Community Structure Recovery and Interaction Probability Estimation for Gossip Opinion Dynamics ⋆

Yu Xing [a], Xingkang He [a,*], Haitao Fang [b], Karl H. Johansson [a]

[a] *Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, and Digital Futures, Stockholm, Sweden.*

[b] *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences; School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P. R. China.*

Abstract

We study how to jointly recover the community structure and estimate the interaction probabilities of gossip opinion dynamics. In this process, agents randomly interact pairwise, and there are stubborn agents never changing their states. Such a model illustrates how disagreement and opinion fluctuation arise in a social network. It is assumed that each agent is assigned with one of two community labels, and the agents interact with probabilities depending on their labels. The considered problem is to jointly recover the community labels of the agents and estimate interaction probabilities between the agents, based on a single trajectory of the model. We first study stability and limit theorems of the model, and then propose a joint recovery and estimation algorithm based on trajectories. It is verified that the community recovery can be achieved in finite time, and the interaction estimator converges almost surely. We derive a sample-complexity result for the recovery, and analyze the estimator's convergence rate. Simulations are presented for illustration of the performance of the proposed algorithm.

Key words: Community structure recovery; opinion dynamics; gossip models; stubborn agents; social networks; Markov chains

1 Introduction

Networks appear across domains from biology to sociology. Real networks often exhibit community structures, where subsets of nodes have dense connections locally but sparse connections globally (Fortunato and Hric, 2016). Community detection is to partition nodes according to the network topology. There is a growing interest in studying community detection based on state observations of dynamics (Prokhorenkova et al., 2019; Schaub et al., 2020). Lacking topology data makes the problem harder than classic ones. Particularly, it is unclear how to recover communities out of a single trajectory of opinion dynamics (Ravazzi et al., 2021).

1.1 Related Work

In this subsection, we first review key community definitions and detection approaches (Fortunato and Hric, 2016), then discuss recovering communities based on state observations, and finally clarify our motivation.

Traditional community detection methods apply classic clustering techniques to node pairs assigned with certain weights (Girvan and Newman, 2002). Newman and Girvan (2004) introduce the concept of modularity to measure the quality of a graph partition. A famous algorithm based on optimizing modularity is the Louvain method (Blondel et al., 2008), which assigns nodes to one of the communities iteratively to achieve the largest modularity gain. Another approach to community detection is based on statistical inference, which introduces generative network models and considers an observed network as a sample. A canonical model is the

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* Corresponding author.

Email addresses: yuxing2@kth.se (Yu Xing), xingkang@kth.se (Xingkang He), htfang@iss.ac.cn (Haitao Fang), kallej@kth.se (Karl H. Johansson).

stochastic block model (SBM). (Abbe, 2017) reviews detectability of the SBM and performance of algorithms. Besides optimization and statistical approaches, another method is based on dynamical processes (e.g. Rosvall and Bergstrom (2008)). Morarescu and Girard (2010) propose a bounded-confidence model, where agents converge to several clusters corresponding to communities.

Recently, the study of community detection for networked dynamics has emerged. The problem asks whether we can recover communities only based on state observations of a dynamical process. The main difference between this problem and the classic ones, especially the dynamic-based methods, is that the network itself is not available. Prokhorenkova et al. (2019) and Ramezani et al. (2018) follow a natural two-step procedure, which first recovers the underlying network and then clusters agents based on network estimates. Wai et al. (2019); Schaub et al. (2020); Roddenberry et al. (2020) introduce the framework of blind community detection, which uses sample covariance matrices of agent states for recovery. Peixoto (2019) investigates simultaneously reconstructing the topology and the community structure for an epidemic model and an Ising model. Recovering communities based on observations from an Ising blockmodel is studied in Berthet et al. (2019).

We study how to jointly recover the community structure and estimate the interaction probabilities of gossip opinion dynamics. The problem arises from recent investigation of learning interpersonal influence from dynamics (Ravazzi et al., 2021). Learning influential networks from dynamics is useful for decision making and intervention, but directly collecting such data may be hard, due to topic specificity (Cowan and Baldassarri, 2018), consistency issues (Netrapalli and Sanghavi, 2012), and privacy concern (De Montjoye et al., 2018). Learning large-scale networks may be computationally expensive, so recovering communities as a coarse description is a good option. The gossip update rule captures the random nature of real individual interactions. It is a fundamental element of other opinion models (Proskurnikov and Tempo, 2017), and has been extensively studied (Boyd et al., 2006). Stubborn agents, such as media and opinion leaders, play a crucial role in opinion formation (Ramos et al., 2015). Acemoğlu et al. (2013) show that the existence of stubborn agents can explain opinion oscillation. An generalization of stubborn agents is to assume that each agent has some level of stubbornness with respect to its initial belief. This generalization is considered by the Friedkin-Johnsen model and its extensions (Proskurnikov et al., 2017; Tian and Wang, 2018).

1.2 Contributions

We consider jointly recovering communities and estimating interaction probabilities for gossip opinion dynamics. Each agent is assigned with one of two community labels, and the agents interact with probabilities depending on their labels. Our contributions are as follows:

1. We study properties of the model by leveraging results on Markov chains and stochastic approximation (SA) (Theorem 1). It is shown that regular-agent states converge in distribution to a unique stationary distribution, and the time average of the agent states converge almost surely. An explicit expression for the mean of the stationary distribution is given (Proposition 2).
2. We develop a joint algorithm (Algorithm 1) to recover the community structure and to estimate the interaction probabilities, based on Polyak averaging and SA techniques. The algorithm is able to recover the communities in finite time, and then able to estimate the interaction probabilities consistently (Theorem 2).
3. We show how to theoretically analyze the developed joint algorithm. A concentration inequality for Markov chains (Lemma 1) is obtained, and it is used in the sample-complexity analysis of the recovery step (Theorem 3). The obtained result shows that the probability of unsuccessful recovery decays exponentially over time. Additionally, we analyze convergence rate of the interaction estimator from an SA argument (Theorem 4).

