$- gain) is equal to the effective resistance between the nodes in the port. Then, an upper bound of the $\mathcal{H}_\infty$- norm is derived, which relates to the algebraic connectivity of the graph on which the distribution network is defined. The results in this paper can be relevant when designing robust multi-agent systems. In particular when considering a malicious attacker, e.g., [19].

The structure of the paper is as follows. Some preliminaries will be given in Section II. The considered class of dynamic flow networks is given and the optimization problem is formulated in Section III. The main results are presented in Section IV. In Section VI there is a numerical example which demonstrates some results from this paper. Conclusions and future work are given in Section VII and VIII, respectively.

Notation. A positive semi-definite (symmetric) matrix M is denoted as $M \succeq 0$. A positive definite (symmetric) matrix M is denoted as $M \succ 0$. The ith row of a matrix M is given by M_i. The element on the ith row and j^{th} column of a matrix M is denoted M_{ij}. The vectors $e_1, e_2, \ldots, e_n$ denotes the canonical basis of $\mathbb{R}^n$, whereas the vectors $\mathbb{1}_n$ and $\mathbf{0}_n$ represent a n-dimensional column vector with each entry being 1 and 0, respectively. We will omit the subscript n when no confusion arises. The euclidean norm is denoted as $|\cdot|_2$, for a vector $x \in \mathbb{R}^n$, $|x|_2 = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

II. PRELIMINARIES

In this section, we briefly review some essentials about graph theory [5] and give some definitions for robust analysis [26].

A. Graph Theory

An undirected graph $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{v_1, ..., v_n\}$, a set of edges $\mathcal{E} = \{\mathcal{E}_1, ..., \mathcal{E}_m\}$ that contains unordered pairs of elements of $\mathcal{V}$, and a set of corresponding edge weights $\mathcal{W} = \{w_1, ..., w_m\}$. Graphs with unit weights, i.e., $w_i = 1$, for $i = 1, ..., m$, are denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The set of neighbours to node i is

$$N_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}.$$

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The graph Laplacian $L \in \mathbb{R}^{n \times n}$ is defined component-wise as

$$L_{ij} = \begin{cases} \sum_{j \in N_i} w_{ij} & \text{if } i = j, \\ -w_{ij} & \text{if } j \in N_i \setminus \{i\}, \, , \\ 0 & \text{if } j \notin N_i. \end{cases}$$

where both positive and negative weights are allowed. Given an orientation for each edge, the incidence matrix $B \in \mathbb{R}^{n \times m}$ is defined as

$$B_{ij} = \begin{cases} 1 & \text{if } \mathcal{E}_j \text{ starts in node } v_i, \\ -1 & \text{if } \mathcal{E}_j \text{ ends in node } v_i, \\ 0 & \text{else.} \end{cases}$$

These two matrices are related by $L = BWB^T$, where $W = \text{diag}(w_1, ..., w_m)$. If $W \geqslant 0$ then the eigenvalues of L_w can be structured as

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_n,$$

where the eigenvector corresponding to $\lambda_1 = 0$ is $\mathbb{1}^\top = [1, ..., 1]^T$. The second smallest eigenvalue, i.e., λ_2, is commonly referred to as the *algebraic connectivity* [10] and is a measure of how connected a graph is. Furthermore, if $\mathcal{G}$ is connected, then $\lambda_2 > 0$.

If $\mathcal{W}$ contains both positive and negative weights, then the Laplacian can be decomposed as

$$L = L_+ + L_- = B_+W_+B_+^T + B_-W_-B_-^T,$$

where B_+ and B_- are incidence matrices corresponding to the positive and negative sub-graphs, respectively. The matrices W_+ and W_- are the weights of the positive and negative sub graphs, respectively. This decomposition is also used in e.g., [7].

For an undirected and connected graph without self-loops and with only positive weights on the edges, a measure of the connectivity between two nodes is the *effective resistance* [9] [14]. The effective resistance between the nodes v_i and v_j is defined as

$$R_{ij} = (e_i - e_j)^T L^\dagger (e_i - e_j),$$

where $L^\dagger$ is the Moore-Penrose pseudo inverse of L.

