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Synchronization of quadratic integrate-and-fire spiking neurons: Constant versus voltage-dependent couplings

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Abstract— **This paper studies synchronization of a network of hybrid quadratic integrate-and-fire spiking neurons communicating over a complete graph and interconnected by means of bidirectional electrical couplings. Synchronization of the network of identical neurons with a common and constant coupling strength is studied using a Lyapunov-based argument for sufficiently large coupling strength. In addition, a voltage-dependent coupling law is proposed. It is assumed that each neuron is coupled to each of its neighbors by a coupling law which depends on the voltage of its neighboring neuron. For the voltage-dependent case, a sufficient condition for synchronization of two interconnected neurons is presented. Moreover, a comparison between the two mechanisms is given. Simulation results are provided to verify the theoretical analysis.**

I. Introduction

Oscillation is the fundamental function behind the operation of many complex networks, including biological and neural networks. An important feature of neural networks is the synchronized operation. Among factors which play a role in neuronal synchronization is the type of interconnections, *i.e.* chemical and electrical synaptic interactions [13]. Electrical synapses allow bidirectional interconnection between neurons. The role of these synaptic interconnections, which also exist in the mammalian brain [1], in synchronization of neural activities have been verified by experiments and analytically. Although electrical couplings do not exist alone between neurons, studying the effects of bidirectional couplings contribute to understanding of neural behaviors [12].

Neural behavior has been mainly studied (e.g. [3], [7]) using conductance-based models which are derived from the Hodgkin and Huxley model [6]. The quadratic integrate-and-fire neuron model ([4], [7]) is a reduced model to represent the detailed dynamics of any type I [7] conductance-based model near the onset of firing [12]. Synchronization of pulse coupled quadratic integrate-and-fire neurons has been studied in [11]. In this setting, each neuron, after firing, sends a pulse to its neighboring neurons which instantaneously adds a constant value to the state of the neighboring neuron. The results of [11] has shown a dichotomic collective behavior (either slow or fast firing) for quadratic integrate-and-fire neurons under an average monotonicity property. The stability of the asynchronous state for a network of identical quadratic integrate-and-fire neurons with symmetric electrical synapses in a fully connected network was studied in [12]. The analysis has been performed using a model of phase-coupled oscillators where phases represent the temporal deviation of voltage trajectories from each other.

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This paper studies synchronization of a network of quadratic integrate-and-fire neurons interconnected by bidirectional electrical couplings (electrical synapses). Considering the hybrid dynamics of each neuron, which is composed of a continuous-time evolution together with a discrete transition of states, the analysis of such a network is challenging. To achieve synchronization, we first present a sufficient condition to bring the maximum relative voltage to a sufficiently small value before the neuron with the maximum voltage spikes. Second, we propose an asymmetric interconnection law such that the coupling of each neuron to each of its neighbors is weighted by the voltage of the neighboring neuron. This asymmetric and dynamic coupling law interconnects each two neurons by injecting a larger coupling current to the neuron whose voltage level is smaller. We provide a comparison between these two mechanisms. Moreover, a sufficient coupling condition (a lower bound) for achieving asymptotic synchronization is presented for a network of two neurons coupled via the voltage-dependent law.
To the best of our knowledge, voltage-dependent coupling and its effects on the synchronization behavior of quadratic integrate-and-fire spiking neurons have not been studied before.

This paper is organized as follows. Section II presents preliminaries and the problem formulation. Section III studies synchronization in a complete graph with common and constant coupling strength. Synchronization in a voltage-dependent network is presented in Section IV. Section V presents simulation results and Section VI concludes the paper.

II. Preliminaries and Problem statement

This section first recalls preliminaries including the quadratic integrate-and-fire model. We then continue with presenting the problem statement.

A. Preliminaries

For a connected undirected graph $G(\mathcal{N}, \mathcal{E})$, the node-set $\mathcal{N}$ corresponds to n nodes and the edge-set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ corresponds to m edges. An undirected graph is called complete if there is an edge between each two nodes. The notation $\mathcal{N}_i$ denotes the set of neighboring nodes of node i.