The obtained results indicate that a Polyak averaging technique can be useful for recovering communities based on a single trajectory. In addition, we establish a sample-complexity result for successful recovery (recovering all community labels correctly), providing a quantitative dependence of the successful recovery probability on model parameters. These two points make our paper different from Wai et al. (2019); Schaub et al. (2020); Roddenberry et al. (2020), which use properties of sample covariance matrices of agent states collected from several trajectories, and different from Wai et al. (2016), which considers learning a sparse characterization of the network from the gossip model. The considered problem is different from classic system identification (e.g., Sarkar et al. (2021)), because stubborn agents normally have fixed states, which does not satisfy input conditions required for system identification, and also because community recovery cannot be obtained directly from parameter estimates. The major differences between this paper and its conference version (Xing et al., 2020) are that we clarify our assumptions in more detail, characterize the sample complexity and the convergence rate of the algorithm, and add more simulations to illustrate its performance.

1.3 Outline

The rest of the paper is organized as follows. Section 2 formulates the problem. Analysis of the model is given in Section 3, and a joint recovery and estimation algorithm is proposed in Section 4. Section 5 presents convergence results of the algorithm, and Section 6 provides several numerical experiments. Finally, Section 7 concludes the paper. Proofs are given in Xing et al. (2021).

Notation. Denote the n-dimensional Euclidean space, the set of $n \times m$ real matrices, and the set of nonnegative integers by $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{N}$. Denote $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$. Let $\mathbf{1}_n$, $\mathbf{e}_i$, I_n, and $\mathbf{0}_{n,m}$ be n-dimensional all-one vector, the unit vector with i-th entry being one, the $n \times n$ identity matrix, and the $n \times m$ all-zero matrix. Let $\mathbf{1}_{n_1,n_2} := \mathbf{1}_{n_1}\mathbf{1}_{n_2}^T$. Denote both the Euclidean norm of a vector and the spectral norm of a square matrix by $\|\cdot\|$. Denote the diagonal matrix with the elements of a vector x on the main diagonal by $\operatorname{diag}\{x\}$. For a vector x, denote its i-th component by x_i, and, for a matrix A, denote its (i,j)-th entry by a_{ij} or $[A]_{ij}$. Denote the spectral radius of A by $\rho(A)$, and the cardinality of a set Ω by $|\Omega|$. $\mathbb{I}_{[\text{property}]}$ is the indicator function. For two sequences $\{a_k\}$ and $\{b_k\}$ with $a_k \in \mathbb{R}^n$ and $0 \neq b_k \in \mathbb{R}$, $k \geq 1$, $a_k = O(b_k)$ means that $\|a_k/b_k\| \leq C$ for all k and some $C > 0$, and $a_k = o(b_k)$ means that $\lim_{k\to\infty} \|a_k/b_k\| = 0$. An event happens almost surely (a.s.) if it happens with probability one.

2 Problem Formulation

This section introduces the considered model and the definition of communities, and formulates the problem.

2.1 Gossip Model with Stubborn Agents

The gossip model is a random process over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the agent set $\mathcal{V}$, the edge set $\mathcal{E}$, and no self-loops. The agents have two types, regular and stubborn, denoted by $\mathcal{V}_r$ and $\mathcal{V}_s$, respectively ($\mathcal{V} = \mathcal{V}_r \cup \mathcal{V}_s$, $\mathcal{V}_r \cap \mathcal{V}_s = \emptyset$). Each agent i has a state $X_i(t) \in \mathbb{R}$, and the state vector at time $t \in \mathbb{N}$ is $X(t) \in \mathbb{R}^n$. Stubborn agents do not change their states during the process.

An interaction probability matrix $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ captures agent interactions, where $w_{ij} = w_{ji} \geq 0$, $w_{ij} > 0 \Leftrightarrow \{i,j\} \in \mathcal{E}$, $i,j \in \mathcal{V}$, and $\mathbf{1}^T W \mathbf{1}/2 = 1$. At time t, edge $\{i,j\}$ is selected with probability w_{ij} independently of previous updates, and agents update as follows, with the averaging weight $q \in [0,1)$,

$$X_k(t+1) = \begin{cases} qX_i(t) + (1-q)X_j(t), & \text{if } k \in \mathcal{V}_r \cap \{i,j\}, \\ X_k(t), & \text{otherwise.} \end{cases} \tag{1}$$

For $1 \leq i < j \leq n$, define a sequence of independent and identically distributed (i.i.d.) n-dimensional random matrices $\{R(t), t \in \mathbb{N}\}$ such that $\mathbb{P}\{R(t) = R^{ij}\} = w_{ij}$, $1 \leq i < j \leq n$, where

$$R^{ij} = \begin{cases} I - (1-q)(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T, & \text{if } i,j \in \mathcal{V}_r, \\ I - (1-q)\mathbf{e}_i(\mathbf{e}_i - \mathbf{e}_j)^T, & \text{if } i \in \mathcal{V}_r, j \in \mathcal{V}_s, \\ I - (1-q)\mathbf{e}_j(\mathbf{e}_j - \mathbf{e}_i)^T, & \text{if } i \in \mathcal{V}_s, j \in \mathcal{V}_r, \\ I, & \text{if } i,j \in \mathcal{V}_s. \end{cases}$$

The update rule (1) can then be written as

$$X(t+1) = R(t)X(t). \tag{2}$$

Since stubborn agents never change their states, we rewrite (2) to end up with the following compact form of *the gossip model with stubborn agents*:

$$X^r(t+1) = A(t)X^r(t) + B(t)X^s(t), \tag{3}$$

where $X^r(t)$ and $X^s(t)$ are the state vectors obtained from stacking the states of regular and stubborn agents, respectively, $X^s(t) \equiv X^s(0)$, and $[A(t)\ B(t)]$ is the matrix obtained from stacking rows of $R(t)$ corresponding to regular agents. So $\{[A(t)\ B(t)], t \in \mathbb{N}\}$ is a sequence of i.i.d. random matrices. Assume that the initial vector $X(0)$ is fixed, for simplicity. If $X(0)$ is random, we can study the model by conditioning on realizations of $X(0)$.

2.2 Communities

We follow the framework of SBMs (Abbe, 2017) and Ising blockmodels (Berthet et al., 2019), and assume that agents have pre-assigned community labels. We define a community as the set of agents that have the same label.