Lemma 1 ([8],Theorem III.3): Assume $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$ is a graph which has exactly one edge with negative weight and all other edges has positive weights. The negative edge is $\mathcal{E}_- = (u, v)$. Let $\mathcal{G}_+$ be the positive sub-graph of $\mathcal{G}$ and assume it is connected. Then the Laplacian of $\mathcal{G}$ is positive semi-definite if and only if

$$|W(\mathcal{E}_-)| \leqslant R_{uv}^{-1}(\mathcal{G}_+),$$

where $W(\mathcal{E}_-)$ is the negative weight and $R_{uv}(\mathcal{G}_+)$ denotes the effective resistance between the nodes u and v in $\mathcal{G}_+$.

B. $\mathcal{L}_2$-Norm and induced $\mathcal{L}_2$-Gain

In this subsection, we recall some definitions from robust control. The notations used in this paper are fairly standard and are consistent with [26], [20]. The space of square-integrable signals $f : [0, \infty) \to \mathbb{R}^n$ is denoted by $L_2[0, \infty)$. For the linear time-invariant system

$$\dot{x} = Ax + Bu, \tag{1}$$
$$y = Cx + Du,$$

the transfer matrix is $\mathbb{G}(s) = C(sI - A)^{-1}B + D$, which has the impulse response

$$g(t) = \mathcal{L}^{-1}\{\mathbb{G}(s)\} = Ce^{At}B\mathbf{1}_+(t) + D\delta(t),$$

where $\delta(t)$ is the unit impulse and $\mathbf{1}_+(t)$ is the unit step defined as

$$\mathbf{1}_+(t) = \begin{cases} 1, t \geqslant 0, \\ 0, t < 0. \end{cases}$$

If $x(0) = 0$, then we have $y(t) = \int_0^t g(t - \tau)u(\tau)d\tau$. Then the induced $\mathcal{L}_2$- gain is defined as

$$\|g\|_{2-ind} = \sup_{u \in L_2[0,\infty)} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in L_2[0,\infty)} \frac{\|g * u\|_2}{\|u\|_2},$$

where $\|u(t)\|_2 = \left(\int_0^\infty |u(t)|_2^2 dt \right)^{\frac{1}{2}}$.

This induced $\mathcal{L}_2$- gain, i.e., $\|g\|_{2-ind}$ or $\|\mathbb{G}\|_{2-ind}$, is often called the $\mathcal{H}_\infty$- norm, denoted as $\|\mathbb{G}\|_\infty$. It is well-know that $\|\mathbb{G}\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}\{\mathbb{G}(j\omega)\}$, where $\bar{\sigma}(A)$ denotes the largest singular value of the matrix A.

For the system (1) with $D = 0$, the bounded real lemma [26] implies that $\|\mathbb{G}\|_\infty \leqslant \gamma$ if and only if there exists $P = P^\top \succ 0$ such that

$$PA + A^\top P + C^\top C + \frac{1}{\gamma^2}PBB^\top P \preccurlyeq 0. \tag{2}$$

III. Problem Formulation

We consider the dynamical distribution network defined on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$. On the vertices, we consider integrators, given as

$$\dot{x} = u, \qquad x, u \in \mathbb{R}^n, \tag{3}$$
$$z = x, \qquad z \in \mathbb{R}^n.$$

Here the ith element of x and u, i.e. x_i and u_i, are the state and input variables associated with the ith vertex of the graph. System (3) defines a port-Hamiltonian system [23], satisfying the energy-balance

$$\frac{d}{dt} \frac{|x|_2^2}{2} = u^T z.$$

As a next step we will extend the dynamical system (3) with an external input d of *inflows and outflows*

$$\dot{x} = u + Ed, \qquad d \in \mathbb{R}^k,$$
$$z = x,$$

where E is a $n \times k$ matrix whose columns consist of one element which is 1 (inflow) and one element -1 (outflow), while the rest of the elements are zero. A port is a set of nodes(terminals) to where the external flow which enter and leave the network sums to zero. Thus, E specifies k ports.