Quadratic integrate-and-fire spiking neuron

The quadratic integrate-and-fire neuron is used as a reduced model of the detailed sub-threshold dynamics of a large class of neurons near the onset of periodic firing [4], [5]. The model obeys

$$\begin{aligned} c\dot{\bar{v}}_i &= \kappa(\bar{v}_i - V^*)^2 + \bar{I}_{ext} - I_c + I_{noise}, \\ \text{if} \quad \bar{v}_i &\geq \bar{V}_T, \quad \text{then} \quad \bar{v}_i \leftarrow \bar{V}_r < V^*, \end{aligned} \tag{1}$$

where $\bar{v}_i \in \mathbb{R}$ is the membrane potential of neuron i, c is the membrane capacitance, $\bar{V}_r < V^*$, V_T, and V^* are constant and represent the resting, instantaneous threshold, and desired potentials respectively. The parameter $\kappa > 0$ is a constant, $\bar{I}_{ext}$, and I_{noise} are the external and noise currents and I_c is a constant current. It is assumed that if $\bar{v}_i$ grows to $\bar{V}_T$, a mechanism is resetting $\bar{v}_i$ to $\bar{V}_r$.

B. Network model and Problem formulation

We consider an undirected and complete graph to model the underlying interconnection topology of a network of neurons modeled as quadratic integrate-and-fire spiking neurons and coupled via electrical synapses. Following [12], we rewrite (1) in terms of dimensionless variables $v_i = \frac{\kappa}{c}\tau(\bar{v}_i - V^*)$, $I_{ext} = \frac{\kappa}{c^2}\bar{I}_{ext}\tau^2$, $V_T = \frac{\kappa\tau}{c}(\bar{V}_T - V^*)$ and $V_r = \frac{\kappa\tau}{c}(\bar{V}_r - V^*)$. We also assume that $I_{noise} = 0$ and $\bar{I}_{ext} - I_c = I_{ext} > 0$. Then, each neuron dynamics follows

$$\begin{aligned} \tau\dot{v}_i &= v_i^2 + I_{ext} + I_s^i, \\ \text{if} \quad v_i &\geq V_T, \quad \text{then} \quad v_i \leftarrow -V_r, \end{aligned} \tag{2}$$

where $I_s^i = g(v_i - v_j)$ is the synaptic current, $\tau > 0$ is a time-constant and $0 \leq V_r < V_T$. Without loss of generality, we assume $\tau = 1$ in the rest of the paper.

Since the voltage of each neuron by reaching to V_T level is reset to $-V_r$, each neuron has a hybrid dynamics. The network dynamics is represented by the following hybrid system

$$\begin{aligned} \dot{v}_i &= v_i^2 - g\sum_{j\in\mathcal{N}_i}(v_i - v_j) + I_{ext}, \forall i \in \mathcal{N}, \\ \text{if } \exists\ v_i &\geq V_T, \text{ then} \quad v_i^+ = -V_r, v_j^+ = v_j. \end{aligned} \tag{3}$$

Our goal is to study relations between the model parameters and the coupling gain g on synchronization, *i.e.* simultaneous spiking [11], of the network with undirected and complete underlying topology and the node dynamics as in (3).

III. Synchronization with common and constant coupling

In this section, we study the network described in Section II assuming a constant and common coupling strength. All neurons are assumed to have identical V_T, V_r and exogenous current I_{ext}. Considering identical neurons, the only factor which affect the synchronized behavior is the differences in the initial conditions.

Assumption 1 The threshold voltages, V_T and $-V_r$, and external currents, $I_{ext} > 0$ are identical for all neurons. Moreover, $V_r < V_T$ and $v_i(0) \in [-V_r, V_T)$.

Example 1 (Two interconnected neurons) As an example, let us first consider a network composed of two neurons and take $\mathcal{V} = |v_1 - v_2|$ as the Lyapunov function candidate. Calculating $\dot{\mathcal{V}}$ during $[t_0, t_1)$, where t_1 is the time at which the maximum voltage reaches V_T, we obtain

$$\dot{\mathcal{V}} = |v_1 - v_2|(v_1 + v_2 - 2g), t \in [t_0, t_1). \tag{4}$$

The above derivative is always negative if $v_1 + v_2 < 2g$ holds for all $t \in [t_0, t_1)$. Since, each v_i takes a value between $-V_r$ to V_T, then $\dot{\mathcal{V}} < 0$ if $g > V_T$ holds.