In particular, we consider the scenario where the network has two disjoint communities, $\mathcal{V}_1$ and $\mathcal{V}_2$. Denote the community label of i by $\mathcal{C}(i)$, so $\mathcal{C}(i) = k$ for $i \in \mathcal{V}_k$, $k = 1,2$. We call $\mathcal{C}$ the community structure of the network. We further assume that the interaction probability of the agents i and j with $i \neq j$ is

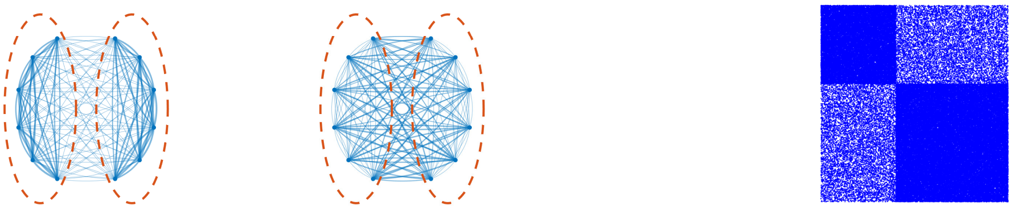
$$w_{ij} = \begin{cases} w_s, & \text{if } \mathcal{C}(i) = \mathcal{C}(j), \\ w_d, & \text{if } \mathcal{C}(i) \neq \mathcal{C}(j), \end{cases} \tag{4}$$

where $w_s, w_d \in (0,1)$ and $w_s \neq w_d$. Thus agents in the same community (different communities) interact with probability w_s (w_d). Fig. 1(a) illustrates two different interaction models, via a simulation where a gossip model defined by (4) is run for 2000 iterations and the number of interactions between agents is counted. To ease notation, we assume that $\mathcal{V}_1 = \{1,\dots,n_1\}$ and $\mathcal{V}_2 = \{n_1+1,\dots,n_1+n_2\}$ with $n_k := |\mathcal{V}_k|$, $k = 1,2$, and $n_1 + n_2 = n$. Thus the interaction probability matrix is

$$W = \begin{bmatrix} w_s\mathbf{1}_{n_1,n_1} - \operatorname{diag}\{w_s\mathbf{1}_{n_1}\} & w_d\mathbf{1}_{n_1,n_2} \\ w_d\mathbf{1}_{n_2,n_1} & w_s\mathbf{1}_{n_2,n_2} - \operatorname{diag}\{w_s\mathbf{1}_{n_2}\} \end{bmatrix} \tag{5}$$

It has a block structure corresponding to the community structure of the network. The following example illustrates how the preceding assumption arises naturally from an SBM. It shows that a graph generated from an SBM defines an interaction probability matrix close to an averaged version with the same structure as (5).

Example 1 *Consider an SBM with two communities, commonly studied in community detection (Abbe, 2017). Such an SBM is a random graph, denoted by* $\text{SBM}(n, \nu_1, \nu_2, p_s, p_d)$*. Here n is the number of agents,* $\nu_1 \in (0,1)$ *(resp.* $\nu_2 \in (0,1)$*) is the portion of agents*



(a) The left (right) graph demonstrates the case where $w_s > w_d$ ($w_s < w_d$), in which agents within the same community interacting more (less) often than agents between communities. The width of edges is proportional to the number of interactions.

(b) The adjacency matrix of a graph generated from $\mathrm{SBM}(n, \nu_1, \nu_2, p_s, p_d)$ with $n = 5000$, $\nu_1 = 0.4$, $\nu_2 = 0.6$, $p_s = 5 \log n/n$, $p_d = \log n/n$. Dots represent nonzero entries, so the block structure of the matrix is clearly visible.

Fig. 1. Illustration of the interaction model (4) and an adjacency matrix generated from an SBM.

with community label 1 *(resp. label* 2*), where* $\nu_1 + \nu_2 = 1$ *and* $\nu_1 n$ *and* $\nu_2 n$ *are integers, and* $p_s, p_d \in (0, 1)$ *are the link probabilities between agents in the same and in different communities. We assume* $\mathcal{C}(i) = 1$, $1 \le i \le \nu_1 n$, *and* $\mathcal{C}(i) = 2$, $\nu_1 n + 1 \le i \le n$.
The $\mathrm{SBM}(n, \nu_1, \nu_2, p_s, p_d)$ *randomly generates an undirected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$*: for* $i \ne j$, $\{i, j\} \in \mathcal{E}$ *with probability* p_s *if* $\mathcal{C}(i) = \mathcal{C}(j)$ *and with probability* p_d *if* $\mathcal{C}(i) \ne \mathcal{C}(j)$*, independently of other edges. The graph* $\mathcal{G}$ *defines a gossip model with the interaction matrix* $\tilde{W} = \mathcal{A}/\alpha$ *and* $\alpha = \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij} = |\mathcal{E}|$*. The inequality* $\|\tilde{W} - \mathbb{E}\{\mathcal{A}\}/\mathbb{E}\{\alpha\}\| \le C/n$ *holds with a constant* C*, except for a probability vanishing as* $n \to \infty$*, if* $\log n/n = O(\min\{p_s, p_d\})$ *(see Xing et al. (2021)). This result implies that, if the network of the gossip model is generated from the SBM, then the interaction probability matrix of the gossip model is close to* $\mathbb{E}\{\mathcal{A}\}/\mathbb{E}\{\alpha\}$ *when* n *is large. Note that* $\mathbb{E}\{\mathcal{A}\}/\mathbb{E}\{\alpha\}$ *has exactly the same structure as* W *in* (5) *with* $n_k = \nu_k n$, $k = 1, 2$, $w_s = p_s/\mathbb{E}\{\alpha\}$*, and* $w_d = p_d/\mathbb{E}\{\alpha\}$*. Fig. 1(b) demonstrates this concentration phenomenon with an obvious two-block structure. The concentration indicates that behavior of the gossip model over a graph generated from the SBM may not deviate too far from the gossip model over the averaged graph, when* n *is large. (Xing and Johansson, 2022) show that the expected stationary states of the two models are close, if* $\log n/n = o(\min\{p_s, p_d\})$*. This result indicates that the gossip model over the averaged graph can be considered as an approximation of the model over the SBM, and results for the former model can be extended to the latter model.*

Remark 1 *A general assumption for community labels in the SBM is that each agent gets a label* k *with probability* ν_k *independently of each other,* $k = 1, 2$*. This is essentially equivalent to the label assignment with deterministic portions of nodes when* $n \to \infty$ *(Remark 3 of Abbe (2017)). Note that it is possible to extend the fixed-label assumption considered in Example 1 to the deterministic-portion assumption, by conditioning on each assignment and using the law of total probability. The condition* $\log n/n = O(\min\{p_s, p_d\})$ *implies that the expected agent degree is at least* $O(\log n)$*. In this case, the SBM generates connected graphs with high probability. The difference between* p_s *and* p_d *has to be large enough to make exact recovery possible (Abbe, 2017). Here we consider the dynamics over the averaged graph, so the detectability only requires* $w_s \ne w_d$ *(Assumption 1 (ii)). Future work will study detectability in the SBM case.*

2.3 Community Recovery and Interaction Estimation

The considered problem is to recover the community structure and to estimate the interaction probabilities based on state observations, as follows.