To achieve a state consensus, many controllers which provide the flows on the edges of $\mathcal{G}$ have been proposed, with the following general form

$$
\begin{aligned}
\dot{\eta}_k &= f_k(\eta_k, \zeta_k), \\
\mu_k &= g_k(\eta_k, \zeta_k), \qquad k = 1, 2, \ldots, m
\end{aligned} \tag{4}
$$

where η_k, ζ_k, μ_k are respectively the states, input and output of the controller on the kth edge of $\mathcal{G}$. Denote the stacked vectors of η_k, ζ_k, μ_k as η, ζ, μ respectively. With the controller (4), the state variables $x_i, i = 1, 2, \ldots, n$, are controlled by the controller output $\mu_k, k = 1, 2, \ldots, m$, in the following manner

$$
u + BW\mu = 0,
$$

where $B \in \mathbb{R}^{n \times m}$ is the incidence matrix of the digraph $\mathcal{G}$, and W is the diagonal matrix corresponding to the gain of the controller to the edges. In addition, the controller is driven by the relative output of the systems (3) on vertices, i.e

$$
\zeta = B^T z.
$$

It is known that, if $d = 0$, the state agreement of the system (3) can be achieved by P-control and PI-control. For the P-control, the closed-loop is,

$$
\begin{aligned}
\dot{x} &= - L_w x + Ed, \\
y &= E^T x,
\end{aligned} \tag{5}
$$

where L_w is the graph Laplacian of $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$ and y is a vector with the components being the state difference at each port. At this point no restriction is made on the sign of the weights of $\mathcal{G}$, only that $L_w \succeq 0$.

Example 1: One physical interpretation of the system (5) is a basic model of a dynamic flow network, where there are water reservoirs on the nodes and pipes on the edges. The reservoirs are identical cylinders and the pipes are horizontal. The state x is constituted by the water levels in the reservoirs and the pressures are proportional to the water levels. The flow in the pipes are passively driven by pressure difference between the reservoirs. The weights $\mathcal{W}$ are representing the capacities of the pipes, in terms of diameter and friction. The passive flow from reservoir i to reservoir j is then $q_{ij} = w_{ij}(x_i - x_j)$. The external input d can e.g. be interpreted as flow in pumps which are distributing water inside the network. The output y is then the difference between water levels of the reservoirs which the pumps are pumping to and the reservoirs which the pumps are pumping from.

There are many other interpretations and applications of the system (5). Others are e.g., mass-damper systems, chemical reaction networks [2] and consensus protocols [21].

In this paper we consider a network design problem. More precisely, we are interested in the following problem: For a given topology, how to achieve the best robust performance of the system (5) by arranging the weights on the edges, i.e.,

$$
\min_W \|\mathbb{G}\|_\infty \tag{6}
$$
$$
s.t., \sum w_i = c, \ w_i \geqslant 0,
$$

where $\mathbb{G}$ is the transfer function of the system (5), $W = \operatorname{diag}(w_1, \ldots, w_m)$ and w_i, for $i = 1, \ldots, m$, are the weights on the edges. The constant c is the constraint on the sum of all edge weights.

Example 2 (flow network continued): For the flow network interpretation of the system (5), the optimization problem above is to allocate capacities of the water pipes such the $\mathcal{H}_\infty$- norm of the flow network is minimized. The constant c represents the total capacity of the pipes.

IV. $\mathcal{H}_\infty$-NORM OF THE DISTRIBUTION NETWORK

A. Optimization problem reformulated with LMI constraints

We start this subsection by reformulating problem (6) as an equivalent optimization problem with LMIs as constraints, which can then be efficiently solved numerically using, e.g., Yalmip [15].