The above implies that if g is sufficiently large, then the error dynamics dissipates energy and the maximum relative voltage is exponentially decreasing in the interval $[t_0, t_1)$. However, this does not guarantee synchronization [11]. We discuss as follows. In the above example, assume that $v_1(0) > v_2(0)$. Denote the time at which $v_1(t)$ reaches the threshold voltage V_T by t^1 and the relative voltage by $\delta^1 v = v_1 - v_2$.At time t^1, v_1 is updated to $-V_r$. The behavior of v_2 depends on the sign of $\dot{v}_2(t^k) = (V_T - \delta^1 v)^2 + I_{ext} - g(V_T - \delta^1 v + V_r)$.

Case I: $\dot{v}_2(t^k) > 0$ (e.g. small g). In this case, after the rest of v_1, v_2 still continues to grow. Depending on the size of $\dot{v}_2(t^k) > 0$, it could grow till reaching V_T. Therefore, neuron v_2 spikes after t^1. The upper bound of $t^2 - t^1$ depends on $\delta^1 v, I_{ext}$ and g.

Case II: With large g and if $\dot{v}_2(t^k) < 0$, then the coupling g is big enough to make v_2 moves towards v_1. In this case, the dynamics between $[0, t^1)$ will be repeated in the interval $[t^1, t^2)$ with the difference that now v_2 is the maximum trajectory. Hence, after $\delta^2 v = v_2 - v_1$ is small enough and hence the derivative of $\dot{v}_2$ will be positive despite the large coupling term, v_2 reaches V_T and spikes.

In both cases, with a sufficiently small relative voltage synchronization can be achieved.

A. Synchronization of n interconnected neurons

From Example 1, if g is sufficiently large, then the error dynamics dissipates energy and the maximum relative voltage is exponentially decreasing in any interval $[t^k, t^{k+1}]$. Thus, the trajectories of the system can be brought sufficiently close to each other before the neuron with the maximum voltage spikes. This allows synchronization [10].

In what follows, we present a sufficient condition on g to guarantee exponential stability of the network during continuous evolution. Then we prove absence of Zeno behavior and show that the trajectories keep order during the continuous flow. Thereafter, we formulate a trade off between V_T, I_{ext}, g and the initial conditions in order to bring voltage trajectories to ε closeness before the maximum neuron spikes.

Lemma 1 Denote the time at which $1 \leq m \leq n$ neurons reach V_T by t^k. The time interval between each two consecutive updates, *i.e.*, $t^{k+1} - t^k$, of the state of each neuron with dynamics in (3) is bounded from below.

Proof: Considering the finite number of neurons, at each jump (update) the number of updated states is smaller

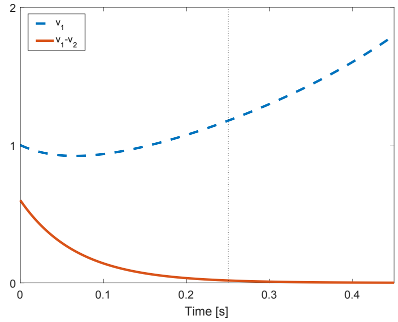

Fig. 1. The growth of the trajectory of the maximum voltage versus the decay of the maximum relative voltage.

or equal to the number of all nodes n. Since the depth $V_T + V_r$ is always positive, the voltage corresponding to each neuron should evolve from $v_i(t^k) < V_T$ to V_T with a bounded velocity smaller than $v_T^2 + I_{ext} + g(n-1)\Delta_{\max}^v$, where $\Delta_{\max}^v < V_T + V_r$ denotes the maximum relative voltage. Hence, the interval between any two jumps is bounded with $\delta t \geq \frac{(V_T - \max\{v_i(t^k)\})}{V_T^2 + I_{ext} + g(n-1)(V_T+V_r)}$. Notice that at each jump, more than one neuron can be updated simultaneously [8], since the number of nodes is finite, the network dynamics will evolve after the update. □

Lemma 2 For the network modeled in (3), assume that $ng > 2V_T$. If $v_i(t^k) - v_j(t^k) \geq 0$ holds for each two nodes i, j at t^k, then $v_i(t) - v_j(t) \geq 0$ holds for $\forall t \in [t^k, t^{k+1})$.