Problem. Given a trajectory of the gossip model with the interaction matrix (5), develop an algorithm to jointly recover the community structure $\mathcal{C}$ and estimate the interaction probabilities w_s and w_d.

Remark 2 *In the problem, we assume that the developed algorithm uses data coming from the gossip model over the averaged graph. A natural question is how this algorithm performs if it uses a trajectory of the gossip model over a graph sampled from an SBM. In Section 6, we illustrate through simulation that the algorithm performs well also in the SBM case. Such performance is guaranteed by that these two processes behave similarly in terms of their stationary states, as explained in Example 1. We use "community recovery" instead of "community detection" to avoid ambiguity, following the terminology of Berthet et al. (2019), because here agent behavior depends directly on the community structure.*

Recall $\mathcal{V}_1 = \{1, \dots, n_1\}$ and $\mathcal{V}_2 = \{n_1 + 1, \dots, n_1 + n_2\}$. We further sort the agents as follows: $\mathcal{V}_{r1} = \{1, \dots, n_{r1}\}$, $\mathcal{V}_{s1} = \{n_{r1} + 1, \dots, n_1\}$, $\mathcal{V}_{r2} = \{n_1 + 1, \dots, n_1 + n_{r2}\}$, and $\mathcal{V}_{s2} = \{n_1 + n_{r2} + 1, \dots, n\}$. Here, $\mathcal{V}_{rk}$ (resp. $\mathcal{V}_{sk}$) is the set of regular (resp. stubborn) agents in the community k, $k = 1, 2$. Denote $n_{rk} := |\mathcal{V}_{rk}|$, $n_{sk} := |\mathcal{V}_{sk}|$, $n_r := |\mathcal{V}_r|$, and $n_s := |\mathcal{V}_s|$. In the considered problem, the total number of agents is known in advance, the network has two communities, and the stubborn-agent states are observable. But difficulty still remains

$$\bar{A} = I_{n_r} - (1-q)\begin{bmatrix} a_1 I_{n_{r1}} - w_s \mathbf{1}_{n_{r1},n_{r1}} & -w_d \mathbf{1}_{n_{r1},n_{r2}} \\ -w_d \mathbf{1}_{n_{r2},n_{r1}} & a_2 I_{n_{r2}} - w_s \mathbf{1}_{n_{r2},n_{r2}} \end{bmatrix}, \quad \bar{B} = (1-q)\begin{bmatrix} w_s \mathbf{1}_{n_{r1},n_{s1}} & w_d \mathbf{1}_{n_{r1},n_{s2}} \\ w_d \mathbf{1}_{n_{r2},n_{s1}} & w_s \mathbf{1}_{n_{r2},n_{s2}} \end{bmatrix}, \tag{6}$$
$$a_k = w_s n_k + w_d n_{3-k},\ k = 1, 2.$$

since n_k, n_{rk}, n_{sk}, $k = 1, 2$, and interaction information are unknown. The interaction information cannot be obtained in general situations (e.g., agent states are only observed at some time steps, or observations are corrupted by noise, as discussed in Remark 7).

3 Model Analysis

This section studies model behavior, and provides an explicit expression for the mean of the stationary distribution. Assumptions are summarized as follows.

Assumption 1
(i.1) The agent set $\mathcal{V}$ consists of two communities, $\mathcal{V}_1 = \{1, \dots, n_1\}$ and $\mathcal{V}_2 = \{n_1+1, \dots, n_1+n_2\}$ with $n_1, n_2 > 0$ and $n_1 + n_2 = n$.
(i.2) Both communities have regular agents, namely, $1 \leq n_{r1} \leq n_1$, $1 \leq n_{r2} \leq n_2$.
(ii) The interaction probability matrix W has a block structure (5) with $w_s, w_d > 0$, $w_s \neq w_d$, and

$$(n_1(n_1-1) + n_2(n_2-1))w_s + 2n_1n_2w_d = 2. \tag{7}$$

(iii) $X(0)$ is deterministic. It holds that $X^r(0) \in \mathcal{S}$ with

$$\mathcal{S} := \{x^r \in \mathbb{R}^{n_r} : x_i^r \in [\underline{s}, \overline{s}], 1 \leq i \leq n_r\}, \tag{8}$$

where $\underline{s} := \min_{1\leq i\leq n_s}\{\mathbf{x}_i^s\}$, $\overline{s} := \max_{1\leq i\leq n_s}\{\mathbf{x}_i^s\}$, $\mathbf{x}^s := X^s(0) = [(\mathbf{x}^{s1})^T\ (\mathbf{x}^{s2})^T]^T$ is the stubborn state vector, and $\mathbf{x}^{sk}$ is the vector for the community k, $k = 1, 2$.

Remark 3 *In Assumption 1 (i.1), the order of agents is sorted for convenience, but we do not know which group each agent belongs to, before community recovery. It is necessary to assume $w_s \neq w_d$. Otherwise, W has no block structure. Regular agents are assumed to start from $\mathcal{S}$, which is reasonable and intuitively means that regular states lie between the extreme stubborn states.*

Before studying model behavior, we explicitly write the block structures of $\bar{A} := \mathbb{E}\{A(t)\}$ and $\bar{B} := \mathbb{E}\{B(t)\}$ as follows. The block structure of W results in similar update rules for agents in the same community.

Proposition 1 *Suppose Assumption 1 holds. Then $\bar{A}$ and $\bar{B}$ have block structures given in (6).*

Now we provide the stability and limit theorems of the gossip model.