Theorem 2: Consider the system (5), where $\mathcal{G}$ is an undirected graph and each port belongs to exactly one connected component of $\mathcal{G}$. First, the $\mathcal{H}_\infty$- norm is finite. Second, the following statements are eqvivalent:

- The $\mathcal{H}_\infty$- norm is less than or equal to γ.
- The following LMI is satisfied,

$$
\begin{bmatrix} L_w & E \\ E^\top & \gamma I_k \end{bmatrix} \succeq 0. \tag{7}
$$

Proof: We show that the theorem is true for the case where $\mathcal{G}$ has exactly one connected component. This is without loss of generality since if $\mathcal{G}$ has more than one connected component, the same procedure can be done for each and the LMIs can be merged with a common γ. Denote

$$
U^\top = [\mathbb{1}_n, u_2^\top, \ldots, u_n^\top] \text{ and } U_2^\top = [u_2^\top, \ldots, u_n^\top],
$$

for which $U L_w U^\top = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_n) =: \Lambda$. Denote $\hat{\Lambda} = \operatorname{diag}(\lambda_2, \ldots, \lambda_n)$. Then the system (5) has equal $\mathcal{H}_\infty$- norm as the system

$$
\begin{aligned}
\dot{\tilde{x}} &= -\Lambda \tilde{x} + UEd, \\
z &= E^\top U^\top \tilde{x}.
\end{aligned}
$$

Notice that the first row of UE is zero, thus the $\mathcal{H}_\infty$- norm of the system (5) equals the $\mathcal{H}_\infty$- norm of the system

$$
\begin{aligned}
\dot{\hat{x}} &= -\hat{\Lambda} \hat{x} + U_2 Ed, \\
z &= E^\top U_2^\top \hat{x}.
\end{aligned} \tag{8}
$$

Due to symmetry of the system and by Theorem 6 in [22], the $\mathcal{H}_\infty$- norm of the system (8) is $\|E^\top U_2^\top \hat{\Lambda}^{-1} U_2 E\|_2$, which

is finite. The $\mathcal{H}_\infty$- norm of the system (5) is then less or equal to γ if and only if

$$\|E^\top U_2^\top \hat{\Lambda}^{-1} U_2 E\|_2 \preccurlyeq \gamma.$$

By the property of real symmetric matrix, we can further rewrite the previous constrain as $E^\top U_2^\top \hat{\Lambda}^{-1} U_2 E \preccurlyeq \gamma I_k$. By Schur complement, we have

$$\begin{bmatrix} \hat{\Lambda} & U_2 E \\ E^\top U_2^\top & \gamma I_k \end{bmatrix} \succcurlyeq 0,$$

which is equivalent to

$$\begin{bmatrix} \Lambda & UE \\ E^\top U^\top & \gamma I_k \end{bmatrix} \succcurlyeq 0.$$

By pre and post multiplication of matrix $\mathrm{diag}(U^\top, I_k)$ and $\mathrm{diag}(U, I_k)$, respectively, the previous inequality is transformed to

$$\begin{bmatrix} L_w & E \\ E^\top & \gamma I_k \end{bmatrix} \succcurlyeq 0.$$

Then the conclusion follows. $\blacksquare$

Remark 1: By Theorem 2, the optimization problem (6) is equivalent to

$$\begin{aligned} \min_W \quad & \gamma \\ s.t., \quad & \begin{bmatrix} L_w & E \\ E^\top & \gamma I_k \end{bmatrix} \succcurlyeq 0, \\ & \sum w_i = c, \ w_i \geqslant 0. \end{aligned} \tag{9}$$

Since the constraints are LMIs, this optimization problem can efficiently be solved with e.g., Yalmip. The set up above is used later in Section VI, there the optimal edge weight allocation is determined for the system which is illustrated in Figure 1a and the optimal $\mathcal{H}_\infty$- norm is verified in a simulation.

In Theorem 2, we proved that the inequality (7) is satisfied if and only if the $\mathcal{H}_\infty$- norm is less than or equal to γ. Moreover, by the bounded real lemma we have that $\|\mathbb{G}\|_\infty \leqslant \gamma$ if and only if there exists $P = P^\top \succ 0$ such that

$$-PL_w - L_w^T P + EE^T + \frac{1}{\gamma^2} PEE^T P \preccurlyeq 0. \tag{10}$$

In the next result, we provide one explicit solution to (10).

Theorem 3: Consider the system (5), where $\mathcal{G}$ is a connected and undirected graph. The two following statements are equivalent:

- The $\mathcal{H}_\infty$- norm is less then or equal to γ.
- The Riccati inequality (10) is satisfied with the solution $P = \gamma I$.