Proof: Consider $v_i(t^k) - v_j(t^k) \geq 0$, writing the derivative of the difference gives

$$\dot{v}_i(t) - \dot{v}_j(t) = (v_i(t) - v_j(t))(v_i(t) + v_j(t) - ng).$$

To have $v_i(t) < v_j(t)$, the error dynamics should cross zero which is an stable equilibrium (for example take $\mathcal{V} = (v_i - v_j)^2$ as the Lyapunov function candidate) for the above dynamics if $ng > 2V_T$ (since $v_i \in [-V_r, V_T)$) which ends the proof. □

As shown in Lemma 1, the relative voltage dynamics is independent of the exogenous current I_{ext}. On the other hand, the speed of spiking depends on I_{ext}. Hence, in order to achieve a small enough error for relative voltage before the maximum neuron spikes, there should be a trade off between the speed of decay of relative voltages and the speed of growth of the maximum trajectory for the given initial conditions. Figure III-A shows the plot of the maximum relative voltage error together with the growth of the maximum voltage trajectory related to one neuron. The following result present a relation between g, I_{ext}, V_T, V_r and the initial conditions.

Proposition 1 For the network with node dynamics as in (3), if the following condition holds

$$\frac{\ln(\frac{v_{\max}(t^k) - v_{\min}(t^k)}{\varepsilon})}{ng - 2V_T} < \frac{1}{\sqrt{I_{ext}}}[\tan^{-1}(\frac{V_T}{\sqrt{I_{ext}}}) - \tan^{-1}(\frac{v_{\max}(0)}{\sqrt{I_{ext}}})] \tag{5}$$

then the maximum relative voltage within the time interval $[t^k, t^{k+1})$ decreases to ε, where ε is a design choice, $v_{\max}(t^k) = \max_i v_i(t = t^k)$, and $v_{\min}(t^k) = \max_i v_i(t = t^k)$.

Proof: Take $\mathcal{V} = \max\{|v_i - v_j| |(i,j) \in \{1, \ldots, n\}\}$ as the Lyapunov function function candidate. Notice that, based on Lemma 2, if $v_i(t^k) \geq v_j(t^{k+1})$ holds, then $v_i(t) \geq v_j(t)$ holds for $t \in [t^k, t^{k+1})$. Thus, $|v_i - v_j| = v_i - v_j$. Since our Lyapunov function candidate is not necessarily continuously differentiable due to the $\max$ operator, we consider its upper Dini derivative $D^+\mathcal{V}$ (see [2]) along solutions of (3) for $t \in [t^k, t^{k+1})$. Hence, $D^+\mathcal{V} = \max(\dot{v}_i - \dot{v}_j)$. Since the graph is complete, the nodes with maximum (minimum) initial voltage, have identical voltage level, we denote the maximum and minimum levels by $v_{\max}$ and $v_{\min}$ respectively. Calculating the latter during $[t^k, t^{k+1})$ where t^{k+1} is the time at which the maximum voltage trajectory reaches V_T (considering the initial times, $t^0 = 0$ and t^1 is the time at which the first neuron spikes), we obtain

$$\begin{aligned} \dot{\mathcal{V}} &= (v_{\max} - v_{\min})(v_{\max} + v_{\min} - ng), \ t \in [t^k, t^{k+1}), \\ &= (v_{\max} + v_{\min} - ng)\mathcal{V}, \ t \in [t^k, t^{k+1}). \end{aligned} \tag{6}$$

The above derivative is negative if $\max(v_i + v_j) < ng$ holds. Since, each v_i takes a value between $-V_r$ to V_T, then $\dot{\mathcal{V}} < 0$ if $g > \frac{2}{n}V_T$. We argue that when the maximum relative voltage is sufficiently small, the role of coupling in each neuron dynamics is negligible. Thereafter, the un-controlled single neuron dynamics is dominant, *i.e.*, the exogenous current I_{ext} will determine the spiking behavior. Thus, the network synchronizes if the error dynamics dissipates sufficiently small before the time at which the neuron with maximum trajectory spikes. Take $V_T^2 + I_{ext}$ as the velocity of the maximum trajectory and denote is spiking time with t_M^k. We obtain

$$t_M^k - t^k \geq \frac{1}{\sqrt{I_{ext}}}[\tan^{-1}(\frac{V_T}{\sqrt{I_{ext}}}) - \tan^{-1}(\frac{v_{\max}(0)}{\sqrt{I_{ext}}})].$$

Also, since the relative voltage is exponentially decreasing, $\forall t \in [t^k, t^{k+1})$ we obtain the upper bound of the relative voltage as follows

$$v_{\max}(t) - v_{\min}(t) \leq e^{(2V_T - ng)(t - t^k)}(v_{\max}(t^k) - v_{\min}(t^k)). \tag{7}$$