Theorem 1 *(Stability and limit theorems) Suppose that Assumption 1 holds and there exists at least one stubborn agent in the network (i.e., $n_r < n$). The following results hold for the gossip model with stubborn agents.*
(i) The model has a unique stationary distribution π with mean $\mathbf{x}^r$, and $X^r(t)$ converges in distribution to π.

(ii) The expectation of the state vector converges to $\mathbf{x}^r$:

$$\mathbf{x}^r = \lim_{t\to\infty} \mathbb{E}\{X^r(t)\} = (I - \bar{A})^{-1}\bar{B}\mathbf{x}^s. \tag{9}$$

(iii) Denote $S^r(t) := \frac{1}{t}\sum_{i=0}^{t-1} X^r(i)$, then

$$\lim_{t\to\infty} S^r(t) = \mathbf{x}^r \quad a.s., \tag{10}$$

Remark 4 *The first two results show that the agent states, though may not converge a.s., converge in distribution to a unique stationary distribution, and their expectations converge to the mean of the stationary distribution. The third result indicates that we can obtain the value of $\mathbf{x}^r$ by computing the state time average.*

The next proposition shows that $\mathbf{x}^r$ also has a block structure, indicating that regular agents in the same community behave similarly on average.

Proposition 2 *Under the conditions of Theorem 1, $\mathbf{x}^r$ given in (9) has the form*

$$\mathbf{x}^r = [\chi_1 \mathbf{1}_{n_{r1}}^T,\ \chi_2 \mathbf{1}_{n_{r2}}^T]^T, \tag{11}$$

where $\chi_k = (\gamma_{kk}\mathbf{1}_{n_{sk}}^T\mathbf{x}^{sk} + \gamma_{k,3-k}\mathbf{1}_{n_{s,3-k}}^T\mathbf{x}^{s,3-k})/\delta$, $\delta = w_s^2 n_{s1}n_{s2} + w_sw_d(n_1n_{s1} + n_2n_{s2}) + w_d^2(n_1n_2 - n_{r1}n_{r2})$, $\gamma_{kk} = w_s^2 n_{s,3-k} + w_sw_dn_k + w_d^2 n_{r,3-k}$, $\gamma_{k,3-k} = w_d\,(w_sn_{3-k} + w_dn_k)$, and $\mathbf{1}_{n_{sk}}^T\mathbf{x}^{sk} := 0$ if $n_{sk} = 0$, $k = 1, 2$.

Remark 5 *Appendix B of Xing et al. (2021) studies a multiple-community case, generalizing Proposition 2.*

The above proposition means that regular agents in the same community have the same limit, which is a weighted average of stubborn states. Hence it is possible to split regular agents by computing the state time average. However, we are unable to do so if only one community has stubborn agents, or the stubborn states are similar. The following condition rules out these cases.

Assumption 2 *Both communities have stubborn agents (i.e., $n_{s1}n_{s2} > 0$), and $\mathbf{x}^s = [(\mathbf{x}^{s1})^T\ (\mathbf{x}^{s2})^T]^T$ satisfies that $\mathbf{1}_{n_{s1}}^T\mathbf{x}^{s1}/n_{s1} \neq \mathbf{1}_{n_{s2}}^T\mathbf{x}^{s2}/n_{s2}$.*

This assumption has a practical meaning: stubborn agents are distributed among communities, and agents

Algorithm 1 (Joint Recovery and Estimation)

Input: $\{X^r(t), t = 0, 1, 2, \dots\}$, $X^s(0)$, step-size parameter a of the interaction estimator with $a > 0$.
Output: $\{\hat{\mathcal{C}}(i,t)\}$, $\hat{w}_s(t)$, $\hat{w}_d(t)$.

1: Randomize $\hat{\mathcal{C}}(i,0)$, $\hat{w}_s(0)$, $\hat{w}_d(0)$, set $S^r(0) = X^r(0)$.
2: **for** $t = 1, \dots$ **do**
3: Compute

$$S^r(t) = \frac{t}{t+1} S^r(t-1) + \frac{1}{t+1} X^r(t),$$
$$\bar{s}^r(t) = \frac{1}{n_r} \mathbf{1}_{n_r}^T S^r(t).$$

4: **Community recovery:**

$$\hat{\mathcal{C}}(i,t) = 2 - \mathbb{I}_{[S_i^r(t) > \bar{s}^r(t)]},\ i \in \mathcal{V}_r,$$
$$\hat{\mathcal{C}}(i,t) = \hat{\mathcal{C}}(j_i,t),\ i \in \mathcal{V}_s,$$

where j_i is defined in Assumption 3.

5: **Interaction estimation:**

$$\hat{w}_s(t) = \hat{w}_s(t-1) - \frac{a}{t}\mathrm{sgn}(g(t))\Big(g(t)\hat{w}_s(t-1) + \frac{h_2(t)}{\hat{n}_1(t)\hat{n}_2(t)}\Big),$$
$$\hat{w}_d(t) = \frac{2 - \hat{w}_s(t)(\hat{n}_1^2(t) + \hat{n}_2^2(t) - \hat{n}_1(t) - \hat{n}_2(t))}{2\hat{n}_1(t)\hat{n}_2(t)},$$

where

$$g(t) = h_1(t) - \frac{\hat{n}_1^2(t) + \hat{n}_2^2(t) - \hat{n}_1(t) - \hat{n}_2(t)}{2\hat{n}_1(t)\hat{n}_2(t)} h_2(t),$$
$$h_1(t) = \frac{|\hat{\mathcal{V}}_{s1}(t)|}{|\hat{\mathcal{V}}_{r1}(t)|} \textstyle\sum_{i \in \hat{\mathcal{V}}_{r1}(t)} S_i^r(t) - \sum_{i \in \hat{\mathcal{V}}_{s1}(t)} X_i^s(0),$$
$$h_2(t) = \frac{\hat{n}_2(t)}{|\hat{\mathcal{V}}_{r1}(t)|} \textstyle\sum_{i \in \hat{\mathcal{V}}_{r1}(t)} S_i^r(t) - \sum_{i \in \hat{\mathcal{V}}_{r2}(t)} S_i^r(t) - \sum_{i \in \hat{\mathcal{V}}_{s2}(t)} X_i^s(0),$$
$$\hat{n}_k(t) = \textstyle\sum_{i \in \mathcal{V}} \mathbb{I}_{[\hat{\mathcal{C}}(i,t) = k]},$$
$$\hat{\mathcal{V}}_{rk}(t) = \{i \in \mathcal{V}_r : \hat{\mathcal{C}}(i,t) = k\},$$
$$\hat{\mathcal{V}}_{sk}(t) = \{i \in \mathcal{V}_s : \hat{\mathcal{C}}(i,t) = k\},\ k = 1, 2.$$

6: **end for**

from different communities are more likely to have distinct opinions. Under Assumption 2, we have the following result, indicating that the presence of stubborn agents enhances the separation of regular agents.