Proof: By Theorem 2, the $\mathcal{H}_\infty$- norm is less then or equal to γ if and only the LMI (7) is satisfied. The LMI (7) is by the Schur complement equivalent to

$$-L_w + \frac{EE^T}{\gamma} \preccurlyeq 0. \tag{11}$$

Furthermore, by choosing $P = \gamma I$, the Riccati inequality (10) is equivalent to (11). Hence the conclusion follows. $\blacksquare$

B. Graphical interpretation of the Riccati inequality for dynamic flow networks

In this subsection, we give a graphical interpretation of the Riccati inequality (7) which is equivalent to (11) (by Schur complement), for a special type of dynamic flow networks. More precisely, we assume that each column of E has exactly two non-zero elements, one is 1 and the other is -1. By this restriction of E, it has the structure of an incidence matrix and EE^T is therefore a Laplacian. For $\gamma > 0$, let us define

$$L_\gamma = -\frac{1}{\gamma} EE^T,$$

and denote the corresponding graph as $\mathcal{G}_\gamma = (\mathcal{W}_\gamma, \mathcal{V}_\gamma, \mathcal{E}_\gamma)$, where $\mathcal{W}_\gamma = \{-\frac{1}{\gamma}, ..., -\frac{1}{\gamma}\}$ and $\mathcal{V}_\gamma = \mathcal{V}$. The set of edges $\mathcal{E}_\gamma$ is determined by E. Recall that $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$ is the graph on which the system (5) is defined. Moreover, we define $\tilde{L} = L_w + L_\gamma$, which is a Laplacian that possibly contains both positive and negative weights on the edges. The inequality (11) then equals to $\tilde{L} \succcurlyeq 0$. Hence the $\mathcal{H}_\infty$- norm of the system (5) is determined by the inverse of the largest magnitude of the negative weights $-\frac{1}{\gamma}$, that yields a positive definite Laplacian $\tilde{L}$.

Example 3: The connection between $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$ and $\mathcal{G}_\gamma = (\mathcal{W}_\gamma, \mathcal{V}_\gamma, \mathcal{E}_\gamma)$ is illustrated in this example. Consider a system as in (5), where

$$L_w = \begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} & 0 \\ -w_{12} & w_{12}+w_{24} & 0 & -w_{24} \\ -w_{13} & 0 & w_{13}+w_{34} & -w_{34} \\ 0 & -w_{24} & -w_{34} & w_{24}+w_{34} \end{bmatrix},$$

$$E^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}. \tag{12}$$

This dynamic flow system is defined on the graph $\mathcal{G}$, which is illustrated in Figure 1a. For this system, the graph $\mathcal{G}_\gamma$ which corresponds to L_γ is illustrated in Figure 1b. The $\mathcal{H}_\infty$- norm is determined by the inverse of the largest magnitude of the weights $-\frac{1}{\gamma}$ that yields a positive definite $\tilde{L}$.

C. Connection between $\mathcal{H}_\infty$- norm and effective resistance

In the previous subsection, we reinterpret the $\mathcal{H}_\infty$- norm of system (5) in the scenario of the Laplacian $\tilde{L}$ with negative weights. Further conclusions can be drawn from the reasoning above about the definiteness of $\tilde{L}$ if we set the restriction to SISO case, i.e., $E = e_i - e_j$. The $\mathcal{H}_\infty$- norm is then shown to coincide with the effective resistance between node i and j.