Considering a desired ε, we denote the time at which it can be obtained from (7) with

$$t_\varepsilon^k - t^k \geq \frac{\ln(\frac{v_{\max}(t^k) - v_{\min}(t^k)}{\varepsilon})}{ng - 2V_T}.$$

If $t_M^k > t_\varepsilon^k$ holds, then the maximum relative voltage of neurons is smaller than a given ε within the time interval $[t^k, t^{k+1})$ and before the maximum neuron spikes. Thus, (5) should hold. $\square$

IV. Voltage-dependent interconnections

As shown in the previous section, the synchronization of the network of quadratic integrate and fire neurons depends on multiple parameters, e.g. the size of the network, the relations between the exogenous excitation (current), coupling strength, threshold voltages and the initial conditions. Due to the hybrid dynamics, resetting the voltage of each neuron will lead to a discontinuous change in the dynamics of all of its neighboring neurons by changing the corresponding synaptic current. This motivates studying coupling rules which bring voltage trajectories closer during the continuous evolution, thus reaching smaller relative error before the maximum neuron spikes, and mitigating the effects of resting the voltage of one neuron on the dynamics of its neighbors. Here, we propose a bidirectional voltage dependent coupling control law and discuss its mechanism in Section IV-A. We assume that the current neuron i dynamics receives from neuron j is equal to $\bar{g}v_j(v_i - v_j)$, hence the coupling is voltage dependent, *i.e.* $\bar{g}v_j$ (see Remark 2). Hence, the dynamics of each neuron obeys

$$\begin{aligned} \dot{v}_i &= v_i^2 - \bar{g}\sum_{j\in\mathcal{N}_i} v_j(v_i - v_j) + I_{ext}, \forall i \in \mathcal{N}, \\ &\text{if } \exists\ v_i \geq V_T, \text{ then } \quad v_i^+ = -V_r, v_j^+ = v_j. \end{aligned} \tag{8}$$

Assumption 2 The lower voltage threshold V_r is equal to zero.

We continue with obtaining coupling condition under which the maximum relative voltage is decreasing within any interval of continuous evolution of the voltages.

Proposition 2 If $\bar{g} > 1$, then $v_i(t^k) - v_j(t^k) \geq 0$ for each two nodes i, j at t^k guarantees $v_i(t) - v_j(t) \geq 0$ for $\forall t \in [t^k, t^{k+1})$. Moreover, for the network with complete graph topology and the node hybrid dynamics as in (8) the maximum relative voltage is exponentially decreasing during each interval $[t^k, t^{k+1})$ provided that $\bar{g} > 1$ holds.

Proof: Consider the dynamics in (8) in the interval $\forall t \in [t^k, t^{k+1})$. Denote $\dot{v}_i$ by $f_i(v_i)$. Since $f_i(0) \geq 0$, then the dynamics of neuron i during the continuous evolution is a positive system. Hence, by Assumption 2, $v_i \geq 0, \forall i, \forall t \in [t^k, t^{k+1})$. For each two nodes i and j, with $v_i(t^k) > v_j(t^k)$, we have

$$\dot{v}_i(t) - \dot{v}_j(t) = (v_i(t) - v_j(t))((1-\bar{g})(v_i + v_j) - \sum_{k\neq i,j} v_k),$$

$\forall t \in [t^k, t^{k+1})$. If $\bar{g} > 1$, the relative voltage is decreasing. Hence, $v_i(t) - v_j(t) \geq 0$ for $\forall t \in [t^k, t^{k+1})$ holds.

Now, Take $\mathcal{V} = \max\{(v_i - v_j)|(i,j) \in \{1,\ldots,n\}\}$ as the Lyapunov function candidate. The function's upper Dini derivative is $D^+\mathcal{V} = \max(\dot{v}_i - \dot{v}_j)$. We obtain

$$\begin{aligned} \dot{\mathcal{V}} =& (v_{\max} - v_{\min})[(v_{\max} + v_{\min}) - \bar{g}\sum_{i=1}^n v_i], \\ =& -(v_{\max} - v_{\min})[(\bar{g}-1)\sum_{i=1}^n v_i + \sum_{\substack{k \\ v_k\neq \max v \\ v_k \neq \min v}} v_k]. \end{aligned} \tag{9}$$

Based on the above, the trajectories of the network with the node dynamics in (8) do not leave the positive orthant. Thus, the sum of nodal voltages is always non-negative. Thus, if $\bar{g} > 1$ holds, $\dot{\mathcal{V}}$ is negative and the relative voltage dynamics is exponentially decreasing in any interval $[t^k, t^{k+1})$. $\square$

Remark 1 (Interpretation of $\bar{g} > 1$) Recall from Section II that we consider a dimension-less model for the neurons.