Proposition 3 *Under the conditions of Theorem 1, $\chi_1 \neq \chi_2$ if and only if Assumption 2 holds.*

This result shows that Assumption 2 is a necessary and sufficient condition for regular agents from different communities having nonidentical expected stationary states. Note that $\mathbf{1}_{n_{s1}}^T \mathbf{x}^{s1}/n_{s1} \neq \mathbf{1}_{n_{s2}}^T \mathbf{x}^{s2}/n_{s2}$ is generic (i.e., it holds for almost all $\mathbf{x}^s \in \mathbb{R}^{n_s}$).

4 Joint Recovery and Estimation Algorithm

In this section, we design a joint recovery and estimation algorithm (Algorithm 1) to address the considered problem. We assume the following connections between stubborn and regular agents. The information means that we have prior knowledge about stubborn agents, which may be gathered from other sources in practice.

Assumption 3 *For every stubborn agent $i \in \mathcal{V}_s$, it is known for Algorithm 1 that there exists a regular agent $j_i \in \mathcal{V}_r$ such that i and j_i are in the same community (i.e, $\mathcal{C}(i) = \mathcal{C}(j_i)$).*

Now we are ready to introduce Algorithm 1, in which we denote the estimates at time t of community label $\mathcal{C}(i)$, interaction probabilities w_s and w_d, by $\hat{\mathcal{C}}(i,t)$, $\hat{w}_s(t)$, and $\hat{w}_d(t)$, respectively. We use $S_i^r(t)$ to represent the $(i - n_1 + n_{r1})$-th entry of $S^r(t)$, $i \in \mathcal{V}_2 = \{n_1 + 1, \dots, n_1 + n_{r2}\}$ for simplicity. Note that both n_1 and n_{r1} are unknown in the algorithm. In the gossip model, agents randomly interact and update states. Algorithm 1 partitions the agents and estimates interaction strength between agents, out of these state observations, without interaction information.

Remark 6 *The difficulty of recovery is to find a quantity revealing the community structure. Algorithm 1 exploits the trajectory data captured by $S^r(t)$ given in* (10)*. From Proposition 2 we know that the entries of $S^r(t)$ converge to two distinct values corresponding to the communities. Hence clustering methods (Line 4 of Algorithm 1, or other methods such as k-means) can be used. For estimation of interaction probabilities, the key is to find consistent parameter equations. Here we use the stationary property $\mathbf{x}^r = \bar{A}\mathbf{x}^r + \bar{B}\mathbf{x}^s$, giving the following equations of $(x\ y)^T$ (see Xing et al. (2021) for the details)*

$$\begin{cases}(n_{s1}\chi_1 - \mathbf{1}_{n_{s1}}^T \mathbf{x}^{s1})x + (n_2\chi_1 - n_{r2}\chi_2 - \mathbf{1}_{n_{s2}}^T \mathbf{x}^{s2})y = 0, \\ (n_1(n_1 - 1) + n_2(n_2 - 1))x + 2n_1 n_2 y = 2.\end{cases}$$

From (9)*,it has a unique solution under Assumptions 1 and 2, for fixed n_k, n_{rk}, and n_{sk}. But these quantities are unknown, so we leverage SA techniques to estimate them, as presented in Line 5 of Algorithm 1. Note that the algorithm does not need to know the averaging weight q.*

5 Convergence Analysis

This section studies the performance of Algorithm 1. We have the following result, meaning community recovery can be done in finite time, and the interaction probability estimates are convergent.

Theorem 2 *(Convergence of Algorithm 1)*
Under Assumptions 1-3, the following holds.
(i) The community recovery is achieved in finite time:

there exists a positive integer-valued random variable T such that $\hat{\mathcal{C}}(i,t) = \mathcal{C}(i)$, for all $i \in \mathcal{V}$ and $t > T$.
(ii) The interaction estimator converges a.s., namely,

$$\mathbb{P}\left\{\lim_{t\to\infty}(\hat{w}_s(t), \hat{w}_d(t)) = (w_s, w_d)\right\} = 1.$$

Remark 7 *Since Algorithm 1 uses the property* (10)*, it can also deal with situations where state observations are corrupted. For example, one cannot observe the whole trajectory but can only sample the states at some time steps. Ergodic property ensures that the time average of the sampled states still converges, if the sampling process is independent of the update, and the number of samples tends to infinity (Wai et al., 2016; Ravazzi et al., 2014). Another situation is that the observations are disturbed by i.i.d. noise with zero-mean and independent of the process. The law of large numbers guarantees that the influence of noise vanishes over time.*

Now we investigate the sample complexity of the community recovery, and the convergence rate of the interaction estimator. The following result is useful for studying the sample complexity of the recovery.