Theorem 4: Consider the system (5) defined on $\mathcal{G} = (\mathcal{W}, \mathcal{V}, \mathcal{E})$, which is undirected, connected and only contains positive edge weights. Moreover, assume that there is exactly one port, i.e., $d \in \mathbb{R}$ and $E = e_i - e_j$. Then the induced $\mathcal{L}_2$-gain from d to y is

$$\gamma = R_{ij}(\mathcal{G}),$$
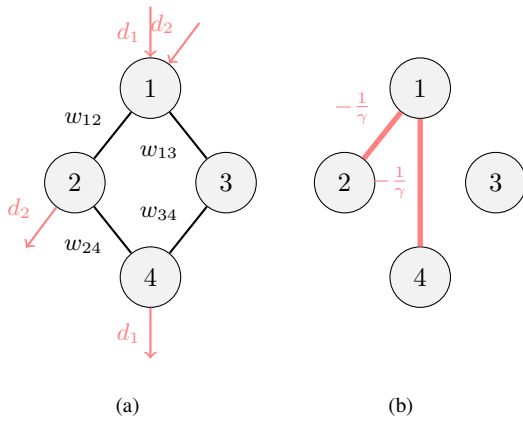
(a)   (b)

Fig. 1: In Subfigure (a), the graph which corresponds to L_w of the system (12) is illustrated. The external inputs to the system are d_1 and d_2. In Subfigure (b), the graph which corresponds to $L_\gamma = -\frac{1}{\gamma} E E^T$ is illustrated. The $\mathcal{H}_\infty$- norm is determined by the inverse of largest magnitude of $-\frac{1}{\gamma}$, which yields a positive definite $\tilde{L} = L_w + L_\gamma$.

where $R_{ij}(\mathcal{G})$ denotes the effective resistance between the nodes i and j in $\mathcal{G}$.

Proof: First note that the left hand side of (11) is composed by a positive and a negative Laplacian. The negative Laplacian has weights $-\frac{1}{\gamma}$. By Lemma 1, the inequality (11) is satisfied if and only if

$$\frac{1}{\gamma} \leqslant R_{ij}^{-1}(\mathcal{G}) \iff \gamma \geqslant R_{ij}(\mathcal{G}).$$

The conclusion follows.

∎

V. $\mathcal{H}_\infty$-NORM BOUNDED BY ALGEBRAIC CONNECTIVITY

In Sections IV-B and IV-C, we showed that the $\mathcal{H}_\infty$- norm has explicit graphical interpretation for a special matrix E. In this section, we focus on general matrices E. Here we provide one preliminary result which relates the $\mathcal{H}_\infty$- norm of system (5) to the algebraic connectivity of the underlying graph. This result can be used if the location of the ports is unknown.

Lemma 5: Consider system (5), where $\mathcal{G}$ is a connected and undirected graph. Then, the $\mathcal{H}_\infty$- norm is bounded by

$$\gamma = \frac{\bar{\lambda}_{EE^T}}{\lambda_2},$$

where λ_2 is the second smallest eigenvalue of the weighted Laplacian L_w and $\bar{\lambda}_{EE^T}$ is the largest eigenvalue of EE^T.

Proof: The result is shown by using that $L\mathbb{1} = 0$ and $E^T\mathbb{1} = 0$ and by applying the Courant-Fischer principle (e.g., [16] and [3]) on inequality (11). ∎

Remark 2: By the previous lemma, maximizing the algebraic connectivity of the graph $\mathcal{G}$ (with respect to the edge weights) is suboptimal to minimizing the $\mathcal{H}_\infty$- norm, for a
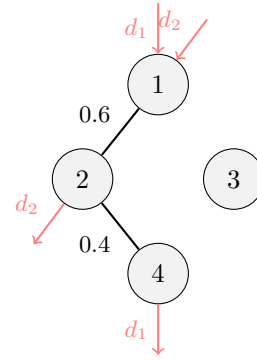

Fig. 2: The flow network (12) with the optimal allocation of pipe capacities. I.e., $w_{12} = 0.6$, $w_{24} = 0.4$, $w_{13} = w_{34} = 0$. The $\mathcal{H}_\infty$- norm corresponding to this allocation is $\gamma^* = 5$.

given E. This result can be relevant for design of robust systems when E is unknown. This is e.g. the scenario if a malicious attacker is considered and the attacked nodes are unknown.