Remark 2 (Synaptic current in the form of $I_s^i = \bar{g}v_i(v_i - v_j)$) Considering the voltage-dependent interactions, another variation is the case where $I_s^i = \bar{g}v_i(v_i - v_j)$ which creates a Lotka-Volterra-based structure [9] with input. The analysis of this case is beyond the scope of the current paper. A numerical example related to this case is included in Section V in order to compare the behavior of the networks with two types of voltage-dependent interconnections, *i.e.* $I_s^i = \bar{g}v_i(v_i - v_j)$ and $I_s^i = \bar{g}v_j(v_i - v_j)$.

Lemma 3 The time interval between each two consecutive updates, *i.e.*, $t^{k+1} - t^k$, of the state of each neuron with dynamics as in (8) is bounded from below.

Proof: The proof follows a similar trend as the proof of Lemma 2. For this system, the interval between any two jumps is lower bounded by $\delta t \geq \frac{(V_T - \max\{v_i(t^k)\})}{V_T^2 + I_{ext} + \bar{g}(n-1)V_T(V_T)}$. $\square$

Based on the above results, if the trajectories are ε close then a similar argument as in Section VI follows. In what follows, we present a comparison of convergence mechanisms between two networks, one with constant coupling, denoted by g, and the other with voltage-dependent coupling, denoted by $\bar{g}v_i$, for a network composed on two nodes.

A. Comparison between constant and voltage-dependent couplings

To guarantee the stability of the relative voltage dynamics, the symmetric and constant coupling strength requires $g > \frac{2V_T}{n}$, which implies dependency of the coupling strength on the threshold voltage and the size of the network, however, the asymmetric voltage-dependent coupling requires $g > 1$. The latter is a local measure independent of the network size and threshold voltages. Here, we compare the dynamics of the difference and the sum of the derivatives of the maximum and minimum voltages for a two-node network for the two design cases. For the constant and symmetric network, we have

$$\begin{aligned} (\dot{v}_{\max} + \dot{v}_{\min}) &= (v_{\max}^2 + v_{\min}^2) + 2I_{ext}, \\ (\dot{v}_{\max} - \dot{v}_{\min}) &= (v_{\max}^2 - v_{\min}^2) - 2g(v_{\max} - v_{\min}). \end{aligned} \tag{10}$$

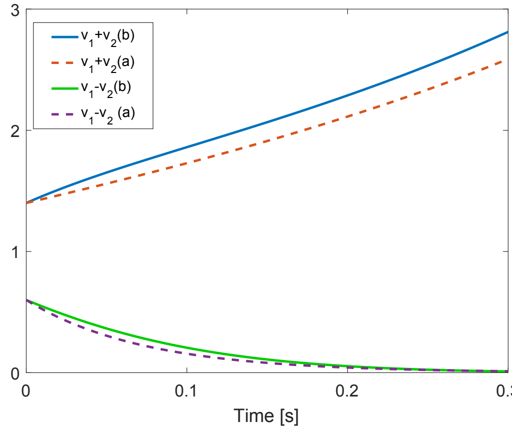

Fig. 2. The growth of the sum and the decay of the relative voltages for a two node graph for constant (a) and voltage-dependent(b)networks.

while the asymmetric and voltage-dependent network gives

$$(\dot v_{\max}+\dot v_{\min}) = (v_{\max}^2+v_{\min}^2)+\bar g(v_{\max}-v_{\min})^2+2I_{ext},$$
$$(\dot v_{\max}-\dot v_{\min}) = -(\bar g-1)(v_{\max}^2-v_{\min}^2). \tag{11}$$