Lemma 1 *Consider a Markov chain $\{X(t)\}$ taking values on a compact state space $\mathcal{X}$ and having a unique stationary distribution π. For a function $f : \mathcal{X} \to \mathbb{R}$ and $\alpha := \int_{\mathcal{X}} f(x)\pi(dx)$, denote $g(x) := \sum_{t=0}^{\infty} \mathbb{E}\{f(X(t)) - \alpha | X(0) = x\}$, and the supremum of $|g|$ on $\mathcal{X}$ by $\|g\|_s := \sup\{|g(x)| : x \in \mathcal{X}\}$. If $\|g\|_s < \infty$, then, for all $\varepsilon > 0$ and $t > 2\|g\|_s/\varepsilon$, it holds for $S_f(t) := \frac{1}{t}\sum_{i=0}^{t-1} f(X(i))$ that*

$$\mathbb{P}\{|S_f(t) - \alpha| \geq \varepsilon\} \leq 2\exp\left\{-\frac{(t\varepsilon - 2\|g\|_s)^2}{2t\|g\|_s^2}\right\}.$$

Remark 8 *Similar concentration results to Lemma 1 have been obtained in the literature for other models. One class of results leverage Markov chain approaches and normally require stability such as uniform ergodicity (Glynn and Ormoneit, 2002; Paulin, 2015) or explicit bounds of the derivative of the initial measure with respect to the stationary measure (Fan et al., 2021). It is hard to derive these properties for Markov chains without continuous distributions (Gibbs and Su, 2002), as in our case. Another line of research studies concentration of Polyak averages, and contains step-size conditions (Mou et al., 2020), which cannot be applied to our problem either.*

Using the preceding lemma, we are able to compute how long it takes for the differences between entries of $S^r(t)$ and $\mathbf{x}^r$ to be small enough, such that agents in different communities have distinct state time averages. As a result, by noting Line 4 of Algorithm 1, we obtain a sample-complexity result for the community recovery. This theorem shows that the probability of recovering communities successfully at time t depends on the network, the interaction probabilities, and the stubborn-agent states. This probability tends to one as t goes to infinity.

Theorem 3 *(Sample complexity)*
Under the conditions of Theorem 2, for the community recovery step of Algorithm 1, it holds that, for $t > t_0$,

$$\mathbb{P}\{\hat{\mathcal{C}}(i,t) = \mathcal{C}(i), \forall i \in \mathcal{V}\} \geq 1 - 2n_r \exp\left\{\frac{-2(t-t_0)^2}{t_0^2 t}\right\},$$

with $t_0 = 4\delta c_{\bar{A}} c_{n_r} c_s / c_w$, where $c_{\bar{A}} = 1/(1-\rho(\bar{A}))$, $c_{n_r} = n_r^{3/2}(n_r+1)$, $c_s = \max\{|\underline{s}|, |\bar{s}|\}/|n_{s1}\mathbf{1}_{n_{s2}}^T\mathbf{x}^{s2} - n_{s2}\mathbf{1}_{n_{s1}}^T\mathbf{x}^{s1}|$, $c_w = |w_s^2 - w_d^2|$, δ is given in Proposition 2, and $\underline{s}$ and $\bar{s}$ are given in (8).

Remark 9 *This result provides a sample complexity characterization for recovering community from a single trajectory, and may be considered as a counterpart of the multiple-trajectory sample complexity given in Schaub et al. (2020). The parameter δ reflects the combined effect of the cardinality of stubborn and regular agents and the interaction probabilities. $c_{\bar{A}}$ captures the "speed" of information diffusion, and increases with $\rho(\bar{A})$. c_{n_r} depends on the number of regular agents. c_s increases with the range of the states and decreases with the difference of averaged stubborn states in different communities, and c_w measures the difference between interaction probabilities within and between communities. Smaller δ, $c_{\bar{A}}$, n_r, and $\max\{|\underline{s}|, |\bar{s}|\}$ would make the recovery easier, and so would larger c_w and $|n_{s1}\mathbf{1}_{n_{s2}}^T\mathbf{x}^{s2} - n_{s2}\mathbf{1}_{n_{s1}}^T\mathbf{x}^{s1}|$. For the gossip model over a graph sampled from an SBM, Example 1 indicates that the algorithm can recover most of the community labels, which is illustrated in Section 6.*

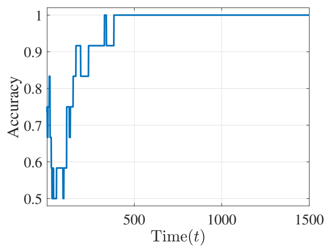
We have the following result for the convergence rate of the interaction estimation. It shows that the convergence rate also depends on the parameters of the gossip model, but a large enough step-size parameter a ensures that the estimator can achieve rate $O(1/\sqrt{t})$.

Theorem 4 *(Convergence rate)*
Under the conditions of Theorem 2, it holds for $d_0 \in [0, \min\{1/2, a|\eta|\})$ that

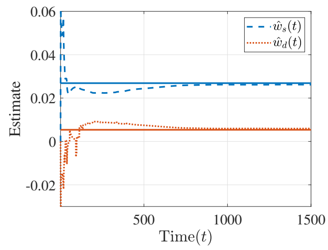
$$(\hat{w}_s(t) - w_s, \hat{w}_d(t) - w_d) = o(t^{-d_0}), \text{ a.s.},$$

where $a > 0$ is the step-size parameter given in Algorithm 1, $\eta = (w_s n_2 + w_d n_1)(n_{s1}\mathbf{1}_{n_{s2}}^T\mathbf{x}^{s2} - n_{s2}\mathbf{1}_{n_{s1}}^T\mathbf{x}^{s1})/(\delta n_1 n_2)$, and δ is given in Proposition 2.

Remark 10 *In the theorem, η increases with the combined effect of the number of agents and the interaction probabilities (i.e., $(w_s n_2 + w_d n_1)/\delta$) and with the disagreement of stubborn agents, and decreases with the cardinality of each community. When $a \geq 1/(2|\eta|)$, the interaction estimator achieves its optimal rate. Larger η provides a wider selection range. Simulation in Section 6 shows that the algorithm using a trajectory from the gossip model over a graph sampled from the SBM can still estimate the ratio of the link probabilities of the SBM.*



(a) Finite-time community recovery. The recovery is achieved after $t = 383$.



(b) Convergence of the interaction estimator. The solid (dashed) lines are true values (estimates).

Fig. 2. Performance of Algorithm 1.

6 Numerical Simulation

This section illustrates the performance of Algorithm 1, conducts an algorithm comparison, and applies Algorithm 1 to the SBM case and a real network.