Example 4: Consider the dynamic flow network (5) and the total capacity of the pipes is to be allocated in order to minimize the $\mathcal{H}_\infty$- norm of the system, i.e., the optimization problem (6). However, the only information about E which is available is the largest eigenvalue of EE^T, i.e., $\bar{\lambda}_{EE^T}$. Since full information about E is not available, it is not possible to minimize the $\mathcal{H}_\infty$- norm. Instead, by Lemma 5, we can minimize an upper bound by

$$\max_W \lambda_2$$
$$s.t., \sum \omega_i = c, \ w_i \geqslant 0.$$

This problem of maximizing algebraic connectivity with respect to the edge weights is well-studied, e.g., [11] and [6].

VI. NUMERICAL EXAMPLE

In this section we will demonstrate the results from Section IV-A. For this purpose the dynamic flow network in Example 3 (Figure 1a) is used. We aim to allocate capacities of the pipes in order to minimize the $\mathcal{H}_\infty$- norm. The optimal allocation of the pipe capacity and the optimal $\mathcal{H}_\infty$- norm is determined numerically by Yalmip and the optimization set up (9).

The total pipe capacity is set to $c = 1$. The optimal allocation of the pipe capacity is $w_{12}^* = 0.6$, $w_{24}^* = 0.4$, $w_{13}^* = w_{34}^* = 0$ and the optimal $\mathcal{H}_\infty$- norm is $\gamma^* = 5$. The flow network with optimally allocated pipe capacities is seen in Figure 2.

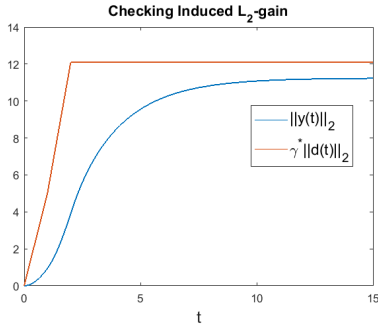Next, we induce an input to the system in order to verify the $\mathcal{H}_\infty$- norm. The input is

Fig. 3: The $\mathcal{L}_2$-norm of the output, i.e., $||y(t)||_2$, is seen together with the $\mathcal{L}_2$-norm of the input scaled with the induced $\mathcal{L}_2$- gain, i.e., $\gamma^*||d(t)||_2$.

$$d(t) = \begin{cases} [1,0]^T & \text{if } 0 \leqslant t < 1, \\ [1,1]^T & \text{if } 1 \leqslant t < 2, \\ [0,0]^T & \text{if } 2 \leqslant t. \end{cases}$$

In Figure 3, the $\mathcal{L}_2$-norm of the output, i.e. $||y(t)||_2$, is seen together with the $\mathcal{L}_2$-norm of the input, scaled with the optimal $\mathcal{H}_\infty$- norm, i.e. $\gamma^*||d(t)||_2$. In the figure it is seen that $||y(t)||_2 \leqslant \gamma^*||d(t)||_2$, hence the $\mathcal{H}_\infty$- norm is verified.

VII. Conclusions

For the dynamic flow networks which we have considered, we have derived an optimization set up with LMIs as constraints, which minimizes the $\mathcal{H}_\infty$- norm with respect to the allocation of the capacity of the pipes. Moreover, for the flow networks, we have interpreted the Riccati inequality as a definiteness criterion of a Laplacian to a graph containing both positive and negative weights on the edges. For flow networks which are SISO, more precisely, $E = e_i - e_j$, we have shown that the $\mathcal{H}_\infty$-norm coincides with the effective resistance between node i and node j. Moreover, we have derived an upper bound of the induced $\mathcal{H}_\infty$-norm of the flow networks. This upper bound relates to the algebraic connectivity on which the flow network is defined. This upper bound can be relevant when full information about the input matrix, i.e E, is not available. Then, the capacities of the pipes can be allocated to get a suboptimal solution which bounds the $\mathcal{H}_\infty$- norm.

VIII. Future work

A related future topic is the problem of minimizing the $\mathcal{H}_\infty$- norm of dynamic flow networks with respect to topology, more precisely, a limited amount of edges is to be allocated in a graph with fixed vertices. Another future topic is to consider a fixed graph (both topology and weights), but consider saturation of the flow on the edges. The problem is then to minimize the induced $\mathcal{L}_2$-gain with respect to allocation of the saturation limits.

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