Based on (10) and (11), the rate of growth of the sum of trajectories with the constant coupling is independent of the coupling, however, the rate of sum for the voltage-dependent case is largest when the relative voltage is maximum. This shows that with $g=\bar g$ the voltage-dependent coupling law bring trajectories closer together. In fact, in the voltage-dependent case, $\dot v_{\max} < \dot v_{\min}$. The coupling law adds more to the minimum trajectory and deduct less from the maximum voltages. This is different from the constant coupling which is a balanced coupling law, *i.e.* the added term to the minimum trajectory is equal to the deduction from the maximum trajectory. In addition, after the spike of the maximum neuron, the coupling current of the minimum one jumps from $-\bar g V_T(V_T-v_{\min})$ to zero. For the constant coupling case, the change of coupling current from $-g(V_T-v_{\min})$ to $g(v_{\min})$. Considering that a smaller coupling strength is needed for stability of relative dynamics and that $v_{\min}$ before jump is close to V_T, the jump in the voltage-dependent case is less disturbing, *i.e.* from $V_T\delta$ to zero compared to $-V_T\delta$ to $V_T v_{\min}$, with $\bar g>1$ and $g>V_T$ respectively. Hence, the mechanism of achieving synchronization is different for these two coupling laws. Figure IV-A shows the plot of the evolution of the sum and relative voltages for voltage-dependent and constant couplings for identical neurons with similar $V_T, V_r, g=\bar g, \Delta_0^v$. We now study a sufficient condition under which two interconnected neurons with the dynamics as in (8) achieve synchronization asymptotically.

Proposition 3 Assume $v_{\min}(0)=0, 0<v_{\max}(0)<\frac{V_T}{2}$. The network of two interconnected neurons with the dynamics as in (8) achieves synchronization asymptotically if $\bar g>1$ and $\bar g V_T^2>I$.

Proof: As proved in Proposition 2, $\bar g>1$ guarantees that if $v_1(0)>v_2(0)$, then the order is preserved for all time before v_1 spikes. Take the ratio $\kappa=\min\frac{\dot v_{\min}}{\dot v_{\max}}$. If $\bar g V_T^2>I$, the latter could be bounded by $\bar g$. Now we calculate the error $d(t_{k-1}^-)=V_T-v_{\max}(t_{k-1}), k\in\{0,1,2,..\}$. Based on this we estimate $v_{\min}(t_k^-)=v_{\max}(t_k^+)$ as $v_{\max}(t_k^+)=\kappa d(t_{k-1})$, where $d(t_0)=V_T-v_1(0)$, $d(t_1)=V_T-v_2(1)$, and $d(t_2)=V_T-v_1(2)$, etc. Now, we calculate the error $e_1=v_1(t_2^+)-v_1(0)=\kappa d(t_1)-v_1(0)$. We obtain $e_1=\kappa(V_T-\kappa(V_T-v_1(0))-v_1(0)$. By $\kappa>1$ ($\bar g>1$), and $v_1(0)<\frac{V_T}{2}$, we obtain $e_1<0$ which implies that the error is deceasing, *i.e.* the initial condition at time t^2 where v_1 is the maximum trajectory is smaller than time zero. Hence the error asymptotically converges to zero, which means that synchronization is achieved. □

Numerical comparison

To provide a comparison related to the above result, we simulate two networks, one with constant and the other with voltage dependent couplings. Our numerical verification on two networks with constant and voltage dependent coupling, with similar initial conditions $v_{\max}(0)=0.9$, $v_{\min}=0$, and $V_T=2$, $I=4$ For $g=\bar g=1.2$ both networks synchronize. However, the network with constant coupling de-synchronize for $4<g<15$, while the voltage dependent case synchronize for all $\bar g>1.2$.

V. Simulations

This section presents the simulation results for a complete graph with 6 nodes. The neurons are identical and the model parameters are set to $V_T=2$, $-V_r=-0.2$, and $I_{ext}=4$. The initial condition is set to $[1.2;1;0.5;0.2;0;-0.1]$. For a network with common and constant coupling, with $g>0.66$, the maximum relative voltage is decreasing. To achieve a sufficiently small error, $\varepsilon=0.01$, the condition in Proposition 1 gives $g>9.6$. Figure 3 shows the voltage trajectories for $g=10$. As shown, synchronization is achieved. The results of the voltage-dependent case is shown in Figure 4 with $\bar g=2.8$. To compare the results of Section IV with the design in Remark 2, Figure 5 shows the results of the design in Remark 2. As shown, both networks synchronize. However, the network with the model in Remark 2 de-synchronizes for $g<2.8$, while the network analyzed in Proposition 2 (with $I_i^s=v_j(v_i-v_j)$) still synchronizes for $\bar g\geq 1.1$. The results of $g<2.8$ are not shown due to space constraints.
Figure 6 shows the voltage trajectories for the network with constant coupling with $g=2.8$. As shown, the network does not achieve synchronization.