To illustrate the performance of Algorithm 1 under Assumptions 1-3, consider a complete graph consisting of twelve agents. The two communities both have five regular agents and one stubborn agent. Set interaction probabilities be $w_s = 5/186$ and $w_d = 1/186$. The stubborn agent in community 1 (resp. community 2) has state 1 (resp. -1). The initial states of regular agents are drawn from uniform distribution on $(-1, 1)$. The averaging weight is set to be $q = 1/2$ in all experiments. Fig. 2(a) shows that Algorithm 1 recovers the communities in finite time, where the accuracy at time t is defined by $\frac{1}{n}(\max_{\sigma \in S_2}\{\sum_{i=1}^{n} \mathbb{I}_{[\sigma(\hat{\mathcal{C}}(i,t))=\mathcal{C}(i)]}\}) \in [0, 1]$. Here $\sigma : \{1, 2\} \to \{1, 2\}$ is a permutation function (to prevent a reverse distribution of labels), S_2 is the group of permutations on $\{1, 2\}$, $\mathcal{C}(i)$ is agent i's community label, $\hat{\mathcal{C}}(i, t)$ is the estimate of agent i's label at time t, and $n = 12$. Consistency of the interaction estimator with step-size parameter $a = 1$ is demonstrated in Fig. 2(b). These results validate Theorem 2.

We now show the sample complexity of the community recovery (Theorem 3) and compare the recovery step with the k-means, k-means++ (Arthur and Vassilvitskii, 2006), and spectral clustering methods (Abbe, 2017). This experiment considers the gossip model under Assumptions 1-3 with $n = 400$, $n_1 = 150$, and $n_{s1} = n_{s2} = 8$. Let $w_s/w_d = 5$ and solve the two parameters from (7). Let stubborn agents in community 1 (resp. community 2) have state 1 (resp. -1), and generate the initial states of other agents from uniform distribution on $(-1, 1)$. By running the algorithms for 200 times, we obtain the relative frequency that the algorithms recover all community labels, defined by $p_t := (\sum_{k=1}^{N} \max_{\sigma \in S_2}\{\mathbb{I}_{[\sigma(\hat{\mathcal{C}}_k(i,t))=\mathcal{C}(i), \forall i \in \mathcal{V}]}\})/N$, where $N = 200$ and $\hat{\mathcal{C}}_k(i, t)$ is the estimate of agent i's label at time t in the k-th run. After computing the time average $S^r(t)$, we use k-means and k-means++ with $k = 2$ instead of Line 4 of Algorithm 1, to recover communities. To implement the spectral clustering method, assume that edge activation is known, and use the activation information to estimate the interaction probability matrix W. Applying spectral clustering to estimates of W obtains estimates of communities. Fig. 3 shows that the probabilities of unsuccessful community recovery of all approaches tends to zero exponentially over time. The spectral clustering method performs better than other algorithms, because it directly uses interaction information, but the required time is still of the same order as the other algorithms. The k-means and k-means++ methods perform similarly to each other, and also similarly to Algorithm 1. This observation indicates that the major challenge of the problem is how to use agent states to recover communities without topological information.

We now consider the case where trajectories of the gossip model over graphs sampled from SBMs are given to Algorithm 1. We use three SBMs with size $n = 100, 300, 900$ and with two equal-sized communities ($\nu_1 = \nu_2 = 0.5$). Set $n_{r1} = n_{r2} = 0.45n$, and $n_{s1} = n_{s2} = 0.05n$. Let the link probability in the same community be $p_s = (\log n)^2/n$ and the link probability between different communities be $p_d = (\log n)/n$. For each SBM, we generate 20 graph samples. For each graph sample, we run Algorithm 1 for 20 times. Regular states are generated the same as earlier and stubborn agents in community 1 (resp. community 2) have state 1 (resp. -1). Fig. 4(a) shows that Algorithm 1 has high community recovery accuracy, increasing with n. This phenomenon results from the concentration discussed in Example 1. Algorithm 1 outputs $\hat{w}_s(t)$ and $\hat{w}_d(t)$ as estimates of the two distinct non-zero values of $\mathbb{E}\{\mathcal{W}\}/\mathbb{E}\{\alpha\}$. Note that $[cp_s, cp_d]$ defines the same $\mathbb{E}\{\mathcal{W}\}/\mathbb{E}\{\alpha\}$ for all $c > 0$, so we can only estimate the ratio p_s/p_d without knowing the expected number of edges of the SBM. Fig. 4(b) shows that the median of the estimation error for trajectory samples from each SBM is close to zero and decreases with n.

Zachary's karate club network (Zachary, 1977), presented in Fig. 5(a), is used to demonstrate an application of Algorithm 1. An edge represents frequent interaction between the two agents. The strength of interactions between agents is modeled by a weighted adjacency matrix. A conflict between agents 1 and 34 results in a

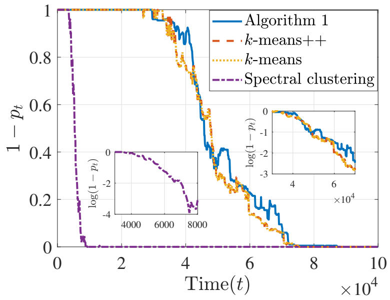


Fig. 3. Performance comparison of four methods.

fission of the club. In the experiment, we assume that only the opinions can be observed, instead of interactions between agents. The process is modeled by the gossip model. Agents 1 and 34 are set to be stubborn agents holding different opinions. In addition, one edge in Fig. 5(a) is selected at each time with a probability proportional to interaction strength given in Zachary (1977). The goal is to partition the agents into communities based on only state observations. The network structure departures from our assumptions, but the result shown in Fig. 5(b) indicates that as time increases, our algorithm can finally recover the community structure, without topological and interaction information.

7 Conclusion and Future Work

In this paper, we developed a joint algorithm to recover the community structure and to estimate the interaction probabilities for gossip opinion dynamics. It was proved that the community recovery is achieved in finite time, and the interaction estimator converges almost surely. We analyzed the sample complexity of the recovery and convergence rate of the estimator. Future work includes to study the case where all regular agents have the same stationary-state expectation, and to analyze the community detection problem for dynamics over the SBM.

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(a) Averaged accuracy of community recovery for each SBM.

(b) Median of estimation error of p_s/p_d for each SBM.

Fig. 4. Performance of Algorithm 1 using trajectories of the gossip model over sampled graphs from SBMs with $n = 100, 300, 900$.

(a) The community structure of Zachary's karate club network. Red squares and green triangles show two communities.

(b) Accuracy of community recovery of Algorithm 1 for the gossip model over Zachary's karate club network.

Fig. 5. Numerical experiment over Zachary's karate club network.

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