VI. Conclusions

This paper has studied synchronization of a network of hybrid quadratic integrate-and-fire neurons over a complete graph topology. Conditions for achieving sufficiently small relative voltage error before the spike of the maximum neuron (the neuron whose voltage is greater than other neurons) has been obtained for a network with constant coupling. In addition, a voltage-dependent coupling law has been proposed and studied. This design represents an

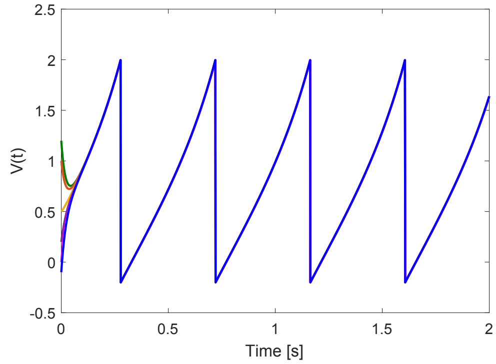

Fig. 3. Voltage trajectories for the network with constant and common coupling with $g = 10$.

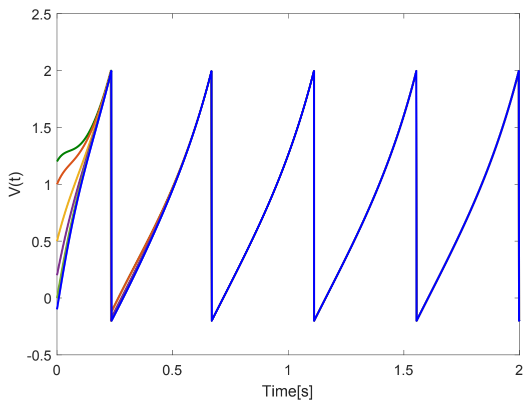

Fig. 4. Voltage trajectories for the network with voltage-dependent coupling with $\bar{g} = 2.8$.

asymmetric and dynamic coupling law which injects a larger coupling current to the neuron with smaller voltage level. A comparison between mechanisms of the two coupling laws (constant and voltage-dependent) has been provided. In addition, a sufficient condition for achieving asymptotic synchronization has been presented for two interconnected neurons with the voltage-dependent law and the results has been numerically compared with the constant coupling law.

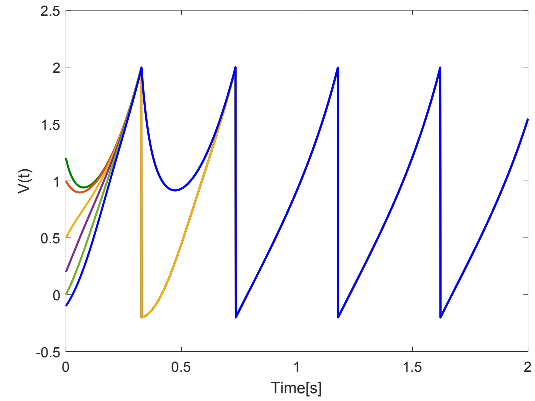

Fig. 5. Voltage trajectories for the network in Remark 2 with $\bar{g} = 2.8$.

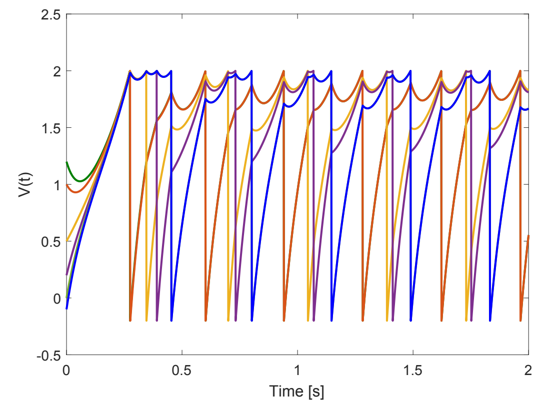

Fig. 6. Voltage trajectories for the network with constant and common coupling with $g = 2.8$.